ω - $\mathcal{N}$ - α -open sets and ω - $\mathcal{N}$ - α -continuity in bitopological spaces

Carlos Granados^{a,*}

^aCorporación Universitaria Latinoamericana, Colombia.

Abstract

The main idea of this article is to introduce and characterize the notions of (i, j) - ω - $\mathcal{N}$ - α -open sets in bitopological spaces. It also introduces and studies the concept of (i, j) - ω - $\mathcal{N}$ - α -continuous functions. Moreover, (i, j) - ω - $\mathcal{N}$ - α -connected and (i, j) - ω - $\mathcal{N}$ - α -set-connected functions are defined in bitopological spaces and some of their properties are shown. ©2019 All rights reserved.

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1. Introduction and preliminaries

The concept of a bitopological spaces is was introduced by Kelly [5]. Besides, Rajesh and Mustafa [8] defined and studied (i, j) - $\mathcal{N}$ - α -open sets and (i, j) - $\mathcal{N}$ - α -continuous functions in bitopological spaces. Furthermore, The notion of ω -closed set is was introduced by Hdeib [2] and studied some properties. On the other hand, many authors have studying the ω -closed sets in bitopological spaces. Recently, Carpintero et al. [1] have introduced the concept of (i, j) - ω -semi open sets, as well as, Hussein et al. [3] where they studied the notion of pairwise $\omega\beta$ -continuous functions.

In this paper, motivated by the authors mentioned in the last paragraph, it took those ideas and it defines (i, j) - ω - $\mathcal{N}$ - α -open sets and (i, j) - ω - $\mathcal{N}$ - α -continuous functions in bitopological spaces where it obtains many properties. Besides, (i, j) - ω - $\mathcal{N}$ - α -connected and (i, j) - ω - $\mathcal{N}$ - α -set-connected functions are defined, it also proves and investigates some of their properties.

Throughout this paper, the space (X, τ_i, τ_j) or simply X always means a bitopological spaces on which no separation axioms area assumed unless otherwise mentioned.

Definition 1.1. A subset A of (X, τ) is said to be ω -closed [2] if it contains all of its condensation points. The complement of a ω -closed is called ω -open.

Definition 1.2. A subset A of (X, τ_i, τ_j) is (i, j) - α -open [4] if $A \subset \text{Int}_{\tau_i}(\text{Cl}_{\tau_j}(\text{Int}_{\tau_i}(A)))$.

Definition 1.3. A subset A is said to be (i, j) - $\mathcal{N}$ - α -open [8] if for each $x \in A$ there exists an (i, j) - α -open set U_x containing x such that $U_x - A$ is a finite set. The complement of an (i, j) - $\mathcal{N}$ - α -open set is called (i, j) - $\mathcal{N}$ - α -close set.

*Corresponding author

Email address: carlosgranadosortiz@outlook.es (Carlos Granados)

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Definition 1.4. A function $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ is said to be (i, j) - $\mathcal{N}$ - α -continuous [8] if the inverse image of every σ_i -open set of Y is (i, j) - $\mathcal{N}$ - α -open set of X , where $i \neq j$.

Definition 1.5. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function. Then, f is said to be (i, j) - α -continuous [4] if $f^{-1}(V)$ is (i, j) - α -open set of X for every σ_i -open set V of Y .

Theorem 1.6. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function. Then, f is said to be pairwise continuous [6] if the induced functions $f_i : (X, \tau_i) \rightarrow (Y, \sigma_i)$ and $f_j : (X, \tau_j) \rightarrow (Y, \sigma_j)$ are both continuous.

2. (i, j) - ω - $\mathcal{N}$ - α -open sets

In this section, it defines (i, j) - ω - α -open sets and it shows some properties.

Definition 2.1. Let (X, τ_i, τ_j) be a bitopological space and $A \subseteq X$. Then A is said to be (i, j) - ω - $\mathcal{N}$ - α -open if for each $x \in A$ there exists an (i, j) - $\mathcal{N}$ - α -open U_x containing x such that $U_x - A$ is a countable set. The complement of an (i, j) - ω - $\mathcal{N}$ - α -open set is an (i, j) - ω - $\mathcal{N}$ - α -closed set.

Remark 2.2. The collection of all (i, j) - ω - $\mathcal{N}$ - α -open sets and (i, j) - ω - $\mathcal{N}$ - α -closed sets are denoted by $\omega\alpha\mathcal{NBO}(X)$ and $\omega\alpha\mathcal{NBC}(X)$.

Lemma 2.3. Every (i, j) - $\mathcal{N}$ - α -open set is (i, j) - ω - $\mathcal{N}$ - α -open set.

Proof. It follows from the Definition 2.1. □

Remark 2.4. The converse of the Lemma 2.3 need not be true as can be seen in the following example:

Let $X = \{1, 2, 3\}$, $\tau_i = \{\emptyset, X, \{1, 2\}\}$, $\tau_j = \{\emptyset, X, \{2, 3\}\}$. Then, $\{1, 3\}$ is an (i, j) - ω - $\mathcal{N}$ - α -open set, but it is not an (i, j) - $\mathcal{N}$ - α -open set.

Lemma 2.5. Let A and Y be subsets of (X, τ_i, τ_j) such that $A \subseteq Y$. If A is (i, j) - ω - $\mathcal{N}$ - α -open set of X , then A is (i, j) - ω - $\mathcal{N}$ - α -open set of $(Y, \tau_i|_Y, \tau_j|_Y)$.

Proof. Let A be an (i, j) - ω - $\mathcal{N}$ - α -open set of X , for every $x \in A$, there exists an (i, j) - ω - $\mathcal{N}$ - α -open set U of X containing x such that $U - A$ is a countable. In consequence, it has that U is (i, j) - ω - $\mathcal{N}$ - α -open set of $(Y, \tau_i|_Y, \tau_j|_Y)$ containing x . This is shown that A is (i, j) - ω - $\mathcal{N}$ - α -open set of $(Y, \tau_i|_Y, \tau_j|_Y)$. □

Theorem 2.6. Let (X, τ_i, τ_j) be a bitopological space and $A \subseteq X$. Then A is said to be (i, j) - ω - $\mathcal{N}$ - α -open if and only if for every $x \in A$, there exists a (i, j) - $\mathcal{N}$ - α -open set U_x containing x and a countable subset B such that $U_x - B \subseteq A$.

Proof. Let A be an (i, j) - ω - $\mathcal{N}$ - α -open set and $x \in A$, then there exists an (i, j) - $\mathcal{N}$ - α -open subset U_x containing x such that $U_x - A$ is countable. Now, let $B = U_x - A = U_x \cap (X - A)$. Then, $U_x - B \subseteq A$. Conversely, let $x \in A$. Then, there exists an (i, j) - ω - $\mathcal{N}$ - α -open subset U_x containing x and a countable subset B such that $U_x - B \subseteq A$. Therefore, $U_x - A \subseteq B$ and $U_x - A$ is countable. □

Definition 2.7. Let $\{U_\delta : \delta \in \Delta\}$ a collection of (i, j) - $\mathcal{N}$ - α -open sets in a bitopological space X is called an (i, j) - $\mathcal{N}$ - α -open cover of a subset A of X if $A \subseteq \bigcup_{\delta \in \Delta} U_\delta$.

Definition 2.8. Let (X, τ_i, τ_j) be a bitopological space. Then, X is said to be (i, j) - $\mathcal{N}$ - α -Lindeloff, if every (i, j) - $\mathcal{N}$ - α -open cover of X has a countable sub-cover.

Theorem 2.9. Let (X, τ_i, τ_j) be a bitopological space. Then, the following statements are equivalent:

1. X is (i, j) - $\mathcal{N}$ - α -Lindeloff.
2. Every countable cover of X by (i, j) - $\mathcal{N}$ - α -open sets has a countable sub-cover.

Proof. (2) $\Rightarrow$ (1) : Since every (i, j) - $\mathcal{N}$ - α -open set is (i, j) - $\omega\mathcal{N}$ - α -open set, the proof follows.

(1) $\Rightarrow$ (2) : Let $\{U_\delta : \delta \in \Delta\}$ be a cover of X by (i, j) - $\omega\mathcal{N}$ - α -open sets of X . Now, for each $x \in X$ there exists an $\delta_x \in \Delta$ such that $x \in U_{\delta_x}$. Since U_{δ_x} is an (i, j) - $\omega\mathcal{N}$ - α -open. Then, there exists an (i, j) - $\mathcal{N}$ - α -open set V_{δ_x} such that $x \in V_{\delta_x}$ and $V_{\delta_x} - U_{\delta_x}$ is countable. Then, the family $\{V_\delta : \delta \in \Delta\}$ is an (i, j) - $\mathcal{N}$ - α cover of X and X is (i, j) - $\mathcal{N}$ - α -Lindeloff. Therefore, there exists a countable sub-cover δ_{x_i} with $i \in I$ such that $X = \bigcup_{i \in I} V_{\delta_{x_i}}$. Since $X = \bigcup_{i \in I} [(V_{\delta_{x_i}} - U_{\delta_{x_i}}) \cup U_{\delta_{x_i}}] = \bigcup_{i \in I} [(V_{\delta_{x_i}} - U_{\delta_{x_i}}) \cup \bigcup_{i \in I} U_{\delta_{x_i}}]$. Since $V_{\delta_{x_i}} - U_{\delta_{x_i}}$ is a countable set, for each $\delta(x_i)$, there exists a countable subset $\Delta_{\delta(x_i)}$ of Δ such that $V_{\delta_{x_i}} - U_{\delta_{x_i}} \subseteq \bigcup_{\delta \in \Delta_{\delta(x_i)}} U_\delta$

and therefore $X \subseteq \bigcup_{i \in I} (\bigcup_{\delta \in \Delta_{\delta(x_i)}} U_\delta) \cup (\bigcup_{i \in I} U_{\delta_{x_i}})$. $\square$

Theorem 2.10. Let (X, τ_i, τ_j) be a bitopological space and $C \subseteq X$. If B is (i, j) - $\omega\mathcal{N}$ - α -closed set. Then, $C \subseteq J \cup B$, for some (i, j) - $\omega\mathcal{N}$ - α -closed subset J and a countable subset B .

Proof. If C is (i, j) - $\omega\mathcal{N}$ - α -closed set. Then, $X - C$ is (i, j) - $\omega\mathcal{N}$ - α -open set and hence by Theorem 2.6, for every $x \in X - C$, there exists an (i, j) - $\omega\mathcal{N}$ - α -open set U containing x and a countable set B such that $U - B \subseteq X - C$. Thus, $C \subseteq X - (U - B) = X - (U \cap (X - B)) = (X - U) \cup B$, let $J = X - U$. Then, J is an (i, j) - $\omega\mathcal{N}$ - α -closed set such that $C \subseteq J \cup B$. $\square$

Theorem 2.11. The union of any family of (i, j) - $\omega\mathcal{N}$ - α -open sets is (i, j) - $\omega\mathcal{N}$ - α -open set.

Proof. Let $\{A_\delta : \delta \in \Delta\}$ is a collection of (i, j) - $\omega\mathcal{N}$ - α -open subsets of X . Then, for every $x \in \bigcup_{\delta \in \Delta} A_\delta$, $x \in A_\delta$, for some $\delta \in \Delta$. Hence, there exists an (i, j) - $\omega\mathcal{N}$ - α -open subset U containing x , such that $U - A_\delta$ is countable. Now, as $U - (\bigcup_{\delta \in \Delta} A_\delta) \subseteq U - A_\delta$, and thus $U - (\bigcup_{\delta \in \Delta} A_\delta)$ is countable. Therefore, $\bigcup_{\delta \in \Delta} A_\delta$ is an (i, j) - $\omega\mathcal{N}$ - α -open set. $\square$

Remark 2.12. Let (X, τ_i, τ_j) be a bitopological space, then the following statements hold:

1. If a subset A of X is (i, j) - $\omega\mathcal{N}$ - α -open and $U \in \omega O(X, \tau_i) \cap \omega O(X, \tau_j)$, then $A \cap U$ is (i, j) - $\omega\mathcal{N}$ - α -open set.
2. The union of the arbitrarily many (i, j) - $\omega\mathcal{N}$ - α -open sets is (i, j) - $\omega\mathcal{N}$ - α -open set.

Definition 2.13. The union of all (i, j) - $\omega\mathcal{N}$ - α -open sets contained in $A \subseteq X$ is called (i, j) - $\omega\mathcal{N}$ - α -interior of A and is denoted by $\omega\alpha\text{NBInt}(A)$.

Definition 2.14. The intersection of all (i, j) - $\omega\mathcal{N}$ - α -closed sets of X containing A is called (i, j) - $\omega\mathcal{N}$ - α -closure of A and is denoted by $\omega\alpha\text{NBCl}(A)$

Remark 2.15. The $\omega\alpha\text{NBInt}(A)$ is an (i, j) - $\omega\mathcal{N}$ - α -open set and the $\omega\alpha\text{NBCl}(A)$ is an (i, j) - $\omega\mathcal{N}$ - α -closed set.

Theorem 2.16. Let (X, τ_i, τ_j) be a bitopological space and $A, B \subseteq X$. Then, the following statements hold:

1. $\omega\alpha\text{NBInt}(\omega\alpha\text{NBInt}(A)) = \omega\alpha\text{NBInt}(A)$.
2. if $A \subset B$, then $\omega\alpha\text{NBInt}(A) \subset \omega\alpha\text{NBInt}(B)$.
3. $\omega\alpha\text{NBInt}(A \cap B) \subset \omega\alpha\text{NBInt}(A) \cap \omega\alpha\text{NBInt}(B)$.
4. $\omega\alpha\text{NBInt}(A) \cup \omega\alpha\text{NBInt}(B) \subset \omega\alpha\text{NBInt}(A \cup B)$.
5. $\omega\alpha\text{NBInt}(A)$ is the largest (i, j) - $\omega\mathcal{N}$ - α -open subset of X . contained in A .
6. A is (i, j) - $\omega\mathcal{N}$ - α -open if and only if $A = \omega\alpha\text{NBInt}(A)$.
7. $\omega\alpha\text{NBCl}(\omega\alpha\text{NBCl}(A)) = \omega\alpha\text{NBCl}(A)$.
8. If $A \subset B$, then $\omega\alpha\text{NBCl}(A) \subset \omega\alpha\text{NBCl}(B)$.

9. $\omega\alpha\text{NBCl}(A) \cup \omega\alpha\text{NBCl}(B) \subset \omega\alpha\text{NBCl}(A \cup B)$.
10. $\omega\alpha\text{NBCl}(A \cap B) \subset \omega\alpha\text{NBCl}(A) \cap \omega\alpha\text{NBCl}(B)$.

Proof. (1), (2), (6), (7) and (8) are follow directly from the Definition 2.1. (3), (4) and (5) are follow from part (2) of this Theorem. (9) and (10) are follow by applying part (8) of this Theorem. $\square$

Theorem 2.17. *Let (X, τ_i, τ_j) be a bitopological space and $A \subseteq X$. Then, $x \in \omega\alpha\text{NBCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in \omega\alpha\text{NBO}(X, x)$.*

Proof. Suppose that $x \in \omega\alpha\text{NBCl}(A)$ and it knows that $U \cap A \neq \emptyset$, for all $U \in \omega\alpha\text{NBO}(X, x)$. Now, suppose the contrary that there exists $U \in \omega\alpha\text{NBO}(X, x)$ such that $U \cap A = \emptyset$, then $A \subseteq X - U$ and $X - U$ is an (i, j) - ω - $\mathcal{N}$ - α -closed set. This follows that $\omega\alpha\text{NBCl}(A) \subseteq \omega\alpha\text{NBCl}(X - U) = X - U$. Since $x \in \omega\alpha\text{NBCl}(A)$, it has $x \in X - U$ and hence $x \notin U$. Which is a contradiction. Therefore, $U \cap A \neq \emptyset$. Conversely, suppose that $U \cap A \neq \emptyset$ for every $U \in \omega\alpha\text{NBO}(X, x)$. Now, it has to prove that $x \in \omega\alpha\text{NBCl}(A)$. Suppose the contrary that $x \notin \omega\alpha\text{NBCl}(A)$. Now, let $U = X - \omega\alpha\text{NBCl}(A)$, then $U \in \omega\alpha\text{NBO}(X, x)$ and $U \cap A = (X - (\omega\alpha\text{NBCl}(A))) \cap A \subseteq (X - A) \cap A = \emptyset$ and this is a contradiction. Therefore, $x \in \omega\alpha\text{NBCl}(A)$. $\square$

Theorem 2.18. *Let (X, τ_i, τ_j) be a bitopological space and $A \subset X$. Then, the following statements hold:*

1. $\omega\alpha\text{NBCl}(X - A) = X - \omega\alpha\text{NBCl}(A)$.
2. $\omega\alpha\text{NBInt}(X - A) = X - \omega\alpha\text{NBInt}(A)$.

Proof. 1. Let $x \in X - \omega\alpha\text{NBCl}(A)$. Then, there exists $U \in \omega\alpha\text{NBO}(X, x)$ such that $U \cap A = \emptyset$ and hence it has $x \in \omega\alpha\text{NBInt}(A)$. This shows that $X - \omega\alpha\text{NBCl}(A) \subset \omega\alpha\text{NBInt}(X - A)$. Now, take $x \in \omega\alpha\text{NBInt}(X - A)$. Since $\omega\alpha\text{NBInt}(X - A) \cap A = \emptyset$, it gets that $x \notin \omega\alpha\text{NBCl}(A)$. In consequence, $\omega\alpha\text{NBCl}(X - A) = X - \omega\alpha\text{NBInt}(A)$.

2. Let $x \in X - \omega\alpha\text{NBInt}(X - A)$. Since $\omega\alpha\text{NBInt}(X - A) \cap A = \emptyset$, it has $x \notin \omega\alpha\text{NBCl}(A)$ and this implies that $x \in X - \omega\alpha\text{NBCl}(A)$. Now, take $x \in X - \omega\alpha\text{NBCl}(A)$. Then, there exist $U \in \omega\alpha\text{NBO}(X, x)$ such that $U \cap A = \emptyset$. Therefore, $\omega\alpha\text{NBInt}(X - A) = X - \omega\alpha\text{NBCl}(A)$. $\square$

Definition 2.19. Let (X, τ_i, τ_j) be a bitopological space and $A \subseteq X$. Then A is said to be (i, j) - ω - $\mathcal{N}$ - α -neighbourhood of a point $x \in X$ if there exists an (i, j) - ω - $\mathcal{N}$ - α -open set J such that $x \in J \subset A$.

Theorem 2.20. *Let (X, τ_i, τ_j) be a bitopological space and $A \subseteq X$. Then, A is (i, j) - ω - $\mathcal{N}$ - α -open set if and only if it is an (i, j) - ω - $\mathcal{N}$ - α -neighbourhood of each of its points.*

Proof. Let A be an (i, j) - ω - $\mathcal{N}$ - α -open set of X . Then by the Definition 2.19 A is (i, j) - ω - $\mathcal{N}$ - α -neighbourhood of each of its points. Conversely, If A is an (i, j) - ω - $\mathcal{N}$ - α -neighbourhood of each of its points. Then, for each $x \in A$, there exists $D_x \in \omega\alpha\text{NB}(X, x)$ such that $D_x \subset A$. In consequence, $A = \bigcup \{D_x : x \in A\}$. Since, each D_x is an (i, j) - ω - $\mathcal{N}$ - α -open and arbitrary union of (i, j) - ω - $\mathcal{N}$ - α -open sets is an (i, j) - ω - $\mathcal{N}$ - α -open set. therefore, A is an (i, j) - ω - $\mathcal{N}$ - α -open set of X . $\square$

Theorem 2.21. *Let (X, τ_i, τ_j) be a bitopological space and for each non-empty (i, j) - ω - $\mathcal{N}$ - α -open set of X is uncountable. Then, $\alpha\text{NBCl}(A) = \omega\alpha\text{NBCl}(A)$, for each subset $A \in \tau_i \cup \tau_j$.*

Proof. The implication $\omega\alpha\text{NBCl}(A) \subseteq \alpha\text{NBCl}(A)$ is clear. On the other hand, let $x \in \alpha\text{NBCl}(A)$ and B be an (i, j) - ω - $\mathcal{N}$ - α -open set containing x . By the Theorem 2.6, there exists an (i, j) - $\mathcal{N}$ - α -open set U containing x and a countable set C such that $U - C \subseteq B$. Then, $(U - C) \subseteq B \cap A$ and $(U \cap A) - C \subseteq B \cap A$. Now, let $x \in U$ and $x \in \alpha\text{NBCl}(A)$ such that $U \cap A \neq \emptyset$ where $U \cap A$ is an (i, j) - ω - $\mathcal{N}$ - α -open set, since U is (i, j) - $\mathcal{N}$ - α -open set and $A \in \tau_i \cup \tau_j$. By hypothesis, each non-empty (i, j) - ω - $\mathcal{N}$ - α -open set of X is uncountable, thus $(U \cap A) - C$. Therefore, $B \cap A$ is uncountable. In consequence, $B \cap A \neq \emptyset$, this implies that $x \in \omega\alpha\text{NBCl}(A)$. $\square$

Theorem 2.22. Let (X, τ_i, τ_j) be a topological space. If every (i, j) - ω - $\mathcal{N}$ - α -open set of X is τ_i -open of X . Then, $(X, \omega\alpha\mathcal{NBO}(X))$ is a topological space.

Proof. 1. $\emptyset, X \in \omega\alpha\mathcal{NBO}(X)$.

2. Let $U, V \in \omega\alpha\mathcal{NBO}(X)$ and $x \in U \cap V$. Then, there exists (i, j) - $\mathcal{N}$ - α -open sets J, K of X containing x such that $J - U$ and $K - V$ are countable. Since $(J \cap K) - (U \cap V) = (J \cap K) \cap ((X - U) \cup (X - V)) \subseteq (J \cap (X - U)) \cup (K \cap (X - V))$, this implies that $(J \cap K) - (U \cap V)$ is a countable set and by hypothesis, the intersection of two (i, j) - $\mathcal{N}$ - α -open sets is an (i, j) - $\mathcal{N}$ - α -open set. Therefore, $U \cap V \in \omega\alpha\mathcal{NBO}(X)$.

3. The union is followed directly.

□

3. (i, j) - ω - $\mathcal{N}$ - α -continuous functions

In this section, it defines the concept of (i, j) - $\mathcal{N}$ - ω - α -continuous functions. Moreover, it proves some properties.

Definition 3.1. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a functions. Then, f is said to be (i, j) - ω - $\mathcal{N}$ - α -continuous if $f^{-1}(V)$ is (i, j) - ω - $\mathcal{N}$ - α -open set of X for every σ_i -open set of Y , where $i \neq j$.

Theorem 3.2. Every (i, j) - $\mathcal{N}$ - α -continuous function is (i, j) - ω - $\mathcal{N}$ - α -continuous functions.

Proof. It follows from the fact that every (i, j) - $\mathcal{N}$ - α -open set is (i, j) - ω - $\mathcal{N}$ - α -open set.

□

Remark 3.3. The converse of the Theorem 3.2 need not be true as can be seen in the following example:

Let $X = \{1, 2, 3\}$, $\tau_i = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}$, $\tau_j = \{\emptyset, X, \{1\}\}$, $\sigma_i = \{\emptyset, X, \{1, 2\}\}$, $\sigma_j = \{\emptyset, X, \{1, 3\}\}$. Then, the identify function $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ is (i, j) - ω - $\mathcal{N}$ - α -continuous, but it is not (i, j) - $\mathcal{N}$ - α -continuous.

Proposition 3.4. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function. If f is (i, j) - ω - $\mathcal{N}$ - α -continuous and $A \in \omega\mathcal{O}(X, \tau_i) \cap \omega\mathcal{O}(X, \tau_j)$, then the restriction $f|_A : (A, \tau_i|_A, \tau_j|_A) \rightarrow (Y, \sigma_i, \sigma_j)$ is (i, j) - ω - $\mathcal{N}$ - α -continuous

Proof. Since f is (i, j) - ω - $\mathcal{N}$ - α -continuous, for any $U \in \sigma_i$ of Y , $f^{-1}(U)$ is (i, j) - ω - $\mathcal{N}$ - α -open set of X . By the Remark 2.12 part (1), $f^{-1}(U) \cap A$ is (i, j) - ω - $\mathcal{N}$ - α -open set of X . Therefore, by the Lemma 2.5 $(f|_A)^{-1}(U)$ is (i, j) - ω - $\mathcal{N}$ - α -open set of $(A, \tau_i|_A, \tau_j|_A)$.

□

Proposition 3.5. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function and $X = \bigcup \{V_\delta \in \tau_i : \delta \in \Delta\}$. If the restriction $f|_{V_\delta} : (V_\delta, \tau_i|_{V_\delta}, \tau_j|_{V_\delta}) \rightarrow (Y, \sigma_i, \sigma_j)$ is (i, j) - ω - $\mathcal{N}$ - α -continuous for each $\delta \in \Delta$, then f is (i, j) - ω - $\mathcal{N}$ - α -continuous.

Proof. Let U be σ_i -open set of Y . Since $f|_{V_\delta}$ is (i, j) - ω - $\mathcal{N}$ - α -continuous for each $\delta \in \Delta$, $(f|_{V_\delta})^{-1}(U) = f^{-1}(U) \cap V_\delta$ is (i, j) - ω - $\mathcal{N}$ - α -open set of V_δ . Now, by the Lemma 2.5, $f^{-1}(U) \cap V_\delta$ is (i, j) - ω - $\mathcal{N}$ - α -open set of X for each $\delta \in \Delta$. Now, taking $f^{-1}(U) = \bigcup_{\delta \in \Delta} (f^{-1}(U) \cap V_\delta)$. Now, by the Remark 2.12 part (2), $f^{-1}(U)$ is (i, j) - ω - $\mathcal{N}$ - α -open set of X . Therefore, f is (i, j) - ω - $\mathcal{N}$ - α -continuous.

□

Theorem 3.6. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function. Then, the following statements are equivalent:

1. f is (i, j) - ω - $\mathcal{N}$ - α -continuous.
2. For each point $x \in X$ and each σ_i -open set J of Y such that $f(x) \in J$, there is an (i, j) - ω - $\mathcal{N}$ - α -open set A of X such that $x \in A$ and $f(A) \subset J$.
3. The inverse image of each σ_i -closed set of Y is an (i, j) - ω - $\mathcal{N}$ - α -closed set of X .
4. For $A \subseteq X$, then $f(\omega\alpha\mathcal{NBCl}(A)) \subset \sigma_i\text{-Cl}(f(A))$.
5. For $A \subseteq Y$, then $\omega\alpha\mathcal{NBCl}(f^{-1}(A)) \subset f^{-1}(\sigma_i\text{-Cl}(A))$.
6. For $A \subseteq Y$, then $f^{-1}(\sigma_i\text{-Int}(A)) \subset \omega\alpha\mathcal{NBInt}(f^{-1}(A))$.

Proof. (1) $\Rightarrow$ (2) : Let $x \in X$ and J be a σ_i -open set of Y containing $f(x)$. By part (1) of this Theorem, $f^{-1}(J)$ is (i, j) - ω - $\mathcal{N}$ - α -open of X . Let $B = f^{-1}(J)$. Then, $x \in B$ and $f(B) \subset J$.

(2) $\Rightarrow$ (1) : Let J be a σ_i -open of Y and $x \in f^{-1}(J)$. Then, $f(x) \in J$. By part (2) of this Theorem, there is (i, j) - ω - $\mathcal{N}$ - α -open set U_x of X such that $x \in U_x$ and $f(U_x) \subset J$. This implies that $x \in U_x \subset f^{-1}(J)$. Therefore, $f^{-1}(J)$ is an (i, j) - ω - $\mathcal{N}$ - α -open set of X .

(1) $\Leftrightarrow$ (3) : It follows from the fact that for any set A of Y , $f^{-1}(Y - B) = X - f^{-1}(B)$.

(3) $\Rightarrow$ (4) : Let $A \subseteq X$. Since $A \subset f^{-1}(f(A))$, it has that $A \subset f^{-1}(\sigma_i\text{-Cl}(f(A)))$. By hypothesis, $f^{-1}(\sigma_i\text{-Cl}(f(A)))$ is an (i, j) - ω - $\mathcal{N}$ - α -closed set of Y and hence $\omega\alpha\text{NBCl}(A) \subset f^{-1}(\sigma_i\text{-Cl}(f(A)))$. Then, $f((\omega\alpha\text{NBCl}(A))) \subset f(f^{-1}(\sigma_i\text{-Cl}(f(A)))) \subseteq \sigma_i\text{-Cl}(f(A))$.

(4) $\Rightarrow$ (3) : Let J be a σ_i -closed set of Y . Then, $f(\omega\alpha\text{NBCl}(f^{-1}(J))) \subset \sigma_i\text{-Cl}(f(f^{-1}(J))) \subset \sigma_i\text{-Cl}(J) = J$. Therefore, $\omega\alpha\text{NBCl}(f^{-1}(J)) \subset f^{-1}(J)$. In consequence, $f^{-1}(J)$ is an (i, j) - ω - $\mathcal{N}$ - α -closed set of X .

(4) $\Rightarrow$ (5) : Let $A \subseteq Y$. Now, $f((\omega\alpha\text{NBCl}(A))) \subset \sigma_i\text{-Cl}(f(f^{-1}(A))) \subset \sigma_i\text{-Cl}(A)$. In consequence, $\omega\alpha\text{NBCl}(f^{-1}(A)) \subset f^{-1}(\sigma_i\text{-Cl}(A))$.

(5) $\Rightarrow$ (4) : Let $A = f(B)$ where $B \subseteq X$. Then, $\omega\alpha\text{NBCl}(A) \subset \omega\alpha\text{NBCl}(B) \subset \omega\alpha\text{NBCl}(f^{-1}(A)) \subset f^{-1}(\sigma_i\text{-Cl}(A)) = f^{-1}(\sigma_i\text{-Cl}(f(A)))$. Therefore, $f(\omega\alpha\text{NBCl}(B)) \subset \sigma_i\text{-Cl}(f(A))$.

(1) $\Rightarrow$ (6) : Let $A \subseteq Y$. It is clear that $f^{-1}(\sigma_i\text{-Int}(A))$ is an (i, j) - ω - $\mathcal{N}$ - α -open and it has $f^{-1}(\sigma_i\text{-Int}(A)) \subset \omega\alpha\text{NBInt}(f^{-1}(\sigma_i\text{-Int}(A))) \subset \omega\alpha\text{NBInt}(f^{-1}(A))$.

(6) $\Rightarrow$ (1) : Let A be a σ_i -open set of Y . Then, $\sigma_i\text{-Int}(A) = A$ and $f^{-1}(A) \subset f^{-1}(\sigma_i\text{-Int}(A)) \subset \omega\alpha\text{NBInt}(f^{-1}(A))$. Therefore, it has $f^{-1}(A) = \omega\alpha\text{NBInt}(f^{-1}(A))$. This implies that $f^{-1}(A)$ is an (i, j) - ω - $\mathcal{N}$ - α -open set of X . $\square$

Definition 3.7. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a functions. Then, f is said to be (i, j) - ω - $\mathcal{N}$ - α -irresolute if $f^{-1}(V)$ is an (i, j) - ω - $\mathcal{N}$ - α -open set of X for every (i, j) - ω - $\mathcal{N}$ - α -open set V of Y , where $i \neq j$.

Theorem 3.8. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ and $g : (Y, \sigma_i, \sigma_j) \rightarrow (Z, \theta_i, \theta_j)$ be two functions. Then, the following statements hold:

1. $g \circ f$ is (i, j) - ω - $\mathcal{N}$ - α -irresolute, if f is (i, j) - ω - $\mathcal{N}$ - α -irresolute and g is (i, j) - ω - $\mathcal{N}$ - α -irresolute.
2. $g \circ f$ is (i, j) - ω - $\mathcal{N}$ - α -continuous, if f is (i, j) - ω - $\mathcal{N}$ - α -continuous and g is pairwise continuous.
3. $g \circ f$ is (i, j) - ω - $\mathcal{N}$ - α -continuous, if f is (i, j) - ω - $\mathcal{N}$ - α -irresolute and g is (i, j) - ω - $\mathcal{N}$ - α -continuous.
4. $g \circ f$ is (i, j) - ω - $\mathcal{N}$ - α -continuous, if f is (i, j) - ω - $\mathcal{N}$ - α -irresolute and g is pairwise continuous.

Proof. It begins proof the part (1): Let V be a (i, j) - ω - $\mathcal{N}$ - α -open set of Z , since g is (i, j) - ω - $\mathcal{N}$ - α -irresolute, then $g^{-1}(V)$ is an (i, j) - ω - $\mathcal{N}$ - α -open set of Y . Now, since f is (i, j) - ω - $\mathcal{N}$ - α -irresolute, $f^{-1}(g^{-1}(V))$ is (i, j) - ω - $\mathcal{N}$ - α -open set of X , therefore $g \circ f$ is (i, j) - ω - $\mathcal{N}$ - α -irresolute.

The proof of (2), (3) and (4) are made in the same way to the part (1). $\square$

Definition 3.9. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a functions. Then, f is said to be (i, j) - ω - $\mathcal{N}$ - α -open if $f(V)$ is an (i, j) - ω - $\mathcal{N}$ - α -open set of Y for every (i, j) - ω - $\mathcal{N}$ - α -open set or τ_i -open V of X , where $i \neq j$.

Proposition 3.10. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a pairwise continuous and pairwise open function. Then, f is (i, j) - ω - $\mathcal{N}$ - α -open.

Proof. Let V be an (i, j) - ω - $\mathcal{N}$ - α -open set of X and $y \in f(V)$. Then, there exists $x \in V$ such that $f(x) = y$. Since V is (i, j) - ω - $\mathcal{N}$ - α -continuous, there exists (i, j) - $\mathcal{N}$ - α -open set V_1 of X containing x such that $U_1 - U \subseteq A$, where A is a countable set. Therefore, $f(U_1) - f(U) \subseteq f(A)$. Since f is pairwise continuous and pairwise open, by the Theorem 1.6, $f(U_1)$ is an (i, j) - $\mathcal{N}$ - α -open set of Y containing $y = f(x)$ and hence $f(U)$ is (i, j) - ω - $\mathcal{N}$ - α -open of Y . $\square$

Definition 3.11. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function. Then, f is said to be (i, j) - ω - $\mathcal{N}$ - α -closed if $f(U)$ is an (i, j) - ω - $\mathcal{N}$ - α -closed set of Y for every τ_i -closed set of X .

Theorem 3.12. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function. Then, the following properties are equivalent:

1. f is an (i, j) - ω - $\mathcal{N}$ - α -open.
2. $f(\tau_i\text{-Int}(U) \subseteq \omega\alpha\text{NBCl}(f(U))$, for each subset U of X .
3. $\tau_i\text{-Int}(f^{-1}(U)) \subseteq \omega\alpha\text{NBInt}(U)$, for each subset U of X .

Proof. (1) $\Rightarrow$ (2) : Let U be a subset of X . Then, $\tau_i\text{-Int}(U)$ is a τ_i -open set of X . Then, $f(\tau_i\text{-Int}(U))$ is (i, j) - ω - $\mathcal{N}$ - α -open set of Y . Since $f(\tau_i\text{-Int}(U)) \subseteq f(U)$, $f(\tau_i\text{-Int}(U)) = \omega\alpha\text{NBInt}(f(\tau_i\text{-Int}(U))) \subseteq \omega\alpha\text{NBInt}(f(U))$.

(2) $\Rightarrow$ (3) : Let U be a subset of Y . Then, $f(\tau_i\text{-Int}(f^{-1}(U))) \subseteq \omega\alpha\text{NBInt}(f(f^{-1}(U)))$. Therefore, $\tau_i\text{-Int}(f^{-1}(U)) \subseteq f^{-1}(\omega\alpha\text{NBInt}(U))$.

(3) $\Rightarrow$ (1) : Let U be a τ_i -open set of X . Then, $\tau - i\text{-Int}(U) = U$. Now, $V = \tau_i\text{-Int}(V) \subseteq \tau_i\text{-Int}(f^{-1}(f(V))) \subseteq f^{-1}(\omega\alpha\text{NBInt}(f(V))) \subseteq \omega\alpha\text{NBInt}(f(V))$. In consequence, $f(V)$ is an (i, j) - ω - $\mathcal{N}$ - α -open set of Y . Therefore, f is (i, j) - ω - $\mathcal{N}$ - α -open. $\square$

Theorem 3.13. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function. Then, f is an (i, j) - ω - $\mathcal{N}$ - α -closed function if and only if for each subset U of X , the $\omega\alpha\text{NBCl}(f(U)) \subseteq f(\tau_i\text{-Cl}(U))$.

Proof. Let f be an (i, j) - ω - $\mathcal{N}$ - α -closed function and U be a subset of X . Then, $f(U) \subseteq f(\tau_i\text{-Cl}(U))$ and $f(\tau_i\text{-Cl}(U))$ is an (i, j) - ω - $\mathcal{N}$ - α -closed set of Y . Therefore, $\omega\alpha\text{NBCl}(f(U)) \subseteq \omega\alpha\text{NBCl}(f(\tau_i\text{-Cl}(U))) = f(\tau_i\text{-Cl}(U))$. Conversely, let U be a τ_i -closed of X . Then, $f(U) \subseteq \omega\alpha\text{NBCl}(f(U)) \subseteq f(\tau_i\text{-Cl}(U)) = f(U)$. Hence, $f(U)$ is an (i, j) - ω - $\mathcal{N}$ - α -closed set of Y . In conclusion, f is an (i, j) - ω - $\mathcal{N}$ - α -closed function. $\square$

Definition 3.14. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function. Then, the graph $G(f)$ of f is said to be (i, j) - ω - $\mathcal{N}$ - α -closed in $X \times Y$ if for each $(x, y) \in (X \times Y) - G(f)$, there exists $U \in \omega\alpha\text{NBO}(X, x)$, and a σ_i -open set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 3.15. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function. Then, the graph $G(f)$ of f is (i, j) - ω - $\mathcal{N}$ - α -closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exists $U \in \omega\alpha\text{NBO}(X, x)$, and a σ_i -open set V of Y containing y such that $f(U) \cap V = \emptyset$.

Proof. It follows by the Definition 3.14. $\square$

Theorem 3.16. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function. If f is (i, j) - ω - $\mathcal{N}$ - α -continuous and (Y, σ_i) is T_1 , then $G(f)$ is (i, j) - ω - $\mathcal{N}$ - α -closed.

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then, $y \neq f(x)$. Since, (Y, σ_i) is T_1 , there exists a σ_i -open sets V, U of Y such that $f(x) \in V$ and $y \notin U$ and $U \cap V = \emptyset$. Since f is (i, j) - ω - $\mathcal{N}$ - α -continuous, there exists $W \in (i, j)$ - ω - $\mathcal{N}$ - α -open, such that $f(W) \subset V$. Therefore, $f(W) \cap U = \emptyset$. In consequence, by the Lemma 3.15, $G(f)$ is (i, j) - ω - $\mathcal{N}$ - α -closed. $\square$

Definition 3.17. Let (X, τ_i, τ_j) be a bitopological space. Then, X is said to be pairwise Lindeloff [1] if each pairwise open cover of X has a countable sub-cover.

Theorem 3.18. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be an (i, j) - ω - $\mathcal{N}$ - α -continuous function. If X is (i, j) - $\mathcal{N}$ - α -Lindeloff, then Y is pairwise Lindeloff.

Proof. Let $\{U_\delta : \delta \in \Delta\}$ be a cover of Y by σ_i -open sets. Then, $\{f^{-1}(U_\delta) : \delta \in \Delta\}$ is an (i, j) - ω - $\mathcal{N}$ - α -open cover of X . Since X is (i, j) - $\mathcal{N}$ - α -Lindeloff and by the Definition 2.8, there exists a countable subset Δ_0 of Δ such that $X = \bigcup_{\delta \in \Delta_0} U_\delta$. Therefore, Y is pairwise Lindeloff. $\square$

Definition 3.19. Let (X, τ_i, τ_j) be a bitopological space and $A \subseteq X$. The (i, j) - ω - $\mathcal{N}$ - α -frontier of A is defined as (i, j) - ω - $\mathcal{N}$ - α -Fr(A) = $\omega\alpha\text{NBCl}(A) \cap \omega\alpha\text{NBCl}(X - A) = \omega\alpha\text{NBCl}(A) - \omega\alpha\text{NBInt}(A)$.

Theorem 3.20. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function. Then, $X - \omega\alpha\text{Nbc}(f) = \bigcup \{(i, j)$ - ω - α -Fr($f^{-1}(V)$) : $V \in \sigma_i, f(x) \in V, x \in X\}$, where $\omega\alpha\text{Nbc}(f)$ denotes the set of points at which f is (i, j) - ω - $\mathcal{N}$ - α -continuous.

Proof. Let $x \in X - \omega\alpha\text{NBC}(f)$. Then, there exists a σ_i -open set V of Y containing $f(x)$ such that $U \cap (X - f^{-1}(V)) \neq \emptyset$ for every (i, j) - $\omega\text{-N-}\alpha$ -open set U of X containing x . Thus, $x \in \omega\alpha\text{NBCl}(X - f^{-1}(V))$. Then, $x \in f^{-1}(V) \cap \omega\alpha\text{NBCl}(X - f^{-1}(V)) \subseteq (i, j)$ - $\omega\text{-N-}\alpha\text{-Fr}(f^{-1}(V))$. Hence, $X - \omega\alpha\text{NBC}(f) \subseteq \bigcup \{(i, j)$ - $\omega\text{-N-}\alpha\text{-Fr}(f^{-1}(V)) : V \in \sigma_i, f(x) \in V, x \in X\}$. Conversely, let $x \notin X - \omega\alpha\text{NBC}(f)$. Then, for each σ_i -open set V of Y containing $f(x)$, there exists an (i, j) - $\omega\text{-N-}\alpha$ -open set U containing x such that $f(U) \subseteq V$ and hence $x \in U \subseteq f^{-1}(V)$. Therefore, $x \in \omega\alpha\text{NBI}(f^{-1}(V))$, in consequence $x \notin (i, j)$ - $\omega\text{-N-}\alpha\text{-Fr}(f^{-1}(V))$ for each σ_i -open set V in Y containing $f(x)$. $\square$

Definition 3.21. Let (X, τ_i, τ_j) be a bitopological space. Then, X is said to be (i, j) - $\omega\text{-N-}\alpha\text{-T}_2$ space if for each pair of distinct points $x, y \in X$, there exists (i, j) - $\omega\text{-N-}\alpha$ -open sets U and V containing x and y , respectively, such that $U \cap V = \emptyset$.

Theorem 3.22. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be an (i, j) - $\omega\text{-N-}\alpha$ -continuous and injective function, and Y is a T_2 space, then X is an (i, j) - $\omega\text{-N-}\alpha\text{-T}_2$ space.

Proof. Let x and y be two distinct points of X . Then, $f(x) \neq f(y)$. Since Y is T_2 , there exist a τ_i -open set U and τ_j -open set V such that $f(x) \in U$, $f(y) \in V$ and $U \cap V = \emptyset$. Therefore, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Since f is (i, j) - $\omega\text{-N-}\alpha$ -continuous, then $f^{-1}(U)$ is (i, j) - $\omega\text{-N-}\alpha$ -open, $f^{-1}(V)$ is (i, j) - $\omega\text{-N-}\alpha$ -open, $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Which implies that X is (i, j) - $\omega\text{-N-}\alpha\text{-T}_2$ space. $\square$

Theorem 3.23. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be an (i, j) - $\omega\text{-N-}\alpha$ -continuous and injective function, and Y is a $\omega\text{-N-}\alpha\text{-T}_2$ space, then X is a $\omega\text{-N-}\alpha\text{-T}_2$ space.

Proof. The proof is similar to the Theorem 3.22. $\square$

4. (i, j) - $\omega\text{-N-}\alpha$ -continuous functions and (i, j) - $\omega\text{-N-}\alpha$ -connected spaces

In this section, it defines the concepts of (i, j) - $\omega\text{-N-}\alpha$ -connected space, (i, j) - $\omega\text{-N-}\alpha$ -set-connected, (i, j) - $\omega\text{-N-}\alpha$ -extremally disconnected and (i, j) - $\omega\text{-N-}\alpha\text{-C}$ -compact space. Besides it proves some properties on (i, j) - $\omega\text{-N-}\alpha$ -continuous functions.

Definition 4.1. Let (X, τ_i, τ_j) be a bitopological space. Then, X is said to be (i, j) - $\omega\text{-N-}\alpha$ -connected if X cannot be expressed as the union of two non-empty disjoint (i, j) - $\omega\text{-N-}\alpha$ -open sets.

Remark 4.2. If A is both (i, j) - $\omega\text{-N-}\alpha$ -open set and (i, j) - $\omega\text{-N-}\alpha$ -closed set of a bitopological space X , then A is called (i, j) - $\omega\text{-N-}\alpha$ -coplen, where $i \neq j$.

Definition 4.3. Let (X, τ_i, τ_j) be a bitopological space. Then, X is said to be pairwise connected [7] if it cannot be expressed as the union of two non-empty disjoint sets U and V such that U is τ_i -open and V is τ_j -open, where $i \neq j$.

Proposition 4.4. If $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ is a (i, j) - $\omega\text{-N-}\alpha$ -continuous and surjection function, besides X is (i, j) - $\omega\text{-N-}\alpha$ -connected, then Y is pairwise connected.

Proof. Suppose that Y is not pairwise connected. Then, there exists $U \in \sigma_i$ and $V \in \sigma_j$ such that $U, V \neq \emptyset$, $U \cap V = \emptyset$ and $U \cup V = Y$. Since f is surjection, it has $f^{-1}(U) \neq \emptyset$ and $f^{-1}(V) \neq \emptyset$. Besides, since f is (i, j) - $\omega\text{-N-}\alpha$ -continuous, $f^{-1}(U)$ is (i, j) - $\omega\text{-N-}\alpha$ -continuous, it has $f^{-1}(U)$ is (i, j) - $\omega\text{-N-}\alpha$ -open and $f^{-1}(V)$ is (i, j) - $\omega\text{-N-}\alpha$ -open such that $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $f^{-1}(U) \cup f^{-1}(V) = X$. This implies that X is not (i, j) - $\omega\text{-N-}\alpha$ -connected, which is a contradiction. In consequence Y is pairwise connected. $\square$

Proposition 4.5. If $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ is a (i, j) - $\omega\text{-N-}\alpha$ -irresolute and surjection function, besides X is (i, j) - $\omega\text{-N-}\alpha$ -connected, then Y is (i, j) - $\omega\text{-N-}\alpha$ -connected.

Proof. The proof is similar to the Proposition 4.4. $\square$

Definition 4.6. A function $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ is said to be (i, j) - ω - $\mathcal{N}$ - α -set-connected if $f(x)$ is (i, j) - ω - $\mathcal{N}$ - α -connected between $f(A)$ and $f(B)$ in the bitopological space X which is (i, j) - ω - $\mathcal{N}$ - α -connected.

Theorem 4.7. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function (i, j) - ω - $\mathcal{N}$ - α -set-connected if and only if $f^{-1}(F)$ is an (i, j) - ω - $\mathcal{N}$ - α -copen set of X for any (i, j) - ω - $\mathcal{N}$ - α -copen F set of Y .

Proof. **NECESSARY:** Let f be (i, j) - ω - $\mathcal{N}$ - α -set-connected and F be (i, j) - ω - $\mathcal{N}$ - α -copen set of Y . Now, suppose that $f^{-1}(F)$ is not (i, j) - ω - $\mathcal{N}$ - α -copen set of X , then X is (i, j) - ω - $\mathcal{N}$ - α -connected between $f^{-1}(F)$ and $X - f^{-1}(F)$. Since f is (i, j) - ω - $\mathcal{N}$ - α -set-connected, Y is (i, j) - ω - $\mathcal{N}$ - α -connected between $f(f^{-1}(F))$ and $f(X - f^{-1}(F))$. But, $f(f^{-1}(F)) = F \cap Y = F$ and $f(X - f^{-1}(F)) = Y - F$, in consequence F is not (i, j) - ω - $\mathcal{N}$ - α -copen set of Y and this is a contradiction. Therefore, $f^{-1}(F)$ is an (i, j) - ω - $\mathcal{N}$ - α -copen set of X .

SUFFICIENCY: Let $f^{-1}(F)$ be an (i, j) - ω - $\mathcal{N}$ - α -copen set of X for any (i, j) - ω - $\mathcal{N}$ - α -copen F set of Y and let X be (i, j) - ω - $\mathcal{N}$ - α -connected between A and B . Now, suppose that Y is not (i, j) - ω - $\mathcal{N}$ - α -connected between $f(A)$ and $f(B)$, then there exists an (i, j) - ω - $\mathcal{N}$ - α -copen F set of Y such that $f(A) \subset F \subset Y - f(B)$. But, $A \subset f^{-1}(F) \subset X - B$ and $f^{-1}(F)$ is an (i, j) - ω - $\mathcal{N}$ - α -copen set of X and this is a contradiction, because X is (i, j) - ω - $\mathcal{N}$ - α -connected. Therefore, f is (i, j) - ω - $\mathcal{N}$ - α -connected. $\square$

Lemma 4.8. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function (i, j) - ω - $\mathcal{N}$ - α -set-connected and $A \subset X$ such that $f(A)$ is an (i, j) -copen set of Y . Then, the restriction $f|_A : A \rightarrow Y$ is (i, j) - ω - $\mathcal{N}$ - α -set-connected.

Proof. Let A be (i, j) - ω - $\mathcal{N}$ - α -connected space between B and C . Then, X is (i, j) - ω - $\mathcal{N}$ - α -connected between B and C of Y is (i, j) - ω - $\mathcal{N}$ - α -connected between $f(B)$ and $f(C)$. Since $f(A)$ is an (i, j) -copen set of Y , then $f(A)$ is an (i, j) - ω - $\mathcal{N}$ - α -connected between $f(B)$ and $f(C)$. $\square$

Definition 4.9. Let (X, τ_i, τ_j) be a bitopological space. Then, X is said to be (i, j) - ω - $\mathcal{N}$ - α -extremally disconnected if the (i, j) - ω - $\mathcal{N}$ - α -closure of any (i, j) - ω - $\mathcal{N}$ - α -open set is (i, j) - ω - $\mathcal{N}$ - α -open set, where $i \neq j$.

Theorem 4.10. Let $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function (i, j) - ω - $\mathcal{N}$ - α -set-connected. If Y is (i, j) - ω - $\mathcal{N}$ - α - T_2 space and (i, j) - ω - $\mathcal{N}$ - α -extremally disconnected, then $f|_A : A \rightarrow Y$ is constant for every (i, j) - ω - $\mathcal{N}$ - α -connected subset A of X .

Proof. Let $x, y \in A$ and $x \neq y$. Suppose that $f(x) \neq f(y)$ in Y . Since Y is (i, j) - ω - $\mathcal{N}$ - α - T_2 space and (i, j) - ω - $\mathcal{N}$ - α -extremally disconnected, there exists (i, j) - ω - $\mathcal{N}$ - α -copen set U of Y such that $f(x) \in U$ and $f(y) \notin U$. Now, since f is (i, j) - ω - $\mathcal{N}$ - α -set-connected, it has $f^{-1}(U)$ is (i, j) - ω - $\mathcal{N}$ - α -copen set of X . And so, by the Lemma 2.5, $f^{-1}(U) \cap A$ is a non-empty proper (i, j) - ω - $\mathcal{N}$ - α -copen set of the subset A , this implies that A is not (i, j) - ω - $\mathcal{N}$ - α -connected space and this is a contradiction. Therefore, $f(x) = f(y)$ and hence $f|_A : A \rightarrow Y$ is constant. $\square$

Definition 4.11. Let (X, τ_i, τ_j) be a bitopological space. Then, X is said to be (i, j) - ω - $\mathcal{N}$ - α - C -compact if given and (i, j) - ω - $\mathcal{N}$ - α -closed set A of X and a cover $\{V_\delta : \delta \in \Delta\}$ of A by (i, j) - ω - $\mathcal{N}$ - α -open sets of X , then there exists a finite subset Δ_0 of Δ such that $A \subset \bigcup \{\omega\alpha\text{BCL}(V_\delta : \delta \in \Delta_0)\}$, where $i \neq j$.

Theorem 4.12. Let Y be (i, j) - ω - $\mathcal{N}$ - α -extremally disconnected, (i, j) - ω - $\mathcal{N}$ - α - C -compact and (i, j) - ω - $\mathcal{N}$ - α - T_2 . Then, $f : (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ is (i, j) - ω - $\mathcal{N}$ - α -irresolute if and only if it is (i, j) - ω - $\mathcal{N}$ - α -set-connected.

Proof. **NECESSITY:** The proof is easy following the Definition.

SUFFICIENCY: Let f be not (i, j) - ω - $\mathcal{N}$ - α -irresolute. Then, there exists an (i, j) - ω - $\mathcal{N}$ - α -closed set J of Y such that $f^{-1}(J)$ is not an (i, j) - ω - $\mathcal{N}$ - α -closed set of X . Now, let $x \in \omega\alpha\text{NBCl}(f^{-1}(J)) - f^{-1}(J)$. Then X is (i, j) - ω - $\mathcal{N}$ - α -connected between $f^{-1}(J)$ and x . Hence, Y is (i, j) - ω - $\mathcal{N}$ - α -connected between $f(f^{-1}(J))$ and $f(x)$. In consequence Y is (i, j) - ω - $\mathcal{N}$ - α -connected between J and $f(x)$. Since Y is (i, j) - ω - $\mathcal{N}$ - α - T_2 , for each $y \in J$ there exists an (i, j) - ω - $\mathcal{N}$ - α -open set U_y containing y in Y such that $f(x) \notin \omega\alpha\text{NBCl}(U_y - y)$. Then, the family $\{U_y : y \in J\}$ is a cover of F by (i, j) - ω - $\mathcal{N}$ - α -open sets of Y . Now, since Y is (i, j) - ω - $\mathcal{N}$ - α - C -compact, there exist a finite number of points $y_1, y_2, \dots, y_n$ in J such that $J \subset \bigcup_{i=1}^n \omega\alpha\text{NBCl}(U_{y_i}) = U$. Then, U

is (i, j) - ω - $\mathcal{N}$ - α -coplen set of Y since Y is (i, j) - ω - $\mathcal{N}$ - α -extremally disconnected. Besides, $f(x) \notin U$ since $f(x) \omega \alpha \text{NBCl}(U_y)$ for any $y \in J$ and this is a contradiction. Hence f is (i, j) - ω - $\mathcal{N}$ - α -irresolute. $\square$

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