Nonlinear and memory boundary feedback stabilization for Schrödinger equations with variable coefficients

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Abstract

In this paper, the boundary stabilization of Schrödinger equations with variable coefficients by nonlinear and memory feedback is considered. The approach adopted uses Riemannian geometry methods and multipliers techniques. ©2020 All rights reserved.

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1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^n$ with boundary $\Gamma := \partial\Omega$. It is assumed that Γ consists of two parts Γ_0 and Γ_1 such that $\Gamma_0, \Gamma_1 \neq \emptyset$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. We consider the mixed problem for Schrödinger equation

$$y_t - iAy = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \quad (1.1a)$$

$$y(0, x) = y_0(x) \quad \text{in } \Omega, \quad (1.1b)$$

$$y = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}_+, \quad (1.1c)$$

$$\frac{\partial y}{\partial \nu_A} = u \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \quad (1.1d)$$

Where

$$Ay = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial y}{\partial x_j} \right).$$

$a_{ij} = a_{ji}$ are C^∞ functions in $\mathbb{R}^n$, $\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial y}{\partial x_i} \nu_j$, $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ is the unit normal of Γ pointing toward the exterior of Ω , $\nu_A = A\nu$, and $A = (a_{ij})$ is a matrix function. We suppose that

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \overline{\xi_j} > \alpha_1 \sum_{i,j=1}^n \xi_i^2, \quad \forall x \in \Omega, \zeta \in \mathbb{C}^n, \zeta \neq 0. \quad (1.2)$$

for some positive constant α_1 .

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Throughout the paper we assume

$$u = - \int_0^t k'(t-s) \frac{F(|y|)}{|y|} (y(s) - y_t(t)) ds - \frac{1}{2}ky. \quad (1.3)$$

where $k : \Gamma_1 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \in C^2(\mathbb{R}^+, L^\infty(\Omega))$, and $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous. we define the corresponding energy functional by

$$E(t) := \frac{1}{2} \int_{\Omega} |\nabla_g y|_g^2 d\Omega + \int_{\Gamma_1} |y|^2 d\Gamma_1 + \frac{1}{2} \int_0^t \int_{\Gamma_1} k(t-s) F(|y(s)|) |y(s)| d\Gamma_1 ds. \quad (1.4)$$

The goal of this work is to stabilize the system (1.1a – 1.1d) and (1.3); to find a suitable feedback $u = F(x, y_t)$ such that the energy (1.3) decays to zero exponentially as $t \rightarrow +\infty$ for every solution y of which $E(0) < +\infty$. The approach adopted uses Riemannian geometry, this method was first introduced into the boundary-control problem by Yao [10] for the exactly controllability of wave equations.

The stabilization of partial differential equations has been considered by many authors.

The asymptotic behaviour of the wave equation with memory and linear feedbacks with constant coefficients has been studied by Guesmia [4], and by Aassila et al [1] in the case nonlinear. This study has been generalised by Chai Guo [2], with variable coefficients by using a very different method, namely, the Riemannian geometry method.

On the other hand, the stabilization of the Schrödinger equation has been studied by Machtyngier Zuazua [9] in the Neumann boundary conditions, and by Cipolatti et al [3] with nonlinear feedbacks, this study has been considered by Lasiecka Triggiani [7] with constant coefficients acting in the Dirichlet boundary conditions.

The objective of this work, we consider the boundary stabilization for system (1.1a – 1.1d) and (1.3) with variable coefficients and memory with nonlinear feedbacks by using multipliers techniques.

Our paper is organized as follows. In subsection 1.1, we introduce some notations and results on Riemannian geometry. Our main results are studied in section 2. Section 3 is devoted to the proof of the main results.

Euclidean metric on $\mathbb{R}^n$

Let $(x_1, \dots, x_n)$ be the natural coordinate system in $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$, denote

$$X \cdot Y = \sum_{i=1}^n \alpha_i \beta_i, \quad |X|_0^2 = X \cdot X, \quad \text{for } X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in T_x \mathbb{R}^n.$$

For $f \in C^1(\overline{\Omega})$ and $X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}$, we denote by

$$\nabla_0 f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \quad \text{and} \quad \text{div}_0(X) = \sum_{i=1}^n \frac{\partial \alpha_i(x)}{\partial x_i} \quad (1.1)$$

the gradient of f and the divergence of X in the Euclidean metric.

Riemannian metric

For each $x \in \mathbb{R}^n$, define the inner product and the corresponding norm on the tangent space $T_x \mathbb{R}^n$ by

$$g(X, Y) = \langle X, Y \rangle_g = X \cdot G(x)Y = \sum_{i,j=1}^n g_{ij}(x) \alpha_i \beta_j$$

$$|X|_g^2 = \langle X, X \rangle_g \quad \text{for } X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in T_x \mathbb{R}^n$$

Then $(\mathbb{R}^n, g)$ is a Riemannian manifold with a Riemannian metric g . Denote the Levi-Cevita connection in metric g by D . Let H be a vector field on $(\mathbb{R}^n, g)$. The covariant differential DH of H determines a bilinear form on $T_x \mathbb{R}^n \times T_x \mathbb{R}^n$. For each $x \in \mathbb{R}^n$, by

$$DH(X, Y) = \langle D_X H, Y \rangle_g, \forall X, Y \in T_x \mathbb{R}^n$$

where $D_X H$ is the covariant derivative of H with respect to X . The following lemma provides some useful equalities.

Lemme 1.1[Yao ,1999, lemma 2.1].

Let $f, h \in C^1(\overline{\Omega})$ and let H, X be vector field on $\mathbb{R}^n$. Then with reference to the above notation, we have

$$(i) \quad \langle H(x), A(x)X(x) \rangle_g = H(x) \cdot X(x), \quad \forall x \in \mathbb{R}^n \quad (1.9)$$

(ii) The gradient $\nabla_g f$ of f in the Riemannian metric g is given by

$$\nabla_g f(x) = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}(x) \frac{\partial f}{\partial x_j} \right) \frac{\partial}{\partial x_i} = A(x) \nabla_0 f. \quad (1.10)$$

(iii)

$$\frac{\partial y}{\partial v_A} = (A(x) \nabla_0 y) \cdot v = \nabla_g y \cdot v. \quad (1.11)$$

(vi)

$$\langle \nabla_g f, \nabla_g H \rangle_g = \nabla_g f(h) = \nabla_0 f \cdot A(x) \nabla_0 h. \quad (1.12)$$

(v)

$$\langle \nabla_g f, \nabla_g H(f) \rangle = DH(\nabla_g f, \nabla_g f) + \frac{1}{2} \operatorname{div}_0 \left(|\nabla_g f|_g^2 H \right)(x) - \frac{1}{2} |\nabla_g f|_g^2 \operatorname{div}_0(H) \quad x \in \mathbb{R}^n. \quad (1.13)$$

2. Statement of main result

To obtain the boundary stabilization of problem (1.1a – 1.1d) and (1.2), the following hypotheses are assumed.

There exists a vector field H on the Riemannian manifold $(\mathbb{R}^n, g)$ such that

$$\forall X \in T_x \mathbb{R}^n, \alpha > 0, \langle D_X H, X \rangle_g \geq \alpha |X|_g^2. \quad (2.1)$$

$$H(x) \cdot v < 0, \quad \text{on } \Gamma_0, \quad (2.2)$$

and

$$H(x) \cdot v \geq 0, \quad \text{on } \Gamma_1. \quad (2.3)$$

We also assumed the feedback function F satisfies

$$F(|y|) \leq |y|, \quad (2.4)$$

and $k \geq 0$ and $k' \leq 0$, on $\Gamma_0 \times \mathbb{R}_+$. Moreover,

$$k(0) = 0 \quad (2.5)$$

$$\varphi = \inf_{(x,t) \in \Gamma_0 \times \mathbb{R}^+} (-k') \neq 0 \quad (2.6)$$

$$k'' \geq 0. \quad (2.7)$$

We have the following result of existence and uniqueness of strong and weak solution to (1.1a – 1.1d) and (1.2).

Theorem 2.1. For all initial data $y_0 \in V = H_{\Gamma_0}^1(\Omega) = \{y \in H^1(\Omega); y = 0 \text{ on } \Gamma_0\}$, problem (1.1a – 1.1d) and (1.3) admits a unique global weak solution $y \in C(\mathbb{R}^+, V)$. Furthermore, if $y_0 \in H^3(\Omega) \cap H_{\Gamma_0}^1(\Omega)$ and $\frac{\partial y_0}{\partial \nu_A} = -\frac{1}{2}ky(0)$ on Γ_1 , then the solution has the regularity

$$y \in C^1(\mathbb{R}^+, V).$$

Proof. We can prove, using Galerkin method [6].

Suppose y is the unique global weak solution of problem (1.1a – 1.1d) and (1.2).

Our main result is the following:

Theorem 2.2

Under the hypothesis (1.2)-(1.4) and (2.1)-(2.7).

The problem

$$\begin{aligned} y_t - iAy &= 0 & \text{on } \Omega \\ y &= 0 & \text{in } \Sigma_0 \\ \frac{\partial y}{\partial \nu_A} &= 0 & \text{in } \Sigma \end{aligned}$$

and $y = 0$ is the unique solution of the problem.

Then for all given initial data $y_0 \in H_{\Gamma_0}^1(\Omega)$, there exist two positive constants $M > 0$ and $\omega > 0$ such that

$$E(t) \leq Me^{-\omega t}E(0), \quad t > 0.$$

3. PROOF OF MAIN RESULT

For simplicity, we assume that y is a strong solution. By a classical density argument, Theorem 2.2 still holds for a weak solution.

Lemma 3.1. The energy defined by (1.4) is decreasing and satisfies

$$\begin{aligned} E(T) - E(0) &= - \int_0^T \int_{\Gamma_1} b |y_t|^2 d\Gamma_1 dt + \frac{1}{2} \int_0^T \int_{\Gamma_1} k' |y|^2 d\Gamma_1 dt \\ &\quad - \frac{1}{2} \int_0^T \int_0^t \int_{\Gamma_1} k'' (t-s) |y(t) - y(s)|^2 d\Gamma_1 ds dt. \end{aligned}$$

where $b = - \int_0^t k' (t-s) \frac{F(|y|)}{|y|} ds$, and whenever $0 < T < \infty$.

Proof. Differentiating the energy $E(\cdot)$ defined by (1.4) and using Green's second theorem, we have

$$\begin{aligned} E'(t) &= -\operatorname{Re} \int_{\Gamma_1} ky \overline{y_t} d\Gamma_1 - \operatorname{Re} \int_{\Gamma_1} \left(\int_0^t k' (t-s) \rho(y(s)) ds \right) \overline{y_t} d\Gamma_1 \\ &\quad + \int_{\Gamma_1} b |y_t|^2 d\Gamma_1 + 2\operatorname{Re} \int_{\Gamma_1} y \overline{y_t} d\Gamma_1 + \int_0^t \int_{\Gamma_1} k' (t-s) F(|y(s)|) |y(s)| d\Gamma_1 ds. \end{aligned}$$

By the boundary condition, and by the assumption (2.5) and several technics we obtain that

$$\begin{aligned} E'(t) &= - \int_{\Gamma_1} b |y_t|^2 d\Gamma_1 + \frac{1}{2} \int_{\Gamma_1} k' |y|^2 d\Gamma_1 \\ &\quad - \frac{1}{2} \int_0^t \int_{\Gamma_1} k'' (t-s) |y(t) - y(s)|^2 d\Gamma_1 ds. \end{aligned}$$

This completes the proof of lemma 3.1.

Lemma 3.2.

For all $0 \leq S < T < \infty$ we have

$$\begin{aligned} & \int_S^T \int_{\Gamma_1} \left(\frac{\partial y}{\partial \nu_A} \right)^2 \frac{H \cdot \nu}{|\nu_A(x)|_g^2} d\Gamma_1 ds - \int_S^T \int_{\Gamma_1} |\nabla_g y|_g^2 H \nu d\Gamma_1 ds \\ & + 2\operatorname{Re} \int_S^T \int_{\Gamma_1} \left(\frac{\partial y}{\partial \nu_A} \right) H(\bar{y}) d\Gamma_1 ds + \operatorname{Im} \int_S^T \int_{\Gamma_1} y \bar{y}_t H \nu d\Gamma_1 ds = \\ & 2\operatorname{Re} \int_S^T \int_{\Omega} DH(\nabla_g y, \nabla_g y) d\Omega ds - \int_S^T \int_{\Omega} |\nabla_g y|_g^2 \operatorname{div} H d\Omega ds \\ & + \operatorname{Im} \int_{\Omega} y_t H(y) d\Omega \Big|_S^T + \operatorname{Im} \int_S^T \int_{\Omega} \bar{y}_t y \operatorname{div} H d\Omega ds \end{aligned}$$

Proof. We multiply (1.1a) by $H \cdot \nabla \bar{y}$ and integrate over $]S, T[\times \Omega$ we obtain

$$\int_S^T \int_{\Omega} y_t H \cdot \nabla \bar{y} d\Omega ds - i \int_S^T \int_{\Omega} A y H \cdot \nabla \bar{y} d\Omega ds = 0 \quad (3.1a)$$

We have

$$\begin{aligned} \int_S^T \int_{\Omega} y_t H \cdot \nabla \bar{y} d\Omega ds &= \int_{\Omega} y H \cdot \nabla \bar{y} d\Omega \Big|_S^T - \int_S^T \int_{\Omega} y H \cdot \nabla \bar{y}_t d\Omega ds. \\ &= \int_{\Omega} y H \cdot \nabla \bar{y} d\Omega \Big|_S^T - \int_S^T \int_{\Gamma} y \bar{y}_t H \cdot \nu d\Gamma ds \\ &\quad + \int_S^T \int_{\Omega} \bar{y}_t y \operatorname{div}(y \cdot H) d\Omega ds. \end{aligned} \quad (3.1b)$$

Substituting (3.1b) into (3.1a), we get

$$\begin{aligned} & \int_{\Omega} y H \cdot \nabla \bar{y} d\Omega \Big|_S^T - \int_S^T \int_{\Gamma} y \bar{y}_t H \cdot \nu d\Gamma ds - i \left[\int_S^T \int_{\Omega} (A \bar{y} H \cdot \nabla y + A y H \cdot \nabla \bar{y}) d\Omega ds \right] \\ & + \int_S^T \int_{\Omega} \bar{y}_t y \operatorname{div} H d\Omega ds = 0. \end{aligned}$$

Hence

$$\begin{aligned} 2\operatorname{Re} \int_S^T \int_{\Omega} A y H \cdot \nabla \bar{y} d\Omega ds &= \operatorname{Im} \int_{\Omega} y H \cdot \nabla \bar{y} d\Omega \Big|_S^T - \operatorname{Im} \int_S^T \int_{\Gamma} y \bar{y}_t H \cdot \nu d\Gamma ds \\ &\quad + \operatorname{Im} \int_S^T \int_{\Omega} \bar{y}_t y \operatorname{div} H d\Omega ds \end{aligned} \quad (3.1c)$$

Using (1.13) of lemma 1.1, we rewrite the integral on the left-hand side of (3.1c) as

$$\begin{aligned} \int_S^T \int_{\Omega} A y H \cdot \nabla \bar{y} d\Omega ds &= \int_S^T \int_{\Gamma} \frac{\partial y}{\partial \nu_A} H(y) d\Gamma ds - \int_S^T \int_{\Omega} DH(\nabla_g y, \nabla_g \bar{y}) d\Omega ds \\ &\quad - \frac{1}{2} \int_S^T \int_{\Omega} \operatorname{div}(|\nabla_g y|^2 \cdot H) d\Omega ds \\ &\quad + \frac{1}{2} \int_S^T \int_{\Omega} |\nabla_g y|_g^2 \operatorname{div} H d\Omega ds \end{aligned} \quad (3.1d)$$

Recalling the boundary condition (1.1b), we have on Γ

$$y = y_t = 0; |\nabla_g y|_g^2 = \frac{1}{|v_A(x)|_g^2} \left(\frac{\partial y}{\partial v_A} \right)^2; H(y) = \frac{H.v}{|v_A(x)|_g^2} \left(\frac{\partial y}{\partial v_A} \right). \quad (3.2)$$

Thus using (1.1c) and (3.2), we find that this simplifies to the sought-after identity .

4. Completion of the proof of theorem 2.1.

Set $C_0 = \left(\alpha_1 \sup_{x \in \overline{\Omega}} \frac{|H|^2}{|H.v|} \right)^2$, $C_1 = \frac{\alpha_1}{2} \sup_{x \in \overline{\Omega}} |\operatorname{div} H|$, $C_2 = \frac{\alpha_1}{2\varepsilon} \sup_{x \in \overline{\Omega}} |\nabla_g (\operatorname{div} H)| + \frac{1}{4} \sup_{x \in \overline{\Omega}} |\operatorname{div} H| + \frac{\varepsilon \alpha_1}{2} \sup_{x \in \overline{\Omega}} |H(x)|$

From assumptions (2.1) , (2.2) and (2.3) , we deduce that

$$\begin{aligned} 2a \int_S^T \int_{\Omega} |\nabla_g y|_g^2 d\Omega ds &\leq - \int_S^T \int_{\Gamma_1} |\nabla_g y|_g^2 H.v d\Gamma_1 ds + 2\operatorname{Re} \int_S^T \int_{\Gamma_1} \left(\frac{\partial y}{\partial v_A} \right) H(\bar{y}) d\Gamma_1 ds \\ &\quad + \operatorname{Im} \int_S^T \int_{\Gamma_1} y \bar{y}_t H.v d\Gamma_1 ds + \int_S^T \int_{\Omega} |\nabla_g y|_g^2 \operatorname{div} H d\Omega ds \\ &\quad - \operatorname{Im} \int_{\Omega} y H(y) d\Omega \Big|_S^T - \operatorname{Im} \int_S^T \int_{\Omega} \bar{y}_t y \operatorname{div} H d\Omega \end{aligned} \quad (3.3a)$$

We also have the estimates, in (3.3c) and (3.3d) ε is an arbitrary positive constant.

$$\begin{aligned} &- \int_S^T \int_{\Gamma_1} |\nabla_g y|_g^2 H.v d\Gamma_1 ds + 2\operatorname{Re} \int_S^T \int_{\Gamma_1} \left(\frac{\partial y}{\partial v_A} \right) H(\bar{y}) d\Gamma_1 ds \\ &\leq C_0 \int_S^T \int_{\Gamma_1} \left(\frac{\partial y}{\partial v_A} \right)^2 d\Gamma_1 ds \end{aligned} \quad (4.1)$$

$$\begin{aligned} &\left| \operatorname{Im} \int_S^T \int_{\Gamma_1} y \bar{y}_t H.v d\Gamma_1 ds - \operatorname{Im} \int_{\Omega} y H(y) d\Omega \Big|_S^T \right| \leq \\ &\frac{\sup_{x \in \overline{\Omega}} |H|}{2\alpha_1} \int_S^T \int_{\Gamma_1} |y|^2 d\Gamma_1 ds + 2\alpha_1 \sup_{x \in \overline{\Omega}} |H| \int_S^T \int_{\Gamma_1} |y_t|^2 d\Gamma_1 ds + \varepsilon \sup_{x \in \overline{\Omega}} |H(x)| E(S) \\ &\quad + \left(\frac{\varepsilon \alpha_1}{2} \sup_{x \in \overline{\Omega}} |H(x)| \right) |y|_{C([S,T], L^2(\Omega))}^2 \end{aligned} \quad (3.3c)$$

$$\begin{aligned} \left| \int_S^T \int_{\Omega} |\nabla_g y|_g^2 \operatorname{div} H d\Omega ds - \operatorname{Im} \int_S^T \int_{\Omega} \bar{y}_t y \operatorname{div} H d\Omega ds \right| &\leq C_1 \int_S^T \int_{\Gamma_1} \left(\frac{\partial y}{\partial v_A} \right)^2 d\Gamma_1 ds \\ &\quad + \frac{\varepsilon}{2} \int_S^T \int_{\Omega} |\nabla_g y|_g^2 d\Omega ds \\ &\quad + \left(\frac{\alpha_1}{2\varepsilon} \sup_{x \in \overline{\Omega}} |\nabla_g (\operatorname{div} H)| + \frac{1}{4} \sup_{x \in \overline{\Omega}} |\operatorname{div} H| \right) \int_S^T \int_{\Gamma_1} |y|^2 d\Gamma_1 ds \end{aligned} \quad (3.3d)$$

$$\left| \int_{\Sigma_1} \left(\frac{\partial y}{\partial \nu_A} \right)^2 d\Sigma_1 \right| \leq 3 \left[\int_S^T \int_{\Gamma_1} \left(- \int_0^t k'(t-s) y(s) ds \right)^2 d\Gamma_1 ds \right. \\ \left. + \int_S^T \int_{\Gamma_1} k(0) |y|^2 d\Gamma_1 ds + \int_S^T \int_{\Gamma_1} b^2 |y_t|^2 d\Gamma_1 ds \right]. \quad (3.3e)$$

Now using a compactness-uniqueness argument in the style of Lasiecka and Triggiani [7], we deduce

$$|y|_{C([S,T],L^2(\Omega))}^2 \leq \int_S^T \int_{\Gamma_1} |by_t|^2 d\Gamma_1 ds \quad (3.3f)$$

Insertion (3.3b), (3.3c), (3.3d) and (3.3f) in (3.3a), leads to

$$2a \int_S^T \int_{\Omega} |\nabla_g y|_g^2 d\Omega ds \leq 3(C_0 + C_1) \left[\int_S^T \int_{\Gamma_1} \left(- \int_0^t k'(t-s) y(s) ds \right)^2 d\Gamma_1 ds \right] \\ 3(C_0 + C_1) \left[\int_S^T \int_{\Gamma_1} k(0) |y|^2 d\Gamma_1 ds + \int_S^T \int_{\Gamma_1} |by_t|^2 d\Gamma_1 ds \right] \\ + C_2 \int_S^T \int_{\Gamma_1} |y|^2 d\Gamma_1 ds + \frac{\varepsilon}{2} \int_S^T \int_{\Omega} |\nabla_g \bar{y}|_g^2 d\Omega ds \\ + \varepsilon \sup_{x \in \bar{\Omega}} |H(x)| E(S) + 2\alpha_1 \sup_{x \in \bar{\Omega}} |H| \int_S^T \int_{\Gamma_1} |y_t|^2 d\Gamma_1 ds \\ + \left(\frac{\varepsilon \alpha_1}{2} \sup_{x \in \bar{\Omega}} |H(x)| \right) |y|_{C([S,T],L^2(\Omega))}^2 \quad (3.4)$$

Choosing ε so that $4a - \varepsilon > 0$, we obtain

$$(4a - \varepsilon) \int_S^T E(t) dt \leq [3C_3(C_0 + C_1) + \varepsilon \sup_{x \in \bar{\Omega}} |H(x)|] E(S) \\ + 2\alpha_1 \sup_{x \in \bar{\Omega}} |H| \frac{1}{\sup_{x \in \bar{\Gamma}_1} |b|} E(S) \\ + [3(C_0 + C_1) |k(0)|_{L^\infty}^2 + C_2] \frac{1}{f} E(S) \\ + [3(C_0 + C_1) + \frac{\varepsilon \alpha_1}{2} \sup_{x \in \bar{\Omega}} |H(x)|] \sup_{x \in \bar{\Gamma}_1} |b| E(S).$$

Where

$$C = \frac{1}{(4a - \varepsilon)} [3C_3(C_0 + C_1) + \varepsilon \sup_{x \in \bar{\Omega}} |H(x)| + 2\alpha_1 \sup_{x \in \bar{\Omega}} |H| \frac{1}{\sup_{x \in \bar{\Gamma}_1} |b|} \\ + [3(C_0 + C_1) + \frac{\varepsilon \alpha_1}{2} \sup_{x \in \bar{\Omega}} |H(x)|] \sup_{x \in \bar{\Gamma}_1} |b| \\ + [3(C_0 + C_1) |k(0)|_{L^\infty}^2 + C_2] \frac{1}{f}].$$

Letting $T \rightarrow +\infty$, we obtain for every $S \in \mathbb{R}_+$.

$$\int_S^{+\infty} E(t) dt \leq CE(S).$$

The desired conclusion follows now from Komornik [6].

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