Quaternion-valued functions in hyperholomorphic $F_G^\alpha(p, q, s)$ spaces

A. Kamal^a, T.I. Yassen^b

^aPort Said University, Faculty of Science, Department of Mathematics
Port Said, Egypt

Current address: Qassim University, College of Sciences and Arts in Muthnib.
Department of Mathematics, KSA.

^bThe Higher Engineering Institute in Al-Minia, Mania, 61111, Egypt.

Abstract

The goal of this article is to define a new class of hyperholomorphic functions, the so called $F_G^\alpha(p, q, s)$ Spaces. For this class, we study some its basic properties. Moreover, the relation which characterized $F_G^\alpha(p, q, s)$ Spaces by spaces $\mathcal{B}^\alpha$ are given. ©2019 All rights reserved.

Keywords: Quaternionic analysis, $F_G^\alpha(p, q, s)$ spaces, hyperholomorphic functions.
2010 MSC: MSC 30G35, MSC 46E15.

1. Introduction

Quaternion analysis is the generalizations of the theory of holomorphic functions in one complex variable to Euclidean space. The study of hyperholomorphic function spaces began with the interesting papers (see [1, 2, 3, 4]) and others. Quaternions are also recognized as a powerful tool for modeling and solving problems in theoretical as well as applied mathematics (see [5]).

1.1. Analytic Function Spaces

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the complex unit disk. The well known Bloch space is defined by:

$$\mathcal{B} = \{f : f \text{ analytic in } \mathbb{D} \text{ and } \mathcal{B}(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty\}.$$

Composing the Möbius transform $\varphi_a(z)$, which maps the unit disk $\mathbb{D}$ onto itself, and the fundamental solution of the two-dimensional real Laplacian on $\mathbb{D}$, we have the Green's function $g(z, a) = \ln \left| \frac{1 - \bar{a}z}{a - z} \right|$ with logarithmic singularity at $a \in \mathbb{D}$. Here, φ_a always stands for the Möbius transformation $\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$. Then, in [6] Zhao gave the following definition:

*Corresponding author

Email addresses: alaa_mohamed1@yahoo.com (A. Kamal), taha_hmour@yahoo.com (T.I. Yassen)

doi:10.31559/glm2020.8.1.3

Definition 1.1. Let f be an analytic function in $\mathbb{D}$ and let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If

$$\|f\|_{F(p,q,s)}^p = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \infty,$$

then $f \in F(p, q, s)$. Moreover, if

$$\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) = 0,$$

then $f \in F_0(p, q, s)$.

1.2. Quaternion function spaces

To introduce the meaning of hyperholomorphic functions, let $\mathbb{H}$ be the skew field of quaternions. This means we can write each element $w \in \mathbb{H}$ in the form

$$w = w_0 + w_1 i + w_2 j + w_3 k, \quad w_0, w_1, w_2, w_3 \in \mathbb{R},$$

where $1, i, j, k$ are the basis elements of $\mathbb{H}$. For these elements we have the multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, kj = -jk = i, ki = -ik = j.$$

The conjugate element $\bar{w}$ is given by $\bar{w} = w_0 - w_1 i - w_2 j - w_3 k$, and we have the property

$$w\bar{w} = \bar{w}w = \|w\|^2 = w_0^2 + w_1^2 + w_2^2 + w_3^2.$$

Moreover, we can identify each vector $\vec{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ with a quaternion x of the form

$$x = x_0 + x_1 i + x_2 j.$$

In follows we will work in $\mathbb{B}_1(0) \subset \mathbb{R}^3$, the unit ball in the real three-dimensional space. We will consider functions f defined on $\mathbb{B}_1(0)$ with values in $\mathbb{H}$. We define a generalized Cauchy-Riemann operator D by

$$Df = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2},$$

and it's conjugate operator by

$$\bar{D}f = \frac{\partial f}{\partial x_0} - i \frac{\partial f}{\partial x_1} - j \frac{\partial f}{\partial x_2}.$$

For these operators, we have that

$$D\bar{D} = \bar{D}D = \Delta_3,$$

where Δ_3 is the Laplacian for functions defined over domains in $\mathbb{R}^3$. For $|a| < 1$, we will denote by

$$\varphi_a(x) = (a - x)(1 - \bar{a}x)^{-1},$$

the Möbius transform, which maps the unit ball onto itself. Furthermore, let

$$g(x, a) = \frac{1}{4\pi} \left(\frac{1}{|\varphi_a(x)|} - 1 \right)$$

be the modified fundamental solution of the Laplacian in $\mathbb{R}^3$ composed with the Möbius transform $\varphi_a(x)$. Especially, we denote for all $p \geq 0$

$$g^p(x, a) = \frac{1}{4^p \pi^p} \left(\frac{1}{|\varphi_a(x)|} - 1 \right)^p.$$

Let $f : \mathbb{B} \mapsto \mathbb{H}$ be a hyperholomorphic function. Then from [3], we have the seminorms

- $\mathcal{B}(f) = \sup_{x \in \mathbb{B}} (1 - |x|^2)^{3/2} |\overline{D}f(x)|,$
- $Q_p(f) = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\overline{D}f(x)|^2 g^p(x, a) d\mathbb{B}_x,$

Definition 1.2. Let $0 < \alpha < \infty$. Recall that the hyperholomorphic α -Bloch space (see [1]) is defined as follows:

$$\mathcal{B}^\alpha = \{f \in \ker D : \sup_{x \in \mathbb{B}} (1 - |x|^2)^{\frac{3\alpha}{2}} |\overline{D}f(x)| < \infty\},$$

the little α -Bloch type space $\mathcal{B}_0^\alpha$ is a subspace of $\mathcal{B}$ consisting of all $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|x| \rightarrow 1^-} (1 - |x|^2)^{\frac{3\alpha}{2}} |\overline{D}f(x)| = 0.$$

Quite recently, El-Sayed Ahmed and Omran in [7], gave the following definition:

Definition 1.3. Let f be quaternion-valued function in $\mathbb{B}$. For $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If

$$\|f\|_{F(p,q,s)}^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3q}{2}} \left(1 - |\varphi_a(x)|^2\right)^s d\mathbb{B}_x < \infty,$$

then $f \in F(p, q, s)$. Moreover, if

$$\lim_{|a| \rightarrow 1} \int_{\mathbb{B}} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3q}{2}} \left(1 - |\varphi_a(x)|^2\right)^s d\mathbb{B}_x = 0,$$

then $f \in F_0(p, q, s)$.

Now, we use the definition of green function in $\mathbb{R}^3$ (see [8])

$$G(x, a) = \frac{1 - |\varphi_a(x)|^2}{|1 - \overline{a}x|}.$$

Then, we introduce the following new definition of the so called the hyperholomorphic $F_G^\alpha(p, q, s)$ spaces.

Definition 1.4. Let $1 < \alpha$, $p < \infty$, $-2 < q < \infty$, and $s > 0$. Assume that f be hyperholomorphic function in the unit ball $\mathbb{B}_1(0)$. Then, $f \in F_G^\alpha(p, q, s)$, if

$$F_G^\alpha(p, q, s) = \left\{ f \in \ker D : \sup_{a \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{3\alpha q/2 + 2s} (G(x, a))^s d\mathbb{B}_x < \infty \right\}.$$

The space $F_{G,0}^\alpha(p, q, s)$ is subspace of $F_G^\alpha(p, q, s)$ consisting of all functions $f \in F_G^\alpha(p, q, s)$, such that

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x = 0.$$

The following lemma, we will need in the sequel:

Lemma 1.5. (see [9]). Let $f : \mathbb{B}_1(0) \rightarrow \mathbb{H}$ be a hyperholomorphic function. Let $0 < R < 1$, $1 < q$. Then for every $a \in \mathbb{B}_1(0)$

$$|\overline{D}f(a)|^q \leq \frac{3 \cdot 4^{2+q}}{\pi R^3 (1 - R^2)^{2q} (1 - |a|^2)^3} \int_{\mathcal{M}(a,R)} |\overline{D}f(x)|^q d\mathbb{B}_x.$$

2. Characterization of $F_G^\alpha(p, q, s)$ spaces in Clifford Analysis

The relations between $F_G^\alpha(p, q, s)$ space and $\mathcal{B}^\alpha$ spaces are given in quaternion sense. The results in this section are extensions and generalize of the results (see [3, 10]).

Proposition 2.1. *Let $\alpha, p \geq 1, -2 < q < \infty, s > 0, 0 < R < 1$, and assume that f be hyperholomorphic function in the unit ball $\mathbb{B}_1(0)$. Then*

$$\begin{aligned} & (1 - |a|^2)^{\frac{3\alpha q}{2} + 2s} |\overline{D}f(a)|^p \\ & \leq \frac{48(2)^{2p}(1+R)^s}{\pi R^3(1-R^2)^{2p+s}(1-|a|^2)^3} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x. \end{aligned}$$

Proof. Let $\mathcal{M}(a, R) = \{x \in \mathbb{B}_1(0) : |\varphi_a(x)| = \frac{|a-x|}{|1-\bar{a}x|} < R\}$ be pseudo-hyperbolic ball with center a and radius R . Then

$$\begin{aligned} & \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x \\ & \geq \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x. \end{aligned}$$

Since

$$|\varphi_a(x)| < R, \quad \forall x \in \mathcal{M}(a, R),$$

and

$$G(x, a) = \frac{1 - |\varphi_a(x)|^2}{|1 - \bar{a}x|}.$$

Then, we have

$$G(x, a) = \frac{1 - R^2}{1 + R}, \quad \text{where } 1 - R \leq |1 - \bar{a}x| \leq 1 + R.$$

Now, for fixed $R \in (0, 1)$ and $a \in \mathbb{B}_1(0)$.

Let $\mathcal{E}(a, R) \subset \mathcal{M}(a, R)$, such that

$$\mathcal{E}(a, R) = \{x \in \mathbb{B}_1(0) : |x - a| < R|1 - a|\}.$$

Then, we deduce that

$$\begin{aligned} & \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x \\ & \geq \left(\frac{1 - R^2}{1 + R} \right)^s \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} d\mathbb{B}_x \\ & \geq \left(\frac{1 - R^2}{1 + R} \right)^s \int_{\mathcal{E}(a, R)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} d\mathbb{B}_x \\ & \geq \left(\frac{1 - R^2}{1 + R} \right)^s (1 - |a|^2)^{\frac{3\alpha q}{2} + 2s} \int_{\mathcal{E}(a, R)} |\overline{D}f(x)|^p d\mathbb{B}_x. \end{aligned}$$

Now, using Lemma 2.1, we obtain

$$\begin{aligned} & \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x \\ & \geq \left(\frac{1 - R^2}{1 + R} \right)^s (1 - |a|^2)^{\frac{3\alpha q}{2} + 2s} \frac{\pi R^3 (1 - R^2)^{2p} (1 - |a|^2)^3}{3(4)^{p+2}} |\overline{D}f(a)|^p \\ & = \frac{\pi R^3 (1 - R^2)^{2p+s} (1 - |a|^2)^{\frac{3\alpha q}{2} + 2s+3}}{3(4)^{p+2} (1 + R)^s} |\overline{D}f(a)|^p, \end{aligned}$$

which implies that,

$$\begin{aligned} & (1 - |a|^2)^{\frac{3\alpha q}{2} + 2s} |\overline{D}f(a)|^p \\ & \leq \frac{48(2)^{2p} (1 + R)^s}{\pi R^3 (1 - R^2)^{2p+s} (1 - |a|^2)^3} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x. \end{aligned}$$

This completes the proof.

Corollary 2.2. From proposition 2.1, we get for $\alpha, p \geq 1, -2 < q < \infty$, and $s > 0$ that

$$F_G^\alpha(p, q, s) \subset \mathcal{B}^{\frac{\alpha q + 2s}{p}}.$$

Proposition 2.3. Let $\alpha, p \geq 1, -2 < q < \infty, s > 2$, and let f be a hyperholomorphic function in $B_1(0)$, $\forall a \in B_1(0), |a| < 1$ and $f \in \mathcal{B}^{\frac{\alpha q + 2s}{p}}$. Then, we have that

$$\int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x \leq k(\mathcal{B}^\gamma(f)),$$

where $\gamma = \frac{\alpha q + 2s}{p}$.

Proof.

$$(1 - |x|^2)^{\frac{3\alpha}{2}} |\overline{D}f(a)|^p \leq \mathcal{B}^\alpha(f).$$

(see [3]). Then,

$$\begin{aligned} & \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x \\ & \leq (\mathcal{B}^\gamma(f)) \int_{\mathbb{B}_1(0)} (G(x, a))^s d\mathbb{B}_x. \end{aligned}$$

Using the equality

$$G(x, a) = \frac{1 - |\varphi_a(x)|^2}{|1 - \bar{a}x|} = \frac{(1 - |a|^2)(1 - |x|^2)}{|1 - \bar{a}x|^3}, \quad (2.1)$$

where

$$1 - |x| \leq |1 - \bar{a}x| \leq 1 + |x|, \quad 1 - |a| \leq |1 - \bar{a}x| \leq 1 + |a| \leq 2. \quad (2.2)$$

Then, we get

$$\begin{aligned}
 & \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x \\
 & \leq (\mathcal{B}^\gamma(f)) \int_{\mathbb{B}_1(0)} \frac{(1 - |a|^2)^s (1 - |x|^2)^s}{|1 - \overline{a}x|^{3s}} d\mathbb{B}_x. \\
 & \leq 2^{2s} (\mathcal{B}^\gamma(f)) \int_{\mathbb{B}_1(0)} \frac{(1 - |a|)^s (1 - |x|)^s}{(1 - |x|^s)(1 - |a|)^{2s}} d\mathbb{B}_x. \\
 & \leq 2^{2s} (\mathcal{B}^\gamma(f)) \int_{\mathbb{B}_1(0)} \frac{1}{(1 - |a|)^s} d\mathbb{B}_x. \\
 & \leq k(\mathcal{B}^\gamma(f)).
 \end{aligned}$$

Therefore, the proof of proposition is complete.

Corollary 2.4. From proposition 2.3, we get for $\alpha, p \geq 1, -2 < q < \infty$, and $s > 2$, that

$$\mathcal{B}^{\frac{\alpha q + 2s}{p}} \subset F_G^\alpha(p, q, s).$$

The results in Corollary 2.2 and Corollary 2.4 prove the following theorem, which give to us the characterization for the hyperholomorphic Bloch space by the integral norms of $F_G^\alpha(p, q, s)$ spaces of hyperholomorphic functions.

Theorem 2.5. Let f be a hyperholomorphic function in $B_1(0)$. Then for $\alpha, p \geq 1, -2 < q < \infty$, and $s > 2$, we have

$$\mathcal{B}^{\frac{\alpha q + 2s}{p}} = F_G^\alpha(p, q, s).$$

For characterization the little hyperholomorphic Bloch space, used the the same arguments in the previous theorem to prove the following theorem.

Theorem 2.6. Let f be a hyperholomorphic function in $B_1(0)$. Then for $\alpha, p \geq 1, -2 < q < \infty$, and $s > 2$, we have

$$\mathcal{B}_0^{\frac{\alpha q + 2s}{p}} = F_{G,0}^\alpha(p, q, s).$$

Theorem 2.7. Let $0 < R < 1$. Then for the hyperholomorphic function f in $\mathbb{B}_1(0)$, the following are equivalent

(a) $f \in \mathcal{B}^{\alpha q + 2s}$.

(b) For each $-2 < q < \infty, 1 \leq \alpha \leq \infty$, and $0 < p < \infty$

$$\sup_{a \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x < +\infty.$$

(c) For each $-2 < q < \infty, 1 \leq \alpha \leq \infty$, and $0 < p < \infty$

$$\sup_{a \in \mathbb{B}_1(0)} \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} d\mathbb{B}_x < +\infty.$$

(d) For each $-2 < q < \infty, 1 \leq \alpha \leq \infty$, and $0 < p < \infty$

$$\sup_{a \in \mathbb{B}_1(0)} |\mathcal{M}(a, R)|^{\frac{\alpha q}{2} + \frac{2s}{3}} \int_{\mathcal{M}(a, R)} |\overline{D}|f(x)|^p d\mathbb{B}_x < +\infty.$$

Proof. To prove (a) implies (b). Using (2.1) and (2.2), we have

$$\begin{aligned} & \sup_{a \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x \\ & \leq \sup_{a \in \mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} \int_{\mathbb{B}_1(0)} (G(x, a))^s d\mathbb{B}_x. \end{aligned}$$

Using the same steps as in Proposition 2.3, we obtain

$$\begin{aligned} & \sup_{a \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x \\ & \leq 2^{2s} \|f\|_{\mathcal{B}^{\frac{\alpha q + 2s}{p}}}^p \int_{\mathbb{B}_1(0)} \frac{1}{(1 - |a|)^s} d\mathbb{B}_x \\ & \leq K_1 \|f\|_{\mathcal{B}^{\frac{\alpha q + 2s}{p}}}^p \\ & < \infty. \end{aligned}$$

(b) implies (c), using the same steps as in Proposition 2.1, we deduce that

$$\begin{aligned} & \sup_{a \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x \\ & \geq \left(\frac{1 - R^2}{2} \right)^s \sup_{a \in \mathbb{B}_1(0)} \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} d\mathbb{B}_x. \end{aligned}$$

For (c) implies (d), we use the fact $(1 - |x|^2)^3 \approx |\mathcal{M}(a, R)|$, $\forall x \in \mathcal{M}(a, R)$ (see [4]). Then

$$\begin{aligned} & \sup_{a \in \mathbb{B}_1(0)} \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} d\mathbb{B}_x \\ & \approx |\mathcal{M}(a, R)|^{\frac{\alpha q}{2} + \frac{2s}{3}} \sup_{a \in \mathbb{B}_1(0)} \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p d\mathbb{B}_x. \end{aligned}$$

For (d) implies (a). Form Lemma 1.5, we have

$$\begin{aligned} & |\overline{D}f(a)|^p (1 - |a|^2)^{\frac{3\alpha q}{2} + 2s} \\ & \leq \frac{3 \cdot 4^{2+p} (1 - |a|^2)^{\frac{3\alpha q}{2} + 2s}}{\pi R^3 (1 - R^2)^{2p} (1 - |a|^2)^3} \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p d\mathbb{B}_x \\ & = \frac{3 \cdot 4^{2+p} (1 - |a|^2)^{\frac{3\alpha q}{2} + 2s}}{\pi R^3 (1 - R^2)^{2p} (1 - |a|^2)^3} \times \frac{(1 - R^2 |a|^2)^{\frac{3\alpha q}{2} + 2s} R^{\frac{3\alpha q}{2} + 2s}}{(1 - R^2 |a|^2)^{\frac{3\alpha q}{2} + 2s} R^{\frac{3\alpha q}{2} + 2s}} \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p d\mathbb{B}_x. \end{aligned}$$

Now, since

$$|\mathcal{M}(a, R)| = \frac{(1 - |a|^2)^3}{(1 - R^2 |a|^2)^3} R^3.$$

Also, we used the following inequalities

$$1 - R^2 \leq 1 - R^2 |a|^2 \leq 1 + R^2 \quad \text{and} \quad 1 - |a|^2 \leq 1 - R^2 |a|^2 \leq 1 + |a|^2.$$

Then, we have

$$\begin{aligned}
 & |\overline{D}f(a)|^p (1 - |a|^2)^{\frac{3\alpha q}{2} + 2s} \\
 \leq & \frac{3 \cdot 4^{2+p} |\mathcal{M}(a, R)|^{\frac{\alpha q}{2} + \frac{2s}{3}}}{\pi R^3 (1 - R^2)^{2p} (1 - |a|^2)^3} \times \frac{(1 - R^2 |a|^2)^{\frac{3\alpha q}{2} + 2s}}{R^{\frac{3\alpha q}{2} + 2s}} \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p d\mathbb{B}_x \\
 \leq & \frac{3 \cdot 4^{2+p} |\mathcal{M}(a, R)|^{\frac{\alpha q}{2} + \frac{2s}{3}}}{\pi R^3 (1 - R^2)^{2p} (1 - R^2)^3} \times \frac{(1 + R^2)^{\frac{3\alpha q}{2} + 2s}}{R^{\frac{3\alpha q}{2} + 2s}} \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p d\mathbb{B}_x \\
 = & \frac{3 \cdot 4^{2+p} (1 + R^2)^{3(\frac{\alpha q}{2} + \frac{2s}{3})}}{\pi R^{3(1 + \frac{\alpha q}{2} + \frac{2s}{3})} (1 - R^2)^{2p+3}} \times |\mathcal{M}(a, R)|^{\frac{\alpha q}{2} + \frac{2s}{3}} \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p d\mathbb{B}_x.
 \end{aligned}$$

Therefore, our theorem is proved.

From Theorem 2.7, using the same arguments, we directly obtain the following theorem.

Theorem 2.8. Let $0 < R < 1$. Then for the hyperholomorphic function f in $\mathbb{B}_1(0)$, the following are equivalent

(a) $f \in \mathcal{B}_0^{\alpha q + 2s}$.

(b) For each $-2 < q < \infty$, $1 \leq \alpha \leq \infty$, and $0 < p < \infty$

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (G(x, a))^s d\mathbb{B}_x = 0.$$

(c) For each $-2 < q < \infty$, $1 \leq \alpha \leq \infty$, and $0 < p < \infty$

$$\lim_{|a| \rightarrow 1^-} \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} d\mathbb{B}_x = 0.$$

(d) For each $-2 < q < \infty$, $1 \leq \alpha \leq \infty$, and $0 < p < \infty$

$$\lim_{|a| \rightarrow 1^-} |\mathcal{M}(a, R)|^{\frac{\alpha q}{2} + \frac{2s}{3}} \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p d\mathbb{B}_x = 0.$$

The following theorem give another relation between the quaternion $\mathcal{B}^\alpha$ space and the quaternion valued-functions space $F_G^\alpha(p, q, s)$.

Theorem 2.9. Let f be a hyperholomorphic function in $B_1(0)$. Then for $\alpha, p \geq 1$, $-2 < q < \infty$, and $\beta, s > 0$, we have

$$\|f\|_{\mathcal{B}^{\frac{\alpha q + 2s}{p}}}^p \approx \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (1 - |\varphi_a(x)|^2)^\beta (G(x, a))^s d\mathbb{B}_x.$$

Proof. From Theorem 2.7, we have

$$\|f\|_{\mathcal{B}^{\frac{\alpha q + 2s}{p}}}^p \approx \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} d\mathbb{B}_x.$$

Let the constant

$$C(R) = (1 - R^2)^\beta \left(\frac{1 - R^2}{1 + R} \right)^s,$$

since $C(R)$ depending on R is finite, then

$$\|f\|_{\mathcal{B}^{\frac{\alpha q + 2s}{p}}}^p \approx \sup_{a \in \mathbb{B}_1(0)} \int_{\mathcal{M}(a, R)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2} + 2s} (1 - R^2)^\beta \left(\frac{1 - R^2}{1 + R} \right)^s d\mathbb{B}_x. \quad (2.3)$$

Since $x \in \mathcal{M}(\alpha, R)$, then $|\varphi_\alpha(x)| < R$, $|\varphi_\alpha(x)|^2 < R^2$, and $1 - |\varphi_\alpha(x)|^2 > 1 - R^2$.

Then, we have

$$\begin{aligned} \|f\|_{\mathcal{B}^{\frac{\alpha q+2s}{p}}}^p &\approx \sup_{\alpha \in \mathbb{B}_1(0)} \int_{\mathcal{M}(\alpha, R)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2}+2s} (1 - R^2)^\beta \left(\frac{1 - R^2}{1 + R}\right)^s d\mathbb{B}_x \\ &\leq \lambda \sup_{\alpha \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2}+2s} (1 - |\varphi_\alpha(x)|^2)^\beta (G(x, \alpha))^s d\mathbb{B}_x. \end{aligned} \quad (2.4)$$

Conversely, we have

$$\begin{aligned} &\sup_{\alpha \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2}+2s} (1 - |\varphi_\alpha(x)|^2)^\beta (G(x, \alpha))^s d\mathbb{B}_x \\ &\leq \|f\|_{\mathcal{B}^{\frac{\alpha q+2s}{p}}}^p \sup_{\alpha \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} (1 - |\varphi_\alpha(x)|^2)^\beta (G(x, \alpha))^s d\mathbb{B}_x. \end{aligned} \quad (2.5)$$

Using (2.1) and (2.2), we obtain

$$\begin{aligned} &\sup_{\alpha \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2}+2s} (1 - |\varphi_\alpha(x)|^2)^\beta (G(x, \alpha))^s d\mathbb{B}_x \\ &\leq \|f\|_{\mathcal{B}^{\frac{\alpha q+2s}{p}}}^p \sup_{\alpha \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} \frac{(1 - |x|^2)^\beta (1 - |\alpha|^2)^\beta}{(1 - |x|)^\beta} \frac{(1 - |x|^2)^s (1 - |\alpha|^2)^s}{(1 - |\alpha|)^{2s} (1 - |x|)^s} d\mathbb{B}_x \\ &\leq 2^{2(\beta+s)} \|f\|_{\mathcal{B}^{\frac{\alpha q+2s}{p}}}^p \sup_{\alpha \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} (1 - |\alpha|)^{\beta-s} d\mathbb{B}_x. \end{aligned} \quad (2.6)$$

Then, we obtain

$$\|f\|_{\mathcal{B}^{\frac{\alpha q+2s}{p}}}^p \geq \lambda \sup_{\alpha \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3\alpha q}{2}+2s} (1 - |\varphi_\alpha(x)|^2)^\beta (G(x, \alpha))^s d\mathbb{B}_x. \quad (2.7)$$

From (2.4) and (2.7) the proof is complete.

3. conclusion

We define a new class of hyperholomorphic functions $F_G^\alpha(p, q, s)$ Spaces. For this class, we study some its basic properties. Moreover, we study the generalized quaternion space $F_G^\alpha(p, q, s)$ and characterize their relations to the quaternion $\mathcal{B}^\alpha$ space.

4. Acknowledgements

The authors would like to thank the referees for their useful comments, which improved the original manuscript.

References

- [1] A. El-Sayed Ahmed, "On weighted α -Besov spaces and α -Bloch spaces of Quaternion-valued functions," *Numerical Functional Analysis and Optimization*, vol. 29, no. 9-10, pp. 1064–1081, 2008. [1, 1.2](#)
- [2] A. El-Sayed Ahmed and F. Asiri, "Characterizations of weighted Bloch space by $Q_{p,\omega}$ -type spaces of Quaternion-valued functions," *Journal of Computational and Theoretical Nanoscience*, vol. 12, no. 11, pp. 4250–4255, 2015. [1](#)
- [3] K. a. K. Gürlebeck, M. Shapiro, and L. Tovar, "On Q_p -spaces of Quaternion-valued functions," *Complex Variables and Elliptic Equations*, vol. 39, no. 2, pp. 115–135, 1999. [1, 1.2, 2, 2](#)
- [4] K. Gürlebeck and A. El-Sayed Ahmed, "On series expansions of hyperholomorphic B^q functions," *Advances in Analysis and Geometry*, pp. 113–129, 2004. [1, 2](#)
- [5] H. Malonek, "Quaternions in applied sciences: A historical perspective of a mathematical concept," *International Conference on the Applications of Computer Science and Mathematics in Architecture and Civil Engineering*, Weimar, 2003. [1](#)
- [6] R. Zhao, "On a general family of function spaces," *Suomalainen Tiedeakatemia*, 1996. [1.1](#)

- [7] A. El-Sayed Ahmed, S. Omran, *et al.*, "Weighted classes of quaternion-valued functions," *Banach Journal of Mathematical Analysis*, vol. 6, no. 2, pp. 180–191, 2012. [1.2](#)
- [8] S. Bernstein, "Harmonic Q_p spaces," *Computational Methods and Function Theory*, vol. 9, no. 1, pp. 285–304, 2009. [1.2](#)
- [9] O. Reséndis and S. Tovar, "Besov-type characterizations for Quaternion Bloch functions," *Finite or Infinite Dimensional Complex Analysis and Applications*, pp. 207–219, 2004. [1.5](#)
- [10] K. Gürlebeck and A. El-Sayed Ahmed, "Integral norms for hyperholomorphic Bloch-functions in the unit ball of $\mathbb{R}^3$," *Progress in Analysis*, vol. 1, no. 1, pp. 253–263, 2003. [2](#)