On lambda-statistical convergence using f -density

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Abstract

We study the concept of f_λ -statistical convergence in a probabilistic normed space, where f is an unbounded modulus function. Also we are trying to investigate some relation between the ordinary convergence and module statistical convergence for every unbounded modulus function. Later on, we study f_λ -statistical convergence with partial average too.

Keywords: f -statistical convergence, Probabilistic Normed Space, f -density, modulus function, λ -statistical convergence, partial average.

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1. Introduction and preliminaries

The study of summability theory and convergence of sequences has been one of the most important and active area of research work in Pure mathematics for the last several decades. Its extensive works are also applicable in Topology, Functional Analysis, Fourier Analysis, Measure Theory, Applied Mathematics, Mathematical Modelling, Computer Science etc. In recent years, the concept of statistical convergence of sequences which was first introduced by Fast [8] as an extension of the usual concept of sequential limits, has also been used as a tool by many mathematicians to solve many open problems in the area of sequence spaces and summability theory and some other applications as well. One may refer to ([6],[16],[12],[14],[17],[19],[21],[22],[23],[25],[31],[10]).

In 2014, A. Aizpuru et al [1] introduced a new concept of density for sets of natural numbers with respect to the modulus function. They studied and characterized the generalization of this notion of f -density with statistical convergence and proved that ordinary convergence is equivalent to the module statistical convergence for every unbounded modulus function. They also worked on double sequence spaces for the results of f -statistical convergence by using unbounded modulus function. Savaş and Borgohain [24] introduced some new spaces of lacunary f -statistical A -convergent sequences of order α .

Menger [13] introduced the notion of metric space under the name of statistical metric space, which is known as probabilistic metric space. The idea of Menger was to use distribution functions instead of nonnegative real numbers. The probabilistic theory has become an area of active research for the last forty years. It has a wide range of applications in functional analysis also[7]. An important family of probabilistic metric spaces are probabilistic normed spaces (briefly, PN-spaces).

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The notion of probabilistic normed spaces was introduced by Sherstnev [30] in 1963 and later on studied by various authors, see ([2],[3], [4], Savas and Mohiuddine [29]).

2. Preliminary Concepts

A sequence (x_i) of real numbers is statistically convergent to L if for arbitrary $\varepsilon > 0$, the set $K(i) = \{i \leq k : |x_i - L| \geq \varepsilon\}$ has natural density zero, i.e.,

$$\lim_k \frac{1}{k} \sum_{j=1}^k \chi_{K(i)}(j) = 0,$$

where $\chi_{K(i)}$ denotes the characteristic function of $K(i)$.

By a modulus function, we mean a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which satisfies ([15]) :

1. $f(x) = 0$ if and only if $x = 0$.
2. $f(x+y) \leq f(x) + f(y)$ for every $x, y \in \mathbb{R}^+$.
3. f is increasing.
4. f is continuous from the right at 0.

Note: A modulus function must be continuous on $\mathbb{R}_+$. Examples of moduli are $f(x) = \frac{x}{1+x}$ and $f(x) = x^p, 0 < p \leq 1$.

Let $A \subseteq \mathbb{N}$, f -density of A is defined as $\delta_f(A) = \lim_k \frac{f(|A(i)|)}{f(k)}$, (in case this limit exists) where $A(i) = \{k \in A : k \leq i\}$ and f is an unbounded modulus function.

Let (x_i) be a sequence in X (X is a normed space). If for each $L > 0$, $A = \{i \leq k : \|x_i - L\| > \varepsilon\}$ has f -density zero, then it is said that (x_i) is f -statistically convergent to $L \in X$, and we write it as $f\text{-stat} \lim x_i = L$. Note that $\delta(A) = 1 - \delta(\mathbb{N} \setminus A)$.

A triangular norm (t -norm) is a continuous mapping $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ such that $([0, 1], *)$ is an abelian monoid with unit one and $c * d \geq a * b$ if $c \geq a$ and $d \geq b$ for all $a, b, c, d \in [0, 1]$. Let X be a real linear space and $N : X \rightarrow D$, where D is the set of all distribution functions $g : \mathbb{R} \rightarrow \mathbb{R}_0^+$ such that it is non-decreasing and left-continuous with $\inf_{t \in \mathbb{R}} g(t) = 0$ and $\sup_{t \in \mathbb{R}} g(t) = 1$.

The probabilistic norm or N -norm is a triangular norm satisfying the following conditions:

1. $N_p(0) = 0$,
2. $N_p(t) = 1$ for all $t > 0$ iff $p = 0$,
3. $N_{\alpha p}(t) = N_p\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$ and for all $t > 0$,
4. $N_{p+q}(s+t) \geq N_p(s) * N_q(t)$ for all $p, q \in X$ and $s, t \in \mathbb{R}_0^+$;

where N_p means $N(p)$ and $N_p(t)$ is the value of N_p at $t \in \mathbb{R}$. $(X, N, *)$ is named as a probabilistic normed space, in short PN-space.

Let $\lambda = (\lambda_i)$ be a non-decreasing sequence of positive numbers such that ,

$$\lambda_1 = 1, \lambda_{i+1} \leq \lambda_i + 1 \text{ and } \lambda_i \rightarrow \infty \text{ as } i \rightarrow \infty.$$

Note: The collection of all such sequences λ will be denoted by Λ .

If $\lambda_i \leq \mu_i, \forall \lambda, \mu \in \Lambda$, we have

$$I_i \subset J_i, \text{ where } I_i = [i - \lambda_i + 1, i] \text{ and } J_i = [i - \mu_i + 1, i]. \quad (2.1)$$

A sequence (x_i) of real numbers is said to be λ -statistically convergent to L if for any $\varepsilon > 0$,

$$\lim_{i \rightarrow \infty} \frac{1}{\lambda_i} |\{i \in I_i : |x_i - L| \geq \varepsilon\}| = 0,$$

where $I_i = [i - \lambda_i + 1, i]$ and $|A|$ denotes the cardinality of $A \subset \mathbb{N}$ (refer [14]).

Savaş ([18]) studied λ -statistical convergence in random 2-normed spaces and I_λ -statistical convergence of sequences in topological groups. In [4], Alotaibi studied the notion of λ -statistical convergence for single sequences in probabilistic normed spaces. Also Savas and Mohiuddine [29] studied $\bar{\lambda}$ -statistically convergent double sequences in probabilistic normed spaces.

In this paper, we study the f_λ -statistical convergence with respect to the probabilistic norm N in the PN-space $(X, N, *)$. We also investigate some results on the new concept of f_λ -statistical convergence with partial average. Moreover, the concepts of f_λ -statistical limits, f_λ -cluster points and f_λ -equivalence are introduced and try to find out the relations among them.

3. Main Results

Definition 3.1. Let $(X, N, *)$ be a PN-space. Then a sequence $x = (x_i)$ is said to be f_λ -statistically convergent to L with respect to the probabilistic norm N provided that, for every $t > 0$ and $\varepsilon > 0$,

$$\delta_{f_\lambda}(|\{i \in I_i : N_{x_i-L}(t) \leq 1 - \varepsilon\}|) = 0, \text{ where } I_i = [i - \lambda_i + 1, i].$$

Definition 3.2. Let $(X, N, *)$ be a PN-space. An element $L \in X$ is said to be an ordinary limit point of the sequence $(x_i) \in X$ with respect to the probabilistic norm N if there is a subsequence of the sequence (x_i) which also converges to L with respect to N . We denote,

$$\Omega_N(x) = \text{set of all ordinary limit points of the sequence w.r.t. } N.$$

Definition 3.3. Let $(X, N, *)$ be a PN-space. An element $L \in X$ is said to be a f_λ -limit point of the sequence (x_i) w.r.t. N if there is a subset $I = \{i_m : i_1 < i_2 < i_3, \dots\}$ of $\mathbb{N}$ such that $\delta_{f_\lambda}(I) = 1$ and (x_{i_m}) is statistically convergent to L with respect to the probabilistic norm N . We denote it as,

$$Y_\lambda^f(x) = \text{set of all } f_\lambda\text{-limit points of } x.$$

Definition 3.4. Let $(X, N, *)$ be a PN-space. An element $L \in X$ is said to be a f_λ -cluster point of (x_i) w.r.t. N if for each $t > 0$ and $\varepsilon > 0$, $\delta_{f_\lambda}(I) = 1$ where $I = \{i \in I_i : N_{x_i-L}(t) > 1 - \varepsilon\}$. We define it as,

$$\Pi_\lambda^f(x) = \text{set of all } f_\lambda\text{-cluster points of } x.$$

Particular case : $Y_\lambda(x) \subset Y_\lambda^f(x)$ and $\Pi_\lambda(x) \subset \Pi_\lambda^f(x)$, since $\delta_f(A) = 0$ implies $\delta(A) = 0$, for some $A \subseteq \mathbb{N}$.

In the following results, we investigate the relations between f -limit points, f -cluster points and ordinary limit points of $(X, N, *)$ with respect to the non-decreasing sequences of positive numbers (λ_i) and (μ_i) where $\lambda_i \leq \mu_i$. Also in Theorem 3.2, we establish the result of f_λ -statistical convergence of (x_i) with respect to f_λ -limit points and f_λ -cluster points of $(X, N, *)$.

Theorem 3.5. Let $(X, N, *)$ be a PN-space. For $\lambda_i \leq \mu_i$, we have f_μ -limit points $\Rightarrow f_\lambda$ -cluster points $\Rightarrow$ ordinary limit points with respect to N if $\liminf_i \frac{f(\lambda_i)}{f(\mu_i)} > 0$

Proof. Let L be a f_μ -limit point of (x_i) . Then for $R = \{i_m : i_1 < i_2 < \dots\}$ such that $\delta_{f_\mu}(R) = 1$ and $\lim_m x_{i_m} = L$ with respect to N . For each $t > 0, \varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that for $i > i_0, N_{x_i-L}(t) > 1 - \varepsilon$.

Hence, $\{i \in J_i : N_{x_i-L}(t) \leq 1 - \varepsilon\} \subset \mathbb{N} \setminus \{i_{i_0+1}, i_{i_0+2}, \dots\}$ which implies that, $\delta_{f_\mu}(|\{i \in J_i : N_{x_i-L}(t) \leq 1 - \varepsilon\}|) = 0$, where $J_i = [i - \mu_i + 1, i]$.

Also

$$\begin{aligned} \{i \in J_i : N_{x_i-L}(t) \leq 1 - \varepsilon\} &\supset \{i \in I_i : N_{x_i-L}(t) \leq 1 - \varepsilon\} \text{ (from (1))} \\ \Rightarrow \{i \in J_i : N_{x_i-L}(t) \leq 1 - \varepsilon\} &\geq \{i \in I_i : N_{x_i-L}(t) \leq 1 - \varepsilon\} \\ \Rightarrow \frac{f(|\{i \in J_i : N_{x_i-L}(t) \leq 1 - \varepsilon\}|)}{f(\mu_i)} &\geq \frac{f(|\{i \in I_i : N_{x_i-L}(t) \leq 1 - \varepsilon\}|)}{f(\mu_i)} \\ &\geq \frac{f(|\{i \in I_i : N_{x_i-L}(t) \leq 1 - \varepsilon\}|) \cdot f(\lambda_i)}{f(\lambda_i) \cdot f(\mu_i)} \end{aligned}$$

Since $\liminf_i \frac{f(\lambda_i)}{f(\mu_i)} > 0$,

$$\delta_{f_\lambda}(|\{i \in I_i : N_{x_i-L}(t) \leq 1 - \varepsilon\}|) = 0, \text{ i.e. } \delta_{f_\lambda}(|\{i \in I_i : N_{x_i-L}(t) > 1 - \varepsilon\}|) = 1,$$

which means that L is a f_λ -cluster point of (x_i) in $(X, N, *)$.

To prove the second part, let us take L to be a f_λ -cluster point of (x_i) in $(X, N, *)$. Then for given $t > 0$ and $\varepsilon > 0$, we have,

$$\delta_{f_\lambda}(|\{i \in I_i : N_{x_i-L}(t) > 1 - \varepsilon\}|) = 1, \text{ where } I_i = [i - \lambda_i + 1, i].$$

Construct a set $I = \{i_m : i_1 < i_2 < i_3 < \dots\}$ such that a subsequence (x_{i_m}) of (x_i) that converges to L with respect to the probabilistic norm N . Thus, L is an ordinary limit point of (x_i) in $(X, N, *)$.

Hence, f_λ -cluster point of $(x_i) \Rightarrow$ ordinary limit point of (x_i) with respect to the probabilistic norm N . $\square$

Theorem 3.6. Let $(X, N, *)$ be a PN-space. Then the f_λ -limit points and f_λ -cluster points of a sequence (x_i) are same, say L , if it is f_λ -statistically convergent to L .

Proof. Let ρ be a f_λ -limit point such that $\rho \neq L$.

Construct two subsets of $\mathbb{N}$ as $K = \{k_m : k_1 < k_2 < k_3 < \dots\}$ and $L = \{l_m : l_1 < l_2 < l_3 < \dots\}$ such that $\delta_{f_\lambda}(K) = 1$ and $\delta_{f_\lambda}(L) = 1$. Also (x_{k_m}) and (x_{l_m}) are convergent to L and ρ respectively with respect to the probabilistic norm N .

We assume that ρ is the f_λ -limit point of X , then for given $t > 0$ and $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that $N_{x_{l_m}-\rho}(t) > 1 - \varepsilon, \forall m \geq m_0$.

Let

$$\begin{aligned} P &= \{l_m \in L : N_{x_{l_m}-\rho}(t) \leq 1 - \varepsilon\} \\ &\subset \mathbb{N} \setminus \{l_{m_0+1}, l_{m_0+2}, l_{m_0+3}, \dots\} \\ \Rightarrow \delta_{f_\lambda}(P) &= 0 \end{aligned}$$

Again, construct $Q = \{l_m \in L : N_{x_{l_m}-\rho}(t) > 1 - \varepsilon\}$ which implies that $\delta_{f_\lambda}(Q) = 1$.

Assume, $L = P \cup Q$, so that if $\delta_{f_\lambda}(Q) = 0 \Rightarrow \delta_{f_\lambda}(P \cup Q) = 0$, which implies that $\delta_{f_\lambda}(L) = 0$. This leads to a contradiction of the fact that $\delta_{f_\lambda}(L) = 1$.

Let $R = \{i \in I_i : N_{x_i-L}(t) \leq 1 - \varepsilon\}$, where $I_i = [i - \lambda_i + 1, i]$.

Since (x_i) is convergent to L with respect to the probabilistic norm N , then $N_{x_i-L}(t) > 1 - \varepsilon$. So we have for each $t > 0$ and $\varepsilon > 0$, the set $\delta_{f_\lambda}(\{i \in I_i : N_{x_i-L}(t) \leq 1 - \varepsilon\}) = 0$, which follows that $\delta_{f_\lambda}(R^c) = 1$ where $R^c = \{i \in I_i : N_{x_i-L}(t) > 1 - \varepsilon\}$.

If $\rho \neq L$, then $Q \cap R^c$ is empty and $Q \subset R$. Also $\delta_{f_\lambda}(R) = 0$ implies $\delta_{f_\lambda}(Q) = 0$, which leads to a contradiction that $\delta_{f_\lambda}(Q) = 1$. Hence L is also a f_λ -limit point of (x_i) in $(X, N, *)$.

To prove the second part, let us take ρ be a f_λ -cluster point such that $L \neq \rho$. Also, for each $t > 0$ and $\varepsilon > 0$, we construct the sets $A = \{i \in I_i : N_{x_i-L}(t) > 1 - \varepsilon\}$ and $B = \{i \in I_i : N_{x_i-\rho}(t) > 1 - \varepsilon\}$ such that $\delta_{f_\lambda}(A) = 1$ and $\delta_{f_\lambda}(B) = 1$ respectively.

We assume that $L \neq \rho$, which implies that $A \cap B = \emptyset$ and therefore $B \subset A^c$ where $A^c = \{i \in I_i : N_{x_i-L}(t) \leq 1 - \varepsilon\}$.

Also, as (x_i) is convergent to L with respect to N , we have $\delta_{f_\lambda}(A^c) = 0 \Rightarrow \delta_{f_\lambda}(B) = 0$, which is a contradiction.

Hence, $\rho = L$, i.e. L is also a f_λ -cluster point. This completes the proof. $\square$

In this section, we define the concept of f_λ -statistically Cauchy in $(X, N, *)$ and establish the relation between f -statistical Cauchy and f -statistical convergence of (x_i) with respect to (λ_i) and (μ_i) where $\lambda_i \leq \mu_i$.

Definition 3.7. Let $(X, N, *)$ be a PN-space. Then (x_i) is said to be f_λ -statistically Cauchy if $\delta_{f_\lambda}(|B_N(t)|) = 0$ where $B_N(t) = \{i, l \in I_i : N_{x_i-x_l}(t) \leq 1 - \varepsilon\}$.

Theorem 3.8. Let $(X, N, *)$ be PN-space. For $\lambda_i \leq \mu_i$, f_μ -statistical Cauchy implies f_λ -statistical convergence if $\lim_i \frac{f(\lambda_i)}{f(\mu_i)} > 0$.

Proof. Let $B_N(t) = \{i, l \in J_i : N_{x_i-x_l}(t) \leq 1 - \varepsilon\}$ such that $\delta_{f_\mu}(|B_N(t)|) = 0$, where $J_i = [i - \mu_i + 1, i]$.

Assume that $x = (x_i)$ is not f_μ -statistically convergent but convergent to L with respect to the probabilistic norm N , so $N_{x_i-L}(\frac{t}{2}) > 1 - \varepsilon$.

For given $t > 0, \varepsilon > 0$, let us take $r > 0$ such that $(1 - r) * (1 - r) \geq 1 - \varepsilon$.

Thus, for $i, l \in I_i = [i - \lambda_i + 1, i]$,

$$\begin{aligned} N_{x_i-x_l}(t) &= N_{x_i-L+L-x_l}\left(\frac{t}{2} + \frac{t}{2}\right) \\ &\geq N_{x_i-L}\left(\frac{t}{2}\right) * N_{x_l-L}\left(\frac{t}{2}\right) \\ &> (1 - r) * (1 - r) \\ &> 1 - \varepsilon \end{aligned}$$

Hence,

$$\begin{aligned} \left\{i \in I_i : N_{x_i-L}\left(\frac{t}{2}\right) > 1 - \varepsilon\right\} &\subset \{i, l \in I_i : N_{x_i-x_l}(t) > 1 - \varepsilon\} \\ \left\{i \in I_i : N_{x_i-L}\left(\frac{t}{2}\right) > 1 - \varepsilon\right\} &\subset \{i, l \in J_i : N_{x_i-x_l}(t) > 1 - \varepsilon\}, \text{ (by using (1))} \\ \Rightarrow \lim_i \frac{f(\{i, l \in J_i : N_{x_i-x_l}(t) \leq 1 - \varepsilon\})}{f(\mu_i)} &\leq \lim_i \frac{f(\{i \in I_i : N_{x_i-L}(\frac{t}{2}) \leq 1 - \varepsilon\})}{f(\mu_i)} \\ &\leq \lim_i \frac{f(\{i \in I_i : N_{x_i-L}(\frac{t}{2}) \leq 1 - \varepsilon\}) f(\lambda_i)}{f(\lambda_i) f(\mu_i)} \end{aligned}$$

Since (x_i) is not f_λ -statistical convergent, so with the help of $\lim_i \frac{f(\lambda_i)}{f(\mu_i)} > 0$, we get

$$\delta_{f_\lambda}\left(\left|\left\{i \in I_i : N_{x_i-l}\left(\frac{t}{2}\right) \leq 1 - \varepsilon\right\}\right|\right) = 1,$$

which leads to a contradiction that

$$\delta_{f_\mu}(|B_N(t)|) = 1, \text{ where } B_N(t) = \{i, l \in J_i : N_{x_i-x_l}(t) \leq 1 - \varepsilon\}.$$

This arrives to the conclusion that (x_i) is f_μ -statistically Cauchy implies that it is f_λ -statistically convergent if $\lim_i \frac{f(\lambda_i)}{f(\mu_i)} > 0$. This completes the proof. $\square$

Corollary 3.9. *Let $(X, N, *)$ be a PN-space. Then if (x_i) is f_λ -statistically Cauchy sequence then it has a Cauchy subsequence with respect to the probabilistic norm N .*

We also examine some results on f_λ -statistical equivalence with f_λ -limit points and f_λ -cluster points. Moreover, in Theorem 3.5., the relation between f -statistical Cauchy, f -statistical convergence and f -statistical equivalence of $(x_i) \in (X, N, *)$ with respect to (λ_i) .

Definition 3.10. Let $(X, N, *)$ be a PN-space. Then $x, y \in X$ is said to be f_λ -statistical equivalent if $\delta_{f_\lambda}(i \in I_i : x_i \neq y_i) = 0$.

Proofs of the following results are routine works, so omitted.

Theorem 3.11. *Let $(X, N, *)$ be a PN-space, If $x, y \in X$ are f_λ -statistical equivalent then f_λ -limit points and f_λ -cluster points of both x and y are same respectively.*

Theorem 3.12. *Let $(X, N, *)$ be a PN-space. Then the following are equivalent to each other:*

1. (x_i) is f_λ -statistically Cauchy.
2. (x_i) is f_λ -statistically convergent to L .
3. There exists $y \in X$ such that x and y are f_λ -statistical equivalent and y is also f_λ -statistically convergent to L .

4. f_λ -statistical convergence with partial average

Partial average/means of sequences has an important role in the theory of ergodic systems [34]. Very recently, Mark Burgin and Oktay Duman [33] studied the statistical convergence of sequences of real numbers in the direction of Statistics. In this section, we define the idea of f -partially statistical convergence and f_λ -partially statistical convergence of (x_i) with respect to the probabilistic norm N and investigate some relations between f_λ -partial statistical convergence and f_λ -statistical convergence.

Definition 4.1. Let $x = (x_i)$ be a sequence of real numbers. The partial average of $x = (x_i)$ is defined as,

$$\mu(x) = \left\{ \mu_i : \frac{1}{i} \sum_{k=1}^i x_k, i = 1, 2, 3, \dots \right\}.$$

Definition 4.2. Let $(X, N, *)$ be a PN-space. A sequence (x_i) is said to be partially convergent to L with respect to the probabilistic norm N , if for $\varepsilon > 0$ and $t > 0$, $\frac{1}{i} \sum_{k=1}^i N_{x_k - L}(t) > 1 - \varepsilon$.

Definition 4.3. Let $(X, N, *)$ be a PN-space. A sequence $x = (x_i)$ is said to be f_λ -partially statistical convergent to L with respect to the probabilistic norm N if its partial average $\mu(x)$ is f_λ -statistically convergent to L with respect to N , i.e. $\lim_i \frac{f(|\mu_i(x)|)}{f(\lambda_i)} = 0$, where $\mu_i(x) = \{i \in I_i : \frac{1}{i} \sum_{k=1}^i N_{x_k - L}(t) \leq 1 - \varepsilon\}$.

We denote it as $f_{\lambda_p} - \text{stat} \lim_i x_i = L$.

Theorem 4.4. *Let $(X, N, *)$ be a PN-space. If a bounded sequence $x = (x_i)$ is f_λ -statistically convergent to L with respect to the probabilistic norm N then it is also f_λ -partially statistical convergent to L with respect to N . But not conversely.*

Proof. Let $x = (x_i)$ be f_λ -statistical convergent to L with respect to N . Then for any $t > 0$ and $\varepsilon > 0$, $\lim_i \frac{f(|A_i(t)|)}{f(\lambda_i)} = 0$ where $A_i(t) = \{i \in I_i : N_{x_i-L}(t) \leq 1 - \frac{1}{\varepsilon}\}$, where $I_i = [i - \lambda_i + 1, i]$. Since (x_i) is a bounded sequence, so there exists $C > 0$ such that for $t > 0$, $N_{x_i-L}(t) \geq 1 - C$.

$$\begin{aligned} \sum_{i \in I_i} N_{x_i-L}(t) &= \sum_{i \in I_i, N_{x_i-L}(t) \leq 1 - \frac{1}{\varepsilon}} N_{x_i-L}(t) + \sum_{i \in I_i, N_{x_i-L}(t) > 1 - \frac{1}{\varepsilon}} N_{x_i-L}(t) \\ &\leq C(|\{i \in I_i : N_{x_i-L}(t) \leq 1 - \frac{1}{\varepsilon}\}|) + \frac{1}{\varepsilon} \end{aligned}$$

Consequently we get,

$$\frac{1}{i} \sum_{i \in I_i} N_{x_i-L}(t) \leq \frac{1}{i} \{C(|\{i \in I_i : N_{x_i-L}(t) \leq 1 - \frac{1}{\varepsilon}\}|) + \frac{1}{\varepsilon}\}$$

Also,

$$\begin{aligned} \{i \in I_i : \frac{1}{i} \sum_{i \in I_i} N_{x_i-L}(t) \leq 1 - \frac{1}{\varepsilon}\} &\leq C \frac{|\{i \in I_i : N_{x_i-L}(t) \leq 1 - \frac{1}{\varepsilon}\}|}{i} \\ &\leq |\{i \in I_i : N_{x_i-L}(t) \leq 1 - \frac{1}{\varepsilon}\}| \\ \Rightarrow \frac{f(\{i \in I_i : \frac{1}{i} \sum_{i \in I_i} N_{x_i-L}(t) \leq 1 - \frac{1}{\varepsilon}\})}{f(\lambda_i)} &\leq \frac{f(|\{i \in I_i : N_{x_i-L}(t) \leq 1 - \frac{1}{\varepsilon}\}|)}{f(\lambda_i)} \end{aligned}$$

By taking $i \rightarrow \infty$, we get f_λ -statistical convergence implies f_λ -partial statistical convergence.

For the converse part, let f be an unbounded modulus function such as $f(x) = \log(1+x)$. Take $(\mathbb{R}, N, *)$ be a PN-space with $a * b = ab$, $N_x(t) = \frac{t}{t+|x|}$. Now we define a sequence ,

$$x_i = \begin{cases} \sqrt{k} & \text{if } i - \sqrt{\lambda_i} + 1 \leq k \leq i; \\ 0, & \text{otherwise.} \end{cases}$$

which shows that (x_i) is f_λ -statistically divergent, but it is f_λ -partial statistical convergent. $\square$

Theorem 4.5. Let $(X, N, *)$ be a PN-space . Then the f_λ - partial statistical limit of $(x_i) \in X$ is unique with respect to the probabilistic norm N .

Proof. Proof is easy, so omitted. $\square$

Corollary 4.6. Let $(X, N, *)$ be a PN-space. If a sequence $(x_i) \in X$ is f_λ -partial statistical convergence then it has also a partial convergent subsequence with respect to N .

Theorem 4.7. Let $(X, N, *)$ be a PN-space. If $\lambda \in \Delta$ with $\liminf_i \frac{f(\lambda_i)}{i} > 0$, then a sequence (x_i) is partially statistical convergent to L with respect to the probabilistic norm N , then it is f_λ -statistically convergent to L with respect to N .

Proof. Let (x_i) be a partially convergent to L with respect to the probabilistic norm N , then for $\varepsilon > 0$ and $t > 0$, we have,

$$\frac{1}{i} \sum_{k=1}^i N_{x_k-L}(t) > 1 - \varepsilon.$$

We can write,

$$\begin{aligned}\sum_{k=1}^i N_{x_k-L}(t) &\geq f\left(\sum_{k=1}^i N_{x_k-L}(t)\right) \\ &\geq f(|\{k \leq i : N_{x_k-L}(t) \leq 1 - \varepsilon\}| \varepsilon) \\ &\geq cf(|\{k \leq i : N_{x_k-L}(t) \leq 1 - \varepsilon\}|)f(\varepsilon)\end{aligned}$$

Consequently we get,

$$\begin{aligned}\frac{1}{i} \sum_{k=1}^i N_{x_k-L}(t) &\geq \frac{c}{i} f(|\{k \leq i : N_{x_k-L}(t) \leq 1 - \varepsilon\}|)f(\varepsilon) \\ &\geq \frac{f(\lambda_i)}{i} \frac{f(|\{k \leq i : N_{x_k-L}(t) \leq 1 - \varepsilon\}|)}{f(\lambda_i)}\end{aligned}$$

Since $\liminf_i f \frac{f(\lambda_i)}{i} > 0$, so it follows that partial convergence implies f_λ -statistical convergence with respect to the probabilistic norm N . This completes the proof. $\square$

5. Conclusion

Convergence of statistical characteristics such as the average/mean and standard deviation are related to statistical convergence as we see in our results. This concept of convergence with respect to means/averages and standard deviations have been studied by many mathematicians. We have studied the convergence of the average/mean in the probabilistic normed spaces. It can be further extended in the direction of fuzzy real numbers and can be found very interesting results in intuitionistic fuzzy normed linear spaces too using various aspects.

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