Initial bounds for analytic and bi-univalent functions by means of (p, q) -Chebyshev polynomials defined by differential operator

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Abstract

In this paper, a subclass $\mathcal{T}_{\Sigma}^{\zeta}(m, \gamma, \lambda, p, q)$ of analytic and bi-univalent functions by means of (p, q) -Chebyshev polynomials is introduced. Certain coefficient bounds for functions belong to this subclass are obtained. In addition, the Fekete-Szegő problem is solved in this subclass.

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1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} , defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq \frac{1}{4})$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

A function $f \in \mathcal{A}$ is said to be in Σ the class of bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Lewin [9] showed that $|a_2| < 1.51$ for every function $f \in \Sigma$ given by (1.1). Posteriorly, Brannan

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and Clunie [3] improved Lewin's result and conjectured that $|a_2| \leq \sqrt{2}$ for every function $f \in \Sigma$ given by (1.1). The coefficient estimate problem for each of the following Taylor Maclaurin coefficients:

$$|a_n| \quad (n \in \mathbb{N}; n \geq 4)$$

It's still an open problem. Since then, there have been many researchers (see [2, 5, 6, 7, 11, 12, 14, 15, 13]) investigated several interesting subclasses of the class Σ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. In fact, its worth to mention that by making use of the Faber polynomial coefficient expansions Jahangiri, Jay M., and Samaneh G. Hamidi [8] have obtained estimates for the general coefficients $|a_n|$ for bi-univalent functions subject to certain gap series.

For any integer $n \geq 2$ and $0 < q < p \leq 1$, (p, q) -Chebyshev polynomials of the second kind is defined by the following recurrence relations:

$$R_n(x, s, p, q) = (p^n + q^n) x R_{n-1}(x, s, p, q) + (pq)^{n-1} s R_{n-2}(x, s, p, q), \quad (1.2)$$

with the initial values $R_0(x, s, p, q) = 1$ and $R_1(x, s, p, q) = (p + q)x$ and s is a variable.

Recently, Kızılateş, Naim and Bayram [16] defined (p, q) -Chebyshev polynomials of the first and second kinds and derived explicit formulas, generating functions and some interesting properties of these polynomials.

The generating function of the (p, q) -Chebyshev polynomials of the second kind is as follows:

$$H_{p,q}(z) = \frac{1}{1 - x p z \tau_p - x q z \tau_q - s p q z^2 \tau_{p,q}} = \sum_{n=0}^{\infty} R_n(x, s, p, q) z^n \quad (z \in \mathbb{U}).$$

where the Fibonacci operator τ_q Mason and Handscomb was introduced [17], by $\tau_q f(z) = f(qz)$, similarly, $\tau_{p,q} f(z) = f(pqz)$.

First off, we present some special cases of the polynomials $H_{p,q}(z)$:

1. For $p = q = 1$ and $s = -1$, we get the Chebyshev polynomials $R_n(x)$ of the second kind.
2. For $p = q = s = 1$ and $x = \frac{x}{2}$, we get the Fibonacci polynomials $F_n(x)$.
3. For $p = q = 1$, $s = 2y$ and $x = \frac{1}{2}$, we get the Jacobsthal polynomials $J_{n+1}(y)$.
4. If $p = q = s = 1$, then we get the Pell polynomials $P_{n+1}(x)$.

Let $w(z)$ and $v(w)$ be two analytic functions in the unit disk $\mathbb{U}$ with $w(0) = v(0) = 0$, $|w(z)| < 1$, $|v(z)| < 1$, and suppose that

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad \text{and} \quad v(w) = d_1 w + d_2 w^2 + d_3 w^3 + \cdots \quad (z, w \in \mathbb{U}). \quad (1.3)$$

Making use of the binomial series

$$(1 - \gamma)^m = \sum_{j=0}^m \binom{m}{j} (-1)^j \gamma^j \quad (m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

recently for $f \in \mathcal{A}$, Frasin [4] defined the differential operator $A_{m,\lambda}^{\zeta} f(z)$ as follows:

$$\begin{aligned} A^0 f(z) &= f(z), \\ A_{m,\gamma}^1 f(z) &= (1 - \gamma)^m f(z) + (1 - (1 - \gamma)^m) z f'(z) = A_{m,\lambda} f(z), \quad \gamma > 0; m \in \mathbb{N}, \\ A_{m,\gamma}^{\zeta} f(z) &= A_{m,\gamma} (A^{\zeta-1} f(z)) \\ &= z + \sum_{n=2}^{\infty} \left[1 + (n-1) C_j^m(\gamma) \right]^{\zeta} a_n z^n; \zeta \in \mathbb{N}, \end{aligned} \quad (1.4)$$

where

$$C_j^m(\gamma) = \sum_{j=1}^m \binom{m}{j} (-1)^{j+1} \gamma^j.$$

Using the relation (1.4), it is easily verified that

$$C_j^m(\gamma) z (A_{m,\gamma}^\zeta f(z))' = A_{m,\gamma}^{\zeta+1} f(z) - (1 - C_j^m(\gamma)) A_{m,\gamma}^\zeta f(z). \quad (1.5)$$

By specializing the parameters we observe that, for $m = 1$, $A_{1,\lambda}^\zeta$ defined by Al-Oboudi [1] and for $m = \gamma = 1$, $A_{1,1}^\zeta$ defined by Sălăgean [10].

2. The function class $\mathcal{T}_\Sigma^\zeta(m, \gamma, \lambda, p, q)$

Definition 2.1. A function $f(z) \in \Sigma$ is said to be in the class $\mathcal{T}_\Sigma^\zeta(m, \gamma, \lambda, p, q)$ if and only if

$$(1 - \lambda) \frac{A_{m,\gamma}^\zeta f(z)}{z} + \lambda (A_{m,\gamma}^\zeta f(z))' \prec H_{p,q}(z) = \frac{1}{1 - x p z \tau_p - x q z \tau_q - s p q z^2 \tau_{p,q}}$$

and

$$(1 - \lambda) \frac{A_{m,\gamma}^\zeta g(w)}{w} + \lambda (A_{m,\gamma}^\zeta g(w))' \prec H_{p,q}(w) = \frac{1}{1 - x p w \tau_p - x q w \tau_q - s p q w^2 \tau_{p,q}}$$

where $0 \leq \lambda \leq 1$, z, w in $\mathbb{U}$ and $g(w) = f^{-1}(w)$.

3. Coefficient bounds for the function class $\mathcal{T}_\Sigma^\zeta(m, \gamma, \lambda, p, q)$

We begin with the following result involving initial coefficient bounds for the function class $\mathcal{T}_\Sigma^\zeta(m, \gamma, \lambda, p, q)$.

Theorem 3.1. If $f(z)$ given by (1.1) is in the class $\mathcal{T}_\Sigma^\zeta(m, \gamma, \lambda, p, q)$. Then

$$|a_2| \leq \frac{(p+q)x\sqrt{(p+q)x}}{\sqrt{\chi}} \quad (3.1)$$

and

$$|a_3| \leq \frac{(p+q)^2 x^2}{\chi} + \frac{(p+q)x}{(1+2\lambda)(1+2C_j^m(\gamma))^\zeta}, \quad (3.2)$$

where

$$\chi = \left| (1 + 2C_j^m(\gamma))^\zeta (1 + 2\lambda) (p+q)^2 x^2 - [(p^2 + q^2)(p+q)x^2 + pqs] \left((1 + C_j^m(\gamma))^\zeta (1 + \lambda) \right)^2 \right|.$$

Proof. Let $f(z) \in \mathcal{T}_\Sigma^\zeta(m, \gamma, \lambda, p, q)$. Then there are analytic functions u and v , with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$, given by (1.3) and satisfying the following conditions:

$$(1 - \lambda) \frac{A_{m,\gamma}^\zeta f(z)}{z} + \lambda (A_{m,\gamma}^\zeta f(z))' = H_{p,q}(w(z)) \quad (3.3)$$

and

$$(1 - \lambda) \frac{A_{m,\gamma}^\zeta g(w)}{w} + \lambda (A_{m,\gamma}^\zeta g(w))' = H_{p,q}(v(w)), \quad (3.4)$$

where $g(w) = f^{-1}(w)$.

or some analytic functions

$$w(z) = c_1z + c_2z^2 + c_3z^3 + \cdots \quad (z \in \mathbb{U}),$$

and

$$v(w) = d_1w + d_2w^2 + d_3w^3 + \cdots \quad (w \in \mathbb{U}),$$

such that $w(0) = v(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$) and $|v(w)| < 1$ ($w \in \mathbb{U}$).

It follows from (3.3) and (3.4) that

$$\begin{aligned} (1 - \lambda) \frac{A_{m,\gamma}^{\zeta} f(z)}{z} + \lambda (A_{m,\gamma}^{\zeta} f(z))' \\ = 1 + R_1(x, s, p, q) c_1 z + [R_1(x, s, p, q) c_2 + R_2(x, s, p, q) c_1^2] z^2 + \cdots \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} (1 - \lambda) \frac{A_{m,\gamma}^{\zeta} g(w)}{w} + \lambda (A_{m,\gamma}^{\zeta} g(w))' \\ = 1 + R_1(x, s, p, q) d_1 w + [R_1(x, s, p, q) d_2 + R_2(x, s, p, q) d_1^2] w^2 + \cdots \end{aligned} \quad (3.6)$$

A short calculation shows that

$$(1 + \lambda)(1 + C_j^m(\gamma))^{\zeta} a_2 = R_1(x, s, p, q) c_1, \quad (3.7)$$

$$(1 + 2\lambda)(1 + 2C_j^m(\gamma))^{\zeta} a_3 = R_1(x, s, p, q) c_2 + R_2(x, s, p, q) c_1^2, \quad (3.8)$$

$$-(1 + \lambda)(1 + C_j^m(\gamma))^{\zeta} a_2 = R_1(x, s, p, q) d_1, \quad (3.9)$$

and

$$(1 + 2\lambda)(1 + 2C_j^m(\gamma))^{\zeta} (2a_2^2 - a_3) = R_1(x, s, p, q) d_2 + R_2(x, s, p, q) d_1^2. \quad (3.10)$$

From (3.7) and (3.9), we get

$$c_1 = -d_1, \quad (3.11)$$

and

$$2[(1 + C_j^m(\gamma))^{\zeta} (1 + \lambda)]^2 a_2^2 = R_1^2(x, s, p, q) (c_1^2 + d_1^2). \quad (3.12)$$

By adding (3.8) to (3.10), we have

$$2(1 + 2C_j^m(\gamma))^{\zeta} (1 + 2\lambda) a_2^2 = R_1(x, s, p, q) (c_2 + d_2) + R_2(x, s, p, q) (c_1^2 + d_1^2). \quad (3.13)$$

Therefore, from equalities (3.12) and (3.13) we find that

$$\begin{aligned} [2(1 + 2C_j^m(\gamma))^{\zeta} (1 + 2\lambda) R_1^2(x, s, p, q) - 2R_2(x, s, p, q) \left((1 + C_j^m(\gamma))^{\zeta} (1 + \lambda) \right)^2] a_2^2 \\ = R_1^3(x, s, p, q) (c_2 + d_2). \end{aligned} \quad (3.14)$$

Then

$$|a_2| \leq \frac{(p+q)x\sqrt{(p+q)x}}{\sqrt{\chi}},$$

where

$$\chi = \left| (1 + 2C_j^m(\gamma))^{\zeta} (1 + 2\lambda) (p+q)^2 x^2 - [(p^2 + q^2)(p+q)x^2 + pqs] \left((1 + C_j^m(\gamma))^{\zeta} (1 + \lambda) \right)^2 \right|.$$

Next, in order to find the bound on $|a_3|$, subtracting (3.10) from (3.8) and using (3.11), we get

$$2(1 + 2\lambda)(1 + 2C_j^m(\gamma))^{\zeta} a_3 = 2(1 + 2\lambda)(1 + 2C_j^m(\gamma))^{\zeta} a_2^2 + R_1(x, s, p, q) (c_2 - d_2). \quad (3.15)$$

Then in view of (3.15) and (3.11), we have

$$2(1+2\lambda)(1+2C_j^m(\gamma))^\zeta |a_3| \leq 2(1+2\lambda)(1+2C_j^m(\gamma))^\zeta |a_2|^2 + 2R_1(x, s, p, q)$$

From (3.7), we immediately have

$$|a_3| \leq |a_2|^2 + \frac{(p+q)x}{(1+2\lambda)(1+2C_j^m(\gamma))^\zeta}.$$

Now the assertion (3.2) follows from (3.1). This evidently completes the proof of Theorem 4.1. $\square$

By taking $\lambda = 1$ in Theorem 3.1, we have

Corollary 3.2. If $f(z)$ given by (1.1) is in the class $\mathcal{T}_\Sigma^\zeta(m, \gamma, p, q)$. Then

$$|a_2| \leq \frac{(p+q)x\sqrt{(p+q)x}}{\sqrt{\left|3(1+2C_j^m(\gamma))^\zeta(p+q)^2x^2 - 4[(p^2+q^2)(p+q)x^2 + pqs] \left((1+C_j^m(\gamma))^\zeta\right)^2\right|}} \quad (3.16)$$

and

$$|a_3| \leq \frac{(p+q)^2x^2}{\tau} + \frac{(p+q)x}{3(1+2C_j^m(\gamma))^\zeta}, \quad (3.17)$$

where

$$\tau = \left|3(1+2C_j^m(\gamma))^\zeta(p+q)^2x^2 - 4[(p^2+q^2)(p+q)x^2 + pqs] \left((1+C_j^m(\gamma))^\zeta\right)^2\right|.$$

4. Fekete-Szegő inequalities for the function class $\mathcal{T}_\Sigma^\zeta(m, \gamma, \lambda, p, q)$

Now, we are ready to find the sharp bounds of Fekete-Szegő functional $a_3 - \delta a_2^2$ defined for $f \in \mathcal{T}_\Sigma^\zeta(m, \gamma, \lambda, p, q)$ given by (1.1).

Theorem 4.1. Let $f(z)$ given by (1.1), be in the class $\mathcal{T}_\Sigma^\zeta(m, \gamma, \lambda, p, q)$. Then

$$|a_3 - \delta a_2^2| \leq \begin{cases} \frac{(p+q)x}{(1+2\lambda)(1+2C_j^m(\gamma))^\zeta} & \text{for } 0 \leq |h(\delta)| < \frac{1}{2(1+2\lambda)(1+2C_j^m(\gamma))^\zeta} \\ 2(p+q)|h(\delta)|x & \text{for } |h(\delta)| \geq \frac{1}{2(1+2\lambda)(1+2C_j^m(\gamma))^\zeta} \end{cases}, \quad (4.1)$$

where

$$h(\delta) = \frac{(p+q)^2x^2(1-\delta)}{2[(1+2C_j^m(\gamma))^\zeta(1+2\lambda)(p+q)^2x^2 - [(p^2+q^2)(p+q)x^2 + pqs] \left((1+C_j^m(\gamma))^\zeta(1+\lambda)\right)^2]}.$$

Proof. From (3.14) and (3.15), we get

$$a_2^2 = \frac{R_1^3(x, s, p, q)(c_2 + d_2)}{2[(1+2C_j^m(\gamma))^\zeta(1+2\lambda)R_1^2(x, s, p, q) - R_2(x, s, p, q) \left((1+C_j^m(\gamma))^\zeta(1+\lambda)\right)^2]} \quad (4.2)$$

and

$$a_3 = \frac{2(1+2\lambda)(1+2C_j^m(\gamma))^\zeta a_2^2 + R_1(x, s, p, q)(c_2 - d_2)}{2(1+2\lambda)(1+2C_j^m(\gamma))^\zeta}. \quad (4.3)$$

From the equations (4.2) and (4.3), it follows that

$$a_3 - \delta a_2^2 = R_1(x, s, p, q) \left[\left(h(\delta) + \frac{1}{2(1+2\lambda)(1+2C_j^m(\gamma))^\zeta} \right) c_2 + \left(h(\delta) - \frac{1}{2(1+2\lambda)(1+2C_j^m(\gamma))^\zeta} \right) d_2 \right],$$

where

$$h(\delta) = \frac{R_1^2(x, s, p, q)(1 - \delta)}{2[(1+2C_j^m(\gamma))^\zeta(1+2\lambda)R_1^2(x, s, p, q) - R_2(x, s, p, q)((1+C_j^m(\gamma))^\zeta(1+\lambda))^2]}.$$

Then, we easily conclude that

$$|a_3 - \delta a_2^2| \leq \begin{cases} \frac{(p+q)x}{(1+2\lambda)(1+2C_j^m(\gamma))^\zeta}, & |h(\delta)| \leq \frac{1}{2(1+2\lambda+6\mu)} \\ 2(p+q)|h(\delta)|x, & |h(\delta)| \geq \frac{1}{2(1+2\lambda+6\mu)} \end{cases}$$

This proves Theorem 4.1. □

By taking $\lambda = 1$ in Theorem 4.1, we have

Corollary 4.2. Let $f(z)$ given by (1.1), be in the class $\mathcal{T}_\Sigma^\zeta(m, \gamma, p, q)$. Then

$$|a_3 - \delta a_2^2| \leq \begin{cases} \frac{(p+q)x}{3(1+2C_j^m(\gamma))^\zeta} & \text{for } 0 \leq |h(\delta)| < \frac{1}{6(1+2C_j^m(\gamma))^\zeta} \\ 2(p+q)|h(\delta)|x & \text{for } |h(\delta)| \geq \frac{1}{6(1+2C_j^m(\gamma))^\zeta} \end{cases}, \quad (4.4)$$

where

$$h(\delta) = \frac{(p+q)^2 x^2 (1 - \delta)}{2[3(1+2C_j^m(\gamma))^\zeta (p+q)^2 x^2 - 4[(p^2 + q^2)(p+q)x^2 + pqs]((1+C_j^m(\gamma))^\zeta)^2]}.$$

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