Spectral Method for the Heat Equation with Axial Symmetry and a Source

A.Boutaghou

département de mathématiques université Dr: Yahia Fares Médéa,
Laboratoire de mécanique, physique et modélisation mathématiques, (LMPMM).

boutaghou_a@yahoo.com

Abstract. In this paper, we present a spectral method for solving the heat equation in cylindrical coordinates in a case where the data are axisymmetric and independent of the z-coordinate at the same time. The spectral method considered is of Galerkin type with a Gauss-Radau numerical quadrature formula, it is based on a weighted weak variational formulation of the continuous problem. The method considered is discret only in r-variable, the time variable remains continuous. Consequently, the discret problem leads to a system of ordinary differential equations, we solve the system and estimate the error, we also give some numerical examples.

Keywords: *Orthogonal polynomials, Spectral method, Error estimate, Condition number, Programming.*
2010 MSC No: 65N35, 41A10, 33D45.

1 Introduction

The heat equation is one of the important partial differential equations. It plays a very important role in all areas of science, such as mathematics, mathematical physics and other sciences. Finding the solution of this equation is therefore a common problem, often difficult or impossible to resonate analytically. In recent years, especially after the emergence of the computer, they have developed many numerical methods to determine an approximate solution instead of the exact solution to this problem. These methods gave these fruits. In this paper, we present a spectral method for solving the equation of heat in cylindrical coordinates in a case where the data (and therefore the solution) are axisymmetric and independent of the z-coordinate at the same time. The spectral method considered is of Galerkin type with a numerical quadrature formula. The study is based on a weak variational formulation of the continuous problem. The numerical quadrature formula used is the Gauss-Radau formula with a weight in the range $[0,1]$. The method considered is discrete only in the r-variable, the time variable remains continuous. As a result, the discrete problem leads to a system of ordinary differential equations. We solve this system by using the diagonalization of the matrix related to the discretization used. Section 2 is devoted to the theoretical description of the method, section 3 is devoted to the rewriting of the method using the nodes of the polynomial defined of the aforementioned Gauss-Radau formula. Section 4 is mainly devoted to analyzing the error, but it also contains numerical examples.

The concerned problem refer to the equation:

$$\begin{cases} u_t - \alpha(u_{rr} + \frac{1}{r}u_r) = f(r, t) & \text{in } \Lambda \times (0, \infty) \\ u(1, t) = 0 & t > 0 \\ u(r, 0) = u_0(r) & \text{in } \bar{\Lambda} \end{cases} \quad (1)$$

where $\Lambda =]0, 1]$ and $\partial\Lambda$ its boundary, α positive real number, $f \in L_1^2(\Lambda)$, where :

$$L_1^2(\Lambda) = \left\{ f : \Lambda \rightarrow \mathbb{R} \text{ measurable} / \int_{\Lambda} (f(r))^2 r dr < \infty \right\} \quad (2)$$

u_0 is smooth on $\bar{\Lambda}$, we denote by $\|\cdot\|_{L_1^2(\Lambda)}$ and $(\cdot, \cdot)$ the norm and inner product in $L_1^2(\Lambda)$ and $H_{11}^1(\Lambda)$ the subspace is the standard Sobolev space $H_1^1(\Lambda)$ satisfying the homogeneous Dirichlet boundary conditions.

In this work we construct approximate solution to the boundary value problem (1) in the form

$$u_N(r, t) = \sum_{n=0}^N a_n(t) l_n(r) \quad (3)$$

but the condition $u(1, t) = 0$ when $t > 0$ leads to $a_{N(t)} = 0$, the approximate solution take the following form

$$u_N(r, t) = \sum_{n=1}^{N-1} a_n(t) l_n(r) \quad (4)$$

Where the Lagrangian interpolates $l_n(r)$, $0 \leq n \leq N$, are defined at the points $r_j \in \bar{\Lambda} = [0, 1]$, $0 \leq j \leq N$. The grid made by r_j , $0 \leq j \leq N$, is denoted by $\sum_{N+1}$.

The choice of the form (3) for the solution, added to some technics lead to a linear system which can be written in a matricial form as $Aa + \Gamma Da = F$, where A is a square symmetric positive defined matrix and Γ is a diagonal invertible matrix and the operator $D = \frac{d}{dt}$. We write $a = Pv$ where P is an orthogonal matrix such that $P^{-1}(\Gamma^{-1}A)P = C$ is a diagonal matrix, then we obtain a system of $N-1$ ordinary differential equations $\frac{d}{dt}v_j(t) + c_j v_j(t) = h_j(t)$, $j = \overline{1, N-1}$, we can use Lagrange's method of undetermined parameters to solve for each component $v_j(t)$ of v [9], finally we conclude the expressions of functions $a_n(t)$ and for which we obtain the approximation solution.

2 Discretization of the problem

2.1 Continuous problem, weak form.

the variational formulation of problem (1) it is written :

$$\begin{cases} \text{find } u \in H_1^1(\Lambda), \text{ such that } u - u_1 \in H_{11}^1(\Lambda) \\ \forall v \in H_{11}^1(\Lambda), (u_t, v) + a_1(u, v) = (f, v) \end{cases} \quad (5)$$

where the bilinear form $a(\cdot, \cdot)$ is given by:

$$a(u, v) = \int_0^1 \partial_r u \partial_r v r dr \quad (6)$$

see[7]. Where the pivot space of the problem (1) is the space $L_1^2(\Lambda)$, and the variational space is

$$H_1^1(\Lambda) = \{v / v, \partial_r v \in L_1^2(\Lambda)\} \quad (7)$$

and the corresponding norm is defined as,

$$\|v\|_{L_1^2(\Lambda)}^2 = \int_{\Lambda} v^2 r dr dt \quad (8)$$

and the semi norm is defined as,

$$|v|_{H_1^1(\Lambda)}^2 = \int_{\Lambda} ((\partial_r v)^2) r dr dt$$

Also we need the subspace of the variational space with zero Dirichlet trace:

$$H_{11}^1(\Lambda) = \{v \in H_1^1(\Lambda); v(1, t) = 0, t > 0\} \quad (9)$$

2.2 Discrete spaces

The approximate spaces is essentially generated by the finite dimensional subspace of $L_1^2(\Lambda)$, $IP_N(\Lambda)$, $IP_N^1(\Lambda)$ are the approximate spaces of the spaces $H_1^1(\Lambda)$, $H_{11}^1(\Lambda)$ respectively where

$$IP_N^1(\Lambda) = \{p_N \in P_N(\Lambda), p_N(1) = 0\}$$

In this work we consider the quadrature formula and introduce a bilinear form a_{1N} which approach the form a_1 and we approximate $(\cdot, \cdot)_N$ for $(\cdot, \cdot)$, where N represent in spectral method the degree of polynomials. Here, the discrete product is defined for all functions g and h continuous on $\bar{\Lambda}$ by

$$(g, h)_N = \sum_{k=0}^N g(r_k) h(r_k) \omega_k \quad (10)$$

2.3 Discrete problem, variational formulation

the variational formulation (5) is written :

$$\begin{cases} \text{find } u_N \in IP_N(\Lambda) \text{ such that} \\ \forall v_N^1 \in IP_N^1(\Lambda), (u_N, v_N^1)_N + a_{1N}(u_N, v_N^1) = (f_N, v_N^1)_N \\ u_{N0} = I_N(u_0) \end{cases} \quad (11)$$

where u_{N0} is the interpolating polynomial of the initial condition u_0 at the Causs-Lobatto nodes and the bilinear form $a(\cdot, \cdot)$ is given by:

$$a_{1N}(u_N, v_N^1) = (\partial_r u_N, \partial_r v_N^1)_N = \sum_{k=0}^N (\partial_r u_N \partial_r v_N^1)(r_k) \omega_k \quad (12)$$

The equation is now what are the necessary tools to insure the existence and uniqueness of the approximate solution which verify the variational formulation (11).

2.4 The spectral method

In this section we describe the spectral element method applied to subsection 2.3 of the algorithm given in the introduction. the spectral method is based on a weak formulation of the considered problem. The approximate solution representation is then given by :

$$\begin{aligned} u_N(r, t) &= u_N^1(r, t) + u_{N1}, \\ u_N^1(r, t) &= \sum_{n=1}^{N-1} a_n(t) l_n(r) \in IP_N^1(\Lambda) \text{ and } u_{N1}(r) = u(1, t) = 0 \end{aligned}$$

the Lagrangian interpolates $l_n(r)$, $0 \leq n \leq N$ are defined on the interval $\bar{\Lambda}$ with $r \in \bar{\Lambda}$, by the relationship

$$l_n(r) = \prod_{k=0, k \neq n}^N \left(\frac{r - r_k}{r_n - r_k} \right)$$

these interplants satisfy the property $l_n(r_m) = \delta_{nm}$, $1 \leq n, m \leq N-1$, where the points r_k are the collocation points on the Gauss-Radau grid, i.e the roots of the polynomial $M_N(x) = L_{N+1}(x) - L_N(x)$, L_N is the Legendre polynomial and $x = 2r - 1$. There also exists a unique set of positive real numbers ω_n , $0 \leq n \leq N$ such that the integral rule

$$\int_0^1 \varphi(r) r dr = \sum_{j=0}^N \varphi(r_j) \omega_j, \quad \omega_j = \frac{r_j^2}{(n+1)^2 L_n^2(2r_j - 1)}$$

is exact for all polynomials φ of degree $2N-1$ or less on the interval $\bar{\Lambda}$ see [5, 10].

2.5 Existence and uniqueness of solution

Proposition 1. *The bilinear form $a_{1N}(\cdot, \cdot)$ satisfies the following properties of continuity:*

$$\forall v_N^1 \in P_N^1(\Lambda), \forall u_N^1 \in P_N^1(\Lambda), |a_{1N}(u_N^1, v_N^1)| \leq |u_N^1|_{H_1^1(\Lambda)} \cdot |v_N^1|_{H_1^1(\Lambda)} \quad (13)$$

and ellipticity

$$\forall u_N^1 \in P_N^1(\Lambda), |a_{1N}(u_N^1, v_N^1)| \geq |u_N^1|_{H_1^1(\Lambda)}^2 \quad (14)$$

Proof. $a_{1N}(u_N^1, v_N^1) = (\partial_r u_N^1, \partial_r v_N^1)_N$ the degree of polynomials $\partial_r u_N^1, \partial_r v_N^1$ is less than or equal to $N - 1$ then $(\partial_r u_N^1, \partial_r v_N^1)_N = \int_0^1 \partial_r u_N^1 \partial_r v_N^1 r dr$ by the Schwarz inequality we obtain the desired results. $\square$

3 Numerical experiment

The variables r and t play different role, to separate these variables we consider the solution $u(r, t)$ and $f(r, t)$ as functions of the variable t its values are in the function space defined in Λ , we consider u defined by:

$$\begin{aligned} u &: [0, \infty) \rightarrow H_1^1(\Lambda) \\ t &\rightarrow u(t) \end{aligned}$$

then we can not $u(r, t) = u(t)(r)$

the variational formulation can be written as:

$$\begin{cases} \text{find } u_N^1 \text{ in } IP_N(\Lambda), \text{ such that} \\ \forall v_N^1 \in IP_N^1(\Lambda), \frac{d}{dt}(u_N^1(t), v_N^1) + a_{1N}(u_N^1(t), v_N) = (f_N, v_N^1)_N \end{cases} \quad (15)$$

the formulation (15) is true for all $v_N \in IP_N^1(\Lambda)$ then it is true for $v_m(r) = l_m(r)$, $m = \overline{1, N-1}$ where $(l_m(r))_{0 \leq m \leq N}$ form a basis to the polynomial space $IP_N(\Lambda)$, the degree of the polynomial $u_N(t)v_N$ is $2N$ with respect the variable r , and the degree of the polynomial $\partial_x u_N^1(t)\partial_x v_N^1$ is $2N - 1$, then we can write (15) as:

$$\begin{cases} \text{find } u_N^1 \in IP_N(\Lambda), \text{ such that} \\ \forall l_m \in IP_N^1(\Lambda), \sum_{k=0}^N \left(\sum_{n=1}^{N-1} a'_n(t) l_n(r_k) l'_m(r_k) \omega_k \right) + \sum_{k=0}^N \left(\sum_{n=1}^{N-1} a_n(t) l'_n(r_k) l'_m(r_k) \omega_k \right) \\ = \sum_{k=0}^N \sum_{n=0}^{N-1} f_N(t, r_k) l_m(r_k) \omega_k - \sum_{k=0}^N u'_0(r_k) l'_m(r_k) \omega_k, \quad m = \overline{1, N-1} \end{cases} \quad (16)$$

(16) is equivalent to,

$$\begin{cases} a_m(t) \omega_m + \sum_{n=1}^{N-1} \left(a_n(t) \sum_{k=0}^N l'_n(r_k) l'_m(r_k) \omega_k \right) = f_N(r_m, t) \omega_m - \sum_{k=0}^N u'_0(r_k) l'_m(r_k) \omega_k \\ , \quad m = \overline{1, N-1} \\ u_N(1, t) = 0, \quad t > 0 \\ u(r_k, 0) = u_0(r_k) \quad k = \overline{1, N} \end{cases} \quad \begin{matrix} \text{in } \Lambda \cap \sum_{N+1} \\ \text{in } \overline{\Lambda} \end{matrix} \quad (17)$$

We obtain a linear system, then we can write this system in a matricial form:

$$Aa + \Gamma Da = F \quad (18)$$

where A is a symmetric positive defined matrix with order $N - 1$, its elements have the form:

$$\alpha_{mn} = \sum_{k=0}^{N-1} l'_n(r_k) l'_m(r_k) \omega_k, \quad n = \overline{0, N-1}, m = \overline{0, N-1}$$

Γ is a diagonal invertible matrix its elements are define as:

$$\gamma_{mn} = \begin{cases} w_m, & n = m \\ 0, & n \neq m \end{cases}, \quad m, n = \overline{0, N-1}$$

F is a known vector where:

$$\begin{aligned} F &= (g_0(t), g_1(t), g_2(t), g_3(t), \dots, g_{N-2}(t), g_{N-1}(t))^t \\ g_m(t) &= f(r_m, t)\omega_m - \sum_{k=0}^N u'_0(r_k)l'_m(r_k)\omega_k, \quad m = \overline{0, N-1} \end{aligned}$$

and the vector a is an unknown vector where

$$a = (a_0(t), a_1(t), a_2(t), a_3(t), \dots, a_{N-1}(t), a_{N-1}(t))^t$$

the operator,

$$D = \frac{d}{dt}$$

multiplying (18) by the invertible matrix Γ^{-1} of Γ then we find

$$\Gamma^{-1}Aa + Da = \Gamma^{-1}F \quad (19)$$

the matrix $\Gamma^{-1}A$ has positive eigenvalues and there exists an orthogonal invertible matrix P such that,

$$P^{-1}(\Gamma^{-1}A)P = C$$

where C is a diagonal matrix, the elements of the diagonal are the eigenvalues $\alpha_m, m = \overline{0, N-1}$ of the matrix $\Gamma^{-1}A$, if we consider the vector v such that

$$a = Pv$$

then the system (19) becomes

$$(\Gamma^{-1}A)Pv + PDv = \Gamma^{-1}F \quad (20)$$

multiplying (20) by the matrix P^{-1} we obtain,

$$Cv + Dv = P^{-1}\Gamma^{-1}F \quad (21)$$

The matricial form (21) has $N-1$ linear equations defined as

$$\begin{aligned} v'_m(t) + \alpha_m v_m(t) &= h_m(t) \\ \text{where } h_m(t) &= \sum_{j=0}^{N-1} p^{-1}(m, j) \Gamma^{-1}(m, j) g_j(t), \quad 0 \leq m \leq N-1 \end{aligned} \quad (22)$$

$p^{-1}(m, j)$ are the elements of the inverse matrix P^{-1} . To solve the equations (22), we may write the solution in the closed form :

$$v_m(t) = \int_0^t \exp(\alpha_m(s-t))h_m(s)ds + d_m e^{-\alpha_m t} \quad (23)$$

where d_m are constants to be determined, using the boundary conditions then (23) may be written in the following form:

$$v_m(t) = \int_0^t \exp(\alpha_m(s-t))h_m(s)ds + \left(\sum_{j=0}^{N-1} p^{-1}(m, j) u_0(r_j) \right) e^{-\alpha_m t} \quad (24)$$

Finally we obtain the functions,

$$a_m(t) = \sum_{j=1}^{N-1} p_{mj} \left(\int_0^t \exp(\alpha_m(s-t))h_m(s)ds + \left(\sum_{j=1}^{N-1} p^{-1}(m, j) u_0(r_j) \right) e^{-\alpha_m t} \right)$$

where p_{nj} , $1 \leq n, j \leq N$ are the elements of the matrix P and the approximation solution is

$$u_N(r, t) = \sum_{m=0}^{N-1} \left(\sum_{j=0}^{N-1} p_{mj} \left(\int_0^t \exp(\alpha_m(s-t)) h_m(s) ds + \left(\sum_{j=1}^{N-1} p^{-1}(m, j) u_0(r_j) \right) e^{-\alpha_m t} \right) \right) l_m(r).$$

see [1, 2].

3.1 Condition number

Definition 2. :The condition number of a $n \times n$ non-singular matrix A is defined by:

$$k_P(A) = \|A\|_P \|A^{-1}\|_P \quad (25)$$

where $\|A\|_P$ is the spectral norm defined by $\rho = (A^t A)^{\frac{1}{2}}$

Remark 3. The condition number of a matrix A gives a measure of how sensitive systems of equations, with coefficients matrix A , are to small perturbations such as those caused by rounding. Then if the condition number of a matrix is large, the effect of rounding error in the solution process may be serious [9].

To compute the condition number of different order of these matrix we use the spectral norm, and all operations are made by the Maple V [8], using [6].

3.2 Figure illustration

The figure 1 present the behavior of the logarithm of the condition number, n vary from 3 to 13, figures 2 and 3 present the true and the approximation solution u and u_N respectively, these plots occur when $N = 14$ and the function test is : $u(r, t) = -\exp(-0.002t)(r^2 - 1) \sin(\pi r)$.

Figure -1-

These plots are occurred when $N = 14$
The true solution u

The approximation solution u_N

Figure -2-

Figure -3-

The error $\|u - u_N\|_{L^2_1(\Omega)}$, $\Omega =]0, 1] \times]0, 3]$ is the difference between the exacte and approximate solution, using

various degrees for the approximate solution.

N	3	4	5	6	7	8	9
Error	$.4499e-1$	$.8387e-2$	$.1615e-2$	$.1608e-3$	$.3144e-4$	$.1743e-5$	$.3404e-6$

Table 1

Conclusion 4. *The main purpose in this work is that we have presented a new approach to the numerical solution to the heat equation with axial symmetry and a source, the result is seem to converge rapidly as the degree of the polynomial creases, The method they handled the error study showed that the error is almost deacreses when the parameter of discratisation N increases.*

References

- [1] G. Allaire, Analyse Numérique et Optimisation. . Une introduction à la modélisation mathématique et à la simulation numérique 4 Janvier 2005.
- [2] A. Boutaghou, On Finite Spectral Method for Helmholtz Problem, Int.J. of Appl. Math and Mech. 7(3): 70-92, 2011.
- [3] A. Boutaghou and F.Z.Nouri, On Finite Spectral Method for Axi-symmetric Elliptic problem, Journal of Analysis and Applications, vol. 4 (2006), No 3, pp.149-168, SAS International Publications.
- [4] C. Bernardy and Y-Maday, Approximation spectrale de problemes aux limites elliptiques, 1992, Springer Verlag France Paris.
- [5] C. Bernardy, M. Dauge and Y. Maday, Numerical algorithms and tests due to Mejd Azaiez, 1999 Spectal Methods for Axisymmetric Domains, Editions scientifiques et médicales Elsevier France.
- [6] D. Richards 2002, Advanced Mathematical Methods with Maple, Cambridge University.
- [7] J. de Frutos, R. Munoz-Sola, Chebyshev pseudospectral collocation for parabolic probléms with nonconstant coefficients. Appear in Houston J. Math (Proccedings ICOSAHOM 95.
- [8] Maple V, Release 4, Waterloo Maple Inc., 1995.
- [9] T. N. Phillips and A. R. Davies, On semi-infinite spectral elements for Poisson problems with re-entrant boundary singularities, 1988, Journal of Computatnl and Apllied Mathematics 21 p173-188.
[https://doi.org/10.1016/0377-0427\(88\)90266-x](https://doi.org/10.1016/0377-0427(88)90266-x)
- [10] R. G. Owens and T.N.Phillips , Steady Viscoelastic Flow past a sphere using Spectral Elements . 1996 Int. J. Numer. Meth. Engng., 39: 1517–1534.
[https://doi.org/10.1002/\(sici\)1097-0207\(19960515\)39:9](https://doi.org/10.1002/(sici)1097-0207(19960515)39:9)