Optimal Investment Policy in a Pension Fund System with Return Clause and Multiple Assets under Volatility Risks

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Abstract. The essence of this work is to study the optimal investment policy in a defined contribution pension scheme with return clause of contributions under volatility risks. In our model, the pension fund managers are mandated to return the accumulated contributions of members who die during the accumulation phase to their next of kin. Also, investment in one risk free asset and two risky assets (stock and loan) are considered such that the stock market price is driven by Heston volatility model and the remaining accumulations are distributed among the remaining members. Using mean variance utility function, game theory and variable separation technique, a closed form solution of the optimal investment policy, the optimal fund size and the efficient frontier were obtained. Furthermore, a sensitivity analysis of the effects of some parameters on the optimal investment policies and efficient frontiers were carried out theoretically.

Keywords: DC. Pension scheme, extended HJB equation, optimal investment policies, return clause of premium, game theoretic approach, mean variance utility, volatility risk.

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1 Introduction

The defined contribution pension scheme is a retirement scheme whereby a certain percentage of an individual's earning are paid into a retirement savings account (RSA), and the contributions thereafter are invested in a financial market such that the returns from such investments (positive or negative) are paid directly into the member's RSA. In this scheme, the only guaranteed amount in the RSA is the member's monthly contributions but the benefits after retirement is uncertain and depends mostly on investment returns; see [3-5]. For the fact that member's retirement benefits in this scheme depend heavily on the size of fund over the accumulation period, pension fund managers must venture into investment in different assets available in the financial markets with the aim of increasing the accumulated funds; this has lead to the study of optimal portfolio problems and has attract so much attention as it has become so crucial to financial institutions such as insurance companies, commercial bank, pension fund system etc.

Based on this revelations, several authors has engaged in the study of optimal investment policies under different assumptions; some of which include [1], who studied the optimal investment strategy and risk measures of the DC pension plan. [2], investigate the optimal dynamic asset allocation for a defined contribution pension scheme with

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stochastic interest rates and inflation in a continuous-time model. [7], studied stochastic strategies for the optimal investment in a DC pension fund with multiple contributors. [8], studied asset allocation problem under a stochastic interest rate. [9], studied the stochastic optimal control problem often DC pension fund in continuous time. [6], studied optimal investment strategies of the DC pension plan under the stochastic interest rate. From the objective function point of view, there are different types of utility functions used in optimization problem; these include the utility function which shows constant relative risk aversion (CRRA); example power and logarithmic utility functions see [3,10], the utility function which shows constant absolute risk aversion (CARA); example exponential utility, see [13]. The mean-variance utility was first developed by Markowitz to investigate optimal portfolio selection problems see [12], but the challenge is that optimal investment policies under the mean-variance utility are not time consistent, since the mean-variance utility function does not have the iterated expectation condition, hence the Bellman's optimality condition does not hold. However, in many situations time consistency of strategies is a basic requirement for rational decision makers. [14], studied the general theory of Markovian time inconsistent stochastic control problems.

Recently, the study of optimal investment policy with return of contributions clause has taken a centre stage since members of the scheme are faced with mortality risk; these include [15], who investigated optimal investment strategy for a defined contribution pension scheme with the return of premiums clauses in a mean-variance utility function. [16], Investigated equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under constant elasticity of variance model. [17], Investigated the optimal time-consistent investment strategy for a DC pension with the return of premiums clauses and annuity contracts; they considered investment in two assets, a risk free asset and a risky asset (stock) and assumed the stock market price to follow Heston volatility model. [18], studied Strategic optimal portfolio management for a DC pension scheme with return of premium clauses; in their work, they considered investment in three assets and assume the risky assets are modelled by geometric Brownian motion.

All through these literatures and to the best of our knowledge, no work have been done on optimal investment policy with return of premium clause that considers investment in a cash, stock and loan such that the stock market price is being modelled by Heston volatility model and as such this form the basis of our discussion in this work.

2 The Model

Consider a financial market which is complete and frictionless and open continuously over a specific time interval $0 \leq t \leq T$, T is the retirement age of any given plan member. Also, consider a market which consist of a risk-free asset (cash) and two risky assets; namely (stock) and loan. Suppose $(\Omega, \mathfrak{H}, P)$ is a complete probability space such that Ω is a real space and P a probability measure satisfying the condition $0 \leq t \leq T$. $\{B_1(t), \dot{B}_2(t), \dot{B}_3(t), \dot{B}_4(t) : t \geq 0\}$ are standard Brownian motions. $\mathfrak{H}$ is the filtration and denotes the information generated by the Brownian motions. Let $\mathfrak{M}_t^1(t)$ denote the price of the risk free asset, $\mathfrak{M}_t^2(t)$ the price of the stock which is modelled by the Heston's stochastic volatility model and $\mathfrak{M}_t^3(t)$ is the price of the loan. Their price models are described as follows:

$$\frac{d\mathfrak{M}_t^1(t)}{\mathfrak{M}_t^1(t)} = \gamma_1 dt, \quad \mathfrak{M}_t^1(0) = 1 \quad (2.1)$$

$$\frac{d\mathfrak{M}_t^2(t)}{\mathfrak{M}_t^2(t)} = (\gamma_1 + d_1 \mathcal{L}(t)) dt + \sqrt{L(t)} d\dot{B}_1 \quad \mathfrak{M}_t^2(0) = \mathfrak{M}_0^2 \quad (2.2)$$

$$\frac{dL_t(t)}{L_t(t)} = h \left(\frac{\varpi}{L_t} - 1 \right) dt + \eta_1 \frac{\sqrt{L_t}}{L_t} d\dot{B}_2, \quad L_t(0) = l_0 \quad (2.3)$$

$$\frac{d\mathfrak{M}_t^3(t)}{\mathfrak{M}_t^3(t)} = (\gamma_1 + d_2) dt + \eta_2 d\dot{B}_3(t) + \eta_3 d\dot{B}_4(t) \quad (2.4)$$

Here γ_1 is the predetermined interest rate of the risk free asset and $h, \varpi, d_1, d_2, \eta_1, \eta_2, \eta_3$ are positive constants and the two Brownian motions $\dot{B}_1(t), \dot{B}_2(t), \dot{B}_3(t)$ and $\dot{B}_4(t)$ are such that $d\dot{B}_1(t)d\dot{B}_2 = \rho$, $d\dot{B}_1(t)d\dot{B}_3 = d\dot{B}_1(t)d\dot{B}_4 = d\dot{B}_2(t)d\dot{B}_3 = d\dot{B}_2(t)d\dot{B}_4 = d\dot{B}_3(t)d\dot{B}_4 = 0$ where ρ represent the correlation coefficient of the two Brownian motions and satisfies the condition $-1 \leq \rho \leq 1$.

Let the premium received at a given time be represented as m , let β_0 represent the initial age of accumulation phase, T , the period of the accumulation phase such that $\beta_0 + T$ is the end age. The actuarial symbol $\kappa \mathfrak{J}_{\beta_0+t}$ is the mortality

rate from time t to $t + \kappa$, bt is the premium accumulated at time t , $tm\kappa\mathfrak{I}_{\beta_0+t}$ is the premium returned to the death members.

Let ℓ_1 , ℓ_2 , and ℓ_3 represent the proportion of the members pension wealth to be invested in cash, stock and loan respectively such that $\ell_1 = 1 - \ell_2 - \ell_3$.

Considering the time interval $[t, t + \kappa]$, the differential form associated with the fund size is given as:

$$\mathcal{W}(t + \kappa) = \left(\mathcal{W}(t) \left(\ell_1 \frac{\mathfrak{M}_{t+\kappa}^1(t)}{\mathfrak{M}_t^1} + \ell_2 \frac{\mathfrak{M}_{t+\kappa}^2(t)}{\mathfrak{M}_t^2} + \ell_3 \frac{\mathfrak{M}_{t+\kappa}^3(t)}{\mathfrak{M}_t^3} \right) + \right) \quad (2.5)$$

$$\mathcal{W}(t + \kappa) = \left(\mathcal{W}(t) \left(\begin{aligned} &1 + (1 - \ell_2 - \ell_3) \left(\frac{\mathfrak{M}_{t+\kappa}^1 - \mathfrak{M}_t^1}{\mathfrak{M}_t^1} \right) \\ &+ \ell_2 \left(\frac{\mathfrak{M}_{t+\kappa}^2 - \mathfrak{M}_t^2}{\mathfrak{M}_t^2} \right) + \ell_3 \left(\frac{\mathfrak{M}_{t+\kappa}^3 - \mathfrak{M}_t^3}{\mathfrak{M}_t^3} \right) \end{aligned} \right) + m(\kappa) - tm\kappa\mathfrak{I}_{\beta_0+t} \right) \quad (2.6)$$

From [11], the conditional death probability $tQ_y = 1 - tP_y = 1 - \exp - \int_0^\kappa \alpha(\beta_0 + t + s)ds$, where $\alpha(t)$ is the force function of the mortality rate at time t , and for $\kappa \rightarrow 0$,

$$\begin{aligned} \kappa\mathfrak{I}_{\beta_0+t} &= 1 - \exp(-\int_0^\kappa \alpha(\beta_0 + t + s)ds) \approx \alpha(\beta_0 + t)\kappa = O(\kappa) \\ \frac{\kappa\mathfrak{I}_{\beta_0+t}}{1 - \kappa\mathfrak{I}_{\beta_0+t}} &= \frac{1 - \exp(-\int_0^\kappa \alpha(\beta_0 + t + s)ds)}{\exp(-\int_0^\kappa \alpha(\beta_0 + t + s)ds)} = \exp\left\{\int_0^\kappa \alpha(\beta_0 + t + s)ds\right\} - 1 \approx \alpha(\beta_0 + t) = O(\kappa) \\ \kappa \rightarrow 0, \quad \frac{\kappa\mathfrak{I}_{\beta_0+t}}{1 - \kappa\mathfrak{I}_{\beta_0+t}} &= \alpha(\beta_0 + t)dt, \quad \kappa\mathfrak{I}_{\beta_0+t} = \alpha(\beta_0 + t)dt \quad m(\kappa) \rightarrow bdt, \\ \frac{\mathfrak{M}_{t+\kappa}^1 - \mathfrak{M}_t^1}{\mathfrak{M}_t^1} &\rightarrow \frac{d\mathfrak{M}_t^1(t)}{\mathfrak{M}_t^1(t)}, \quad \frac{\mathfrak{M}_{t+\kappa}^2 - \mathfrak{M}_t^2}{\mathfrak{M}_t^2} \rightarrow \frac{d\mathfrak{M}_t^2(t)}{\mathfrak{M}_t^2(t)}, \quad \frac{\mathfrak{M}_{t+\kappa}^3 - \mathfrak{M}_t^3}{\mathfrak{M}_t^3} \rightarrow \frac{d\mathfrak{M}_t^3(t)}{\mathfrak{M}_t^3(t)}. \end{aligned} \quad (2.7)$$

Substituting (2.7) into (2.6), we have

$$\begin{aligned} \mathcal{W}(t + \kappa) &= \mathcal{W}(t) \left(1 + (1 - \ell_2 - \ell_3) \frac{d\mathfrak{M}_t^1(t)}{\mathfrak{M}_t^1} + \ell_2 \frac{d\mathfrak{M}_t^2(t)}{\mathfrak{M}_t^2} + \ell_3 \frac{d\mathfrak{M}_t^3(t)}{\mathfrak{M}_t^3} \right) \\ &\quad + mdt - tm\alpha(\beta_0 + t)dt \end{aligned} \quad (2.8)$$

$$\begin{aligned} d\mathcal{W}(t) &= (w(t)(\ell_2 d_1 L(t) + \ell_3 d_2 + \gamma_1 + m(\frac{\beta - \beta_0 - 2t}{\beta - \beta_0 - t}))dt) \\ &\quad + w(t)(\ell_3(\eta_2 d\dot{B}_3(t) + \eta_3 d\dot{B}_4(t)) + \ell_2 \sqrt{L(t)} d\dot{B}_1)w(0) \\ &= w_0 \end{aligned} \quad (2.9)$$

Where β is the maximal age of the life table and $\alpha(t)$ is the force function given by $\alpha(t) = \frac{1}{\beta - t}$, $0 \leq t < \beta$

$$\alpha(\beta_0 + t) = \frac{1}{\beta - \beta_0 - t} \quad (2.10)$$

3 Mean Variance Utility and Extended HJB equation

Considering the fact that pension accumulations are very volatile, hence there is need for the pension fund managers to formulate the optimal investment problem under the mean-variance criterion as follows:

$$\sup_{\ell} \{E_{t,w,l} \mathcal{W}^\ell(T) - Var_{t,w,l} \mathcal{W}^\ell(T)\} \quad (3.1)$$

Applying the game theoretic method described in [14], the mean-variance control problem in (3.1) is equivalent to the following Markovian time inconsistent stochastic optimal control problem with value function $X(t, w, l)$.

$$\begin{cases} Y(t, w, l, \ell) = E_{t,w,l}[\mathcal{W}^\ell(T)] - \frac{\gamma}{2} Var_{t,w,l}[\mathcal{W}^\ell(T)] \\ Y(t, w, l, \ell) = E_{t,w,l}[\mathcal{W}^\ell(T)] - \frac{\gamma}{2} \left(E_{t,w,l}[\mathcal{W}^\ell(T)^2] - (E_{t,w,l}[\mathcal{W}^\ell(T)])^2 \right) \\ X(t, w, l) = \sup_{\ell} Y(t, w, l, \ell) \end{cases}$$

From [14-15], the optimal portfolio policy ℓ^* satisfies:

$$X(t, w, l) = \sup_{\ell} Y(t, w, l, \ell^*)$$

γ is a constant representing risk aversion coefficient of the members.

Let $p^\ell(t, w, l) = E_{t,w,l}[\mathcal{W}^\ell(T)]$, $q^\ell(t, w, l) = E_{t,w,l}[\mathcal{W}^\ell(T)^2]$ then

$$X(t, w, l) = \sup_{\ell} a(t, w, l, p^\ell(t, w, l), q^\ell(t, w, l))$$

Where,

$$a(t, w, l, p, q) = p - \frac{\gamma}{2}(q - p^2)$$

Theorem 3.1 (verification theorem). If there exist three real functions $\mathcal{A}, \mathcal{B}, \mathcal{C} : [0, T] \times \mathcal{R} \rightarrow \mathcal{R}$ satisfying the following extended Hamilton Jacobi Bellman equations:

$$\left\{ \sup_{\ell} \left\{ \begin{aligned} & \mathcal{A}_t - a_t \\ & + (\mathcal{A}_w - a_w) \left[\begin{aligned} & w(\ell_2 d_1 l + \ell_3 d_2 + \gamma_1) \\ & + m \left(\frac{\beta - \beta_0 - 2t}{\beta - \beta_0 - t} \right) \end{aligned} \right] + (\mathcal{A}_l - a_l) h(\varpi - l) \\ & + \frac{1}{2} (\mathcal{A}_{ww} - P_{ww}) [\ell_2^2 w^2 l + \ell_3^2 w^2 (\eta_2^2 + \eta_3^2)] \\ & + \frac{1}{2} (\mathcal{A}_{ll} - P_{ll}) \eta_1^2 l + (\mathcal{A}_{wl} - P_{wl}) (w \rho l \eta_1 \ell_2) \\ & \mathcal{A}(T, w, l) = a(t, w, l, w^2) \end{aligned} \right\} \right\} = 0 \quad (3.2)$$

Where,

$$\left\{ \begin{aligned} P_{ww} &= a_{ww} + 2a_{wp}p_w + 2a_{wq}q_w + a_{pp}p_w^2 + 2a_{pq}p_w q_w + a_{qq}q_w^2 = \gamma \mathcal{B}_w^2 \\ P_{ll} &= a_{ll} + 2a_{lp}p_l + 2a_{lq}q_l + a_{pp}p_l^2 + 2a_{pq}p_l q_l + a_{qq}q_l^2 = \gamma \mathcal{B}_l^2 \\ P_{wl} &= a_{wl} + a_{wp}p_l + a_{wq}q_l + a_{pl}p_w + a_{ql}q_w + a_{pp}p_w p_l \\ & \quad + a_{pq}p_w q_l + a_{pq}p_l q_w + a_{qq}q_l q_w = \gamma \mathcal{B}_w \mathcal{B}_l \end{aligned} \right\} \quad (3.3)$$

$$\left\{ \begin{aligned} & \mathcal{B}_t \\ & + \mathcal{B}_w \left[\begin{aligned} & w(\ell_2 d_1 l + \ell_3 d_2 + \gamma_1) \\ & + m \left(\frac{\beta - \beta_0 - 2t}{\beta - \beta_0 - t} \right) \end{aligned} \right] + \mathcal{B}_l h(\varpi - l) \\ & + \frac{1}{2} \mathcal{B}_{ww} [\ell_2^2 w^2 l + \ell_3^2 w^2 (\eta_2^2 + \eta_3^2)] \\ & + \frac{1}{2} \mathcal{B}_{ll} \eta_1^2 l + \mathcal{B}_{wl} w \rho l \eta_1 \ell_2 \\ & \mathcal{B}(T, w, l) = w \end{aligned} \right\} = 0 \quad (3.4)$$

$$\left\{ \begin{aligned} & \mathcal{C}_t \\ & + \mathcal{C}_w \left[\begin{aligned} & w(\ell_2 d_1 l + \ell_3 d_2 + \gamma_1) \\ & + m \left(\frac{\beta - \beta_0 - 2t}{\beta - \beta_0 - t} \right) \end{aligned} \right] + \mathcal{C}_l h(\varpi - l) \\ & + \frac{1}{2} \mathcal{C}_{ww} [\ell_2^2 w^2 l + \ell_3^2 w^2 (\eta_2^2 + \eta_3^2)] \\ & + \frac{1}{2} \mathcal{C}_{ll} \eta_1^2 l + \mathcal{C}_{wl} w \rho l \eta_1 \ell_2 \\ & \mathcal{C}(T, w, l) = w^2 \end{aligned} \right\} = 0 \quad (3.5)$$

Then $X(t, w, l) = \mathcal{A}(t, w, l)$, $p^{\ell^*} = \mathcal{B}(t, w, l)$, $q^{\ell^*} = \mathcal{C}(t, w, l)$ for the optimal investment policy ℓ^* . See the proof in [19-21]

4 Optimal Investment Policies and Efficient frontier

In this section, we will attempt to solve equations (3.2), (3.4), and (3.5) to obtain the optimal investment policies and also the efficient frontier.

Proposition 4.1

The optimal investment policy for the three assets are given as

$$\ell_1^* = 1 - \frac{d_1 e^{\gamma_1(t-T)}}{\gamma w(h + \rho \eta_1 d_1)} \left[h + \rho \eta_1 d_1 e^{(h + \rho \eta_1 d_1)(t-T)} \right] - \frac{d_2 e^{\gamma_1(t-T)}}{\gamma w(\eta_2^2 + \eta_3^2)} \quad (4.1)$$

$$\ell_2^* = \frac{d_1 e^{\gamma_1(t-T)}}{\gamma w(h + \rho \eta_1 d_1)} \left[h + \rho \eta_1 d_1 e^{(h + \rho \eta_1 d_1)(t-T)} \right] \quad (4.2)$$

$$\ell_3^* = \frac{d_2 e^{\gamma_1(t-T)}}{\gamma w(\eta_2^2 + \eta_3^2)} \quad (4.3)$$

Proof

Recall that $a(t, w, l, p, q) = p - \frac{\gamma}{2}(q - p^2)$

$$a_t = a_w = a_l = a_{ww} = a_{ll} = a_{wl} = a_{pl} = a_{ql} = a_{wp} = a_{wq} = a_{pq} = a_{qq} = 0, a_p = 1 + \gamma p, a_{pp} = \gamma, a_q = -\frac{\gamma}{2} \quad (4.4)$$

Substituting (4.4) into (3.2) and differentiating it with respect to ℓ_2 and ℓ_3 , and solving for ℓ_2 and ℓ_3 , we have:

$$\ell_2^* = - \left[\frac{d_1 \mathcal{A}_w + (\mathcal{A}_{wl} - \gamma \mathcal{B}_w \mathcal{B}_l) \rho \eta_1}{w (\mathcal{A}_{ww} - \gamma \mathcal{B}_w^2)} \right] \quad (4.5)$$

$$\ell_3^* = - \left[\frac{d_2 \mathcal{A}_w}{w (\mathcal{A}_{ww} - \gamma \mathcal{B}_w^2) (\eta_2^2 + \eta_3^2)} \right] \quad (4.6)$$

Substituting (4.5) and (4.6) into (3.2) and (3.4) we have

$$\begin{aligned} \mathcal{A}_t + \mathcal{A}_w \left[\gamma_1 w + m \left(\frac{\beta - \beta_0 - 2t}{\beta - \beta_0 - t} \right) \right] + \mathcal{A}_l h (\varpi - l) + \frac{1}{2} (\mathcal{A}_{ll} - \gamma \mathcal{B}_l^2) \eta_1^2 l \\ - \frac{1}{2} \frac{\mathcal{A}_w^2 d_2^2}{(\mathcal{A}_{ww} - \gamma \mathcal{B}_w^2) (\eta_2^2 + \eta_3^2)} - \frac{l}{2} \left(\frac{(d_1 \mathcal{A}_w + (\mathcal{A}_{wl} - \gamma \mathcal{B}_w \mathcal{B}_l) \rho \eta_1)^2}{(\mathcal{A}_{ww} - \gamma \mathcal{B}_w^2)} \right) = 0 \end{aligned} \quad (4.7)$$

$$\begin{aligned} \mathcal{B}_t + \mathcal{B}_w \left[\gamma_1 w + m \left(\frac{\beta - \beta_0 - 2t}{\beta - \beta_0 - t} \right) \right] + \mathcal{B}_l h (\varpi - l) + \frac{1}{2} \mathcal{B}_{ll} \eta_1^2 l \\ - (lk_1 \mathcal{B}_w + \rho \sigma_1 l \mathcal{B}_{wl}) \left(\frac{d_1 \mathcal{A}_w + (\mathcal{A}_{wl} - \gamma \mathcal{B}_w \mathcal{B}_l) \rho \eta_1}{(\mathcal{A}_{ww} - \gamma \mathcal{B}_w^2)} \right) \\ - \frac{\mathcal{B}_w \mathcal{A}_w d_2^2}{(\mathcal{A}_{ww} - \gamma \mathcal{B}_w^2) (\eta_2^2 + \eta_3^2)} + \frac{1}{2} l \mathcal{B}_{ww} \left(\frac{d_1 \mathcal{A}_w + (\mathcal{A}_{wl} - \gamma \mathcal{B}_w \mathcal{B}_l) \rho \eta_1}{(\mathcal{A}_{ww} - \gamma \mathcal{B}_w^2)} \right)^2 = 0 \end{aligned} \quad (4.8)$$

Next, we assume a solution for $\mathcal{A}(t, w, l)$ and $\mathcal{B}(t, w, l)$ as follows:

$$\begin{cases} \mathcal{A}(t, z, l) = w f_1(t) + \frac{l}{\gamma} f_2(t) + \frac{1}{\gamma} f_3(t), & f_1(T) = 1, f_2(T) = 0, f_3(T) = 0 \\ \mathcal{B}(t, z, l) = w g_1(t) + \frac{l}{\gamma} g_2(t) + \frac{1}{\gamma} g_3(t), & g_1(T) = 1, g_2(T) = 0, g_3(T) = 0 \\ \mathcal{A}_t = w \frac{df_1(t)}{dt} + \frac{l}{\gamma} \frac{df_2(t)}{dt} + \frac{1}{\gamma} \frac{df_3(t)}{dt}, & \mathcal{A}_w = f_1(t), \mathcal{A}_{ww} = 0, \mathcal{A}_l = \frac{f_2(t)}{\gamma}, \mathcal{A}_{ll} = 0 \\ \mathcal{B}_t = w \frac{dg_1(t)}{dt} + \frac{l}{\gamma} \frac{dg_2(t)}{dt} + \frac{1}{\gamma} \frac{dg_3(t)}{dt}, & \mathcal{B}_w = g_1(t), \mathcal{B}_{ww} = 0, \mathcal{B}_l = \frac{g_2(t)l}{\gamma}, \mathcal{B}_{ll} = 0 \end{cases} \quad (4.9)$$

Substituting (4.9) into (4.7) and (4.8), we have

$$\begin{cases} \frac{df_1(t)}{dt} + \gamma_1 f_1(t) = 0 \\ \frac{df_2(t)}{dt} - h f_2 - \frac{\sigma_1^2 g_2^2}{2} + \frac{k_1^2 f_1^2}{2 g_1^2} + \frac{\rho^2 \sigma_1^2 g_2^2}{2} - \frac{\rho \sigma_1 k_1 f_1 g_2}{g_1} = 0 \\ \frac{df_3(t)}{dt} + h \varpi f_2 + f_1 m \gamma \left(\frac{\beta - \beta_0 - 2t}{\beta - \beta_0 - t} \right) + \frac{k_2^2 f_1^2 g_1}{2 (\eta_2^2 + \eta_3^2) g_1^2} = 0 \end{cases} \quad (4.10)$$

$$\begin{cases} \frac{dg_1(t)}{dt} + \gamma_1 g_1(t) = 0 \\ \frac{dg_2(t)}{dt} - h g_2 + \frac{k_1^2 f_1}{g_1} - \frac{\rho \sigma_1 k_1 f_1 g_2}{g_1} = 0 \\ \frac{dg_3(t)}{dt} + h \varpi g_2 + g_1 m \gamma \left(\frac{\beta - \beta_0 - 2t}{\beta - \beta_0 - t} \right) + \frac{f_1 g_1 k_2^2}{(\eta_2^2 + \eta_3^2) g_1^2} = 0 \end{cases} \quad (4.11)$$

Solving (4.10) and (4.11), we have

$$\begin{aligned} f_1(t) &= e^{\gamma_1(T-t)} \\ f_2(t) &= \frac{\rho \eta_1 d_1^3}{h + \rho \eta_1 d_1} \left\{ + \frac{1}{\rho \eta_1 d_1} (e^{\frac{1}{h}(t-T)} - 1) \right\} + \frac{d_1^2}{2h} (1 - e^{h(t-T)}) \\ &\quad + \frac{\eta_1^2 d_1^3 (\rho^2 - 1)}{2(h + \rho \eta_1 d_1)^2} \left\{ + \frac{2e^{h(t-T)}}{\rho \eta_1 d_1} (1 - e^{\rho \eta_1 d_1(t-T)}) + \right. \\ &\quad \left. + \frac{e^{h(t-T)}}{h + 2\rho \eta_1 d_1} (e^{(h + \rho \eta_1 d_1)(t-T)} - 1) \right\} \end{aligned}$$

$$\begin{aligned}
f_3(t) &= -h\varpi \int_t^T f_2(\tau) d\tau + m\gamma \int_t^T \frac{\tau e^{\gamma_1(T-\tau)}}{\beta - \beta_0 - \tau} d\tau + \frac{m\gamma}{r_1} (e^{\gamma_1(t-T)} - 1) + \frac{d_2^2}{2(\eta_2^2 + \eta_3^2)}(T - t) \\
g_1(t) &= e^{\gamma_1(T-t)} \\
g_2(t) &= \frac{d_1^2}{h + \rho\eta_1 d_1} (1 - e^{(h+\rho\eta_1 d_1)(t-T)}) \\
g_3(t) &= -h\varpi \int_t^T g_2(\tau) d\tau + m\gamma \int_t^T \frac{\tau e^{\gamma_1(T-\tau)}}{\beta - \beta_0 - \tau} d\tau + \frac{m\gamma}{r_1} (e^{\gamma_1(t-T)} - 1) + \frac{d_2^2}{2(\eta_2^2 + \eta_3^2)}(T - t) \\
\mathcal{A}(t, w, l) &= \left(\begin{aligned} &we^{\gamma_1(T-t)} \\ &+ \frac{l}{\gamma} \left[\begin{aligned} &\frac{\rho\eta_1 d_1^3}{h + \rho\eta_1 d_1} \left\{ + \frac{1}{\rho\eta_1 d_1} (e^{\frac{1}{h}(t-T)} - 1) \right. \\ &\quad \left. + \frac{d_1^2}{2h} (1 - e^{h(t-T)}) \right\} \\ &+ \frac{\eta_1^2 d_1^3 (\rho^2 - 1)}{2(h + \rho\eta_1 d_1)^2} \left\{ + \frac{2e^{\frac{1}{h}(t-T)}}{h + 2\rho\eta_1 d_1} (1 - e^{\rho\sigma_1 d_1(t-T)}) + \right. \\ &\quad \left. + \frac{e^{h(t-T)}}{h + 2\rho\eta_1 d_1} (e^{(h+\rho\eta_1 d_1)(t-T)} - 1) \right\} \\ &+ \frac{1}{\gamma} \left[\begin{aligned} &h\varpi \int_t^T f_2(\tau) d\tau \\ &+ m\gamma \int_t^T \frac{\tau e^{\gamma_1(T-\tau)}}{\beta - \beta_0 - \tau} d\tau \\ &+ \frac{m\gamma}{\gamma_1} (e^{\gamma_1(t-T)} - 1) \\ &+ \frac{d_2^2}{2(\eta_2^2 + \eta_3^2)} (T - t) \end{aligned} \right] \end{aligned} \right] \end{aligned} \right) \quad (4.12)
\end{aligned}$$

$$\mathcal{B}(t, w, l) = \left(\begin{aligned} &we^{\gamma_1(T-t)} \\ &+ \frac{l}{\gamma} \left\{ \frac{d_1^2}{h + \rho\eta_1 d_1} (1 - e^{(h+\rho\eta_1 d_1)(t-T)}) \right\} \\ &+ \frac{1}{\gamma} \left\{ \begin{aligned} &-h\varpi \int_t^T g_2(\tau) d\tau \\ &+ m\gamma \int_t^T \frac{\tau e^{\gamma_1(T-\tau)}}{\beta - \beta_0 - \tau} d\tau \\ &+ \frac{m\gamma}{\gamma_1} (e^{\gamma_1(t-T)} - 1) \\ &+ \frac{d_2^2}{2(\eta_2^2 + \eta_3^2)} (T - t) \end{aligned} \right\} \end{aligned} \right) \quad (4.13)$$

$$\mathcal{A}_w = f_1(t), \mathcal{B}_w = g_1(t), \mathcal{A}_{ww} = 0, \text{ and } \mathcal{B}_l = g_2(t) \quad (4.14)$$

Substituting (4.14) into (4.5) and (4.6), we have

$$\begin{aligned}
\ell_1^* &= 1 - \ell_2^* - \ell_3^* = 1 - \frac{d_1 e^{\gamma_1(t-T)}}{\gamma w (h + \rho\eta_1 d_1)} \left[h + \rho\eta_1 d_1 e^{(h+\rho\eta_1 d_1)(t-T)} \right] - \frac{d_2 e^{\gamma_1(t-T)}}{\gamma w (\eta_2^2 + \eta_3^2)} \\
\ell_2^* &= \frac{d_1 e^{\gamma_1(t-T)}}{\gamma w (h + \rho\eta_1 d_1)} \left[h + \rho\eta_1 d_1 e^{(h+\rho\eta_1 d_1)(t-T)} \right] \\
\ell_3^* &= \frac{d_2 e^{\gamma_1(t-T)}}{\gamma w (\eta_2^2 + \eta_3^2)}
\end{aligned}$$

Proposition 4.2

The efficient frontier of the pension members is given by

$$E_{t,w,l}[\mathcal{W}^{\ell^*}(T)] = we^{\gamma_1(T-t)} + \sqrt{\frac{Var_{t,w,l}[\mathcal{W}^{\ell^*}(T)]}{I(t)}} \left\{ \begin{aligned} &\frac{ld_1^2}{h + \rho\eta_1 d_1} (1 - e^{(h+\rho\eta_1 d_1)(t-T)}) \\ &-h\varpi \int_t^T g_2(\tau) d\tau \\ &+ m\gamma \int_t^T \frac{\tau e^{\gamma_1(T-\tau)}}{\beta - \beta_0 - \tau} d\tau \\ &+ \frac{m\gamma}{\gamma_1} (e^{\gamma_1(t-T)} - 1) \\ &+ \frac{d_2^2}{2(\eta_2^2 + \eta_3^2)} (T - t) \end{aligned} \right\}$$

Proof

$$Var_{t,w,l}[\mathcal{W}^{\ell^*}(T)] = E_{t,w,l}[\mathcal{W}^{\ell}(T)^2] - (E_{t,w,l}[\mathcal{W}^{\ell}(T)])^2 = \frac{2}{\gamma} (\mathcal{B}(t, w, l) - \mathcal{A}(t, w, l))$$

Substituting (4.12) and (4.13) into the above equation, we have

$$Var_{t,w,l}[\mathcal{W}^{\ell*}(T)] = \frac{1}{\gamma^2} \left(2l \left\{ \begin{aligned} & \left(\frac{d_1^2}{h+\rho\eta_1 d_1} + \frac{\eta_1^2 d_1^3 (\rho^2-1)}{2(h+\rho\eta_1 d_1)^2} + \frac{e^{h(t-T)}}{h+2\rho\eta_1 d_1} \right) (1 - e^{(h+\rho\eta_1 d_1)(t-T)}) \\ & + \frac{d_1^2}{2h} (1 - e^{h(t-T)}) \\ & - \left[\frac{\rho\eta_1 d_1^3}{h+\rho\eta_1 d_1} \left\{ + \frac{1}{\rho\eta_1 d_1} (e^{h(t-T)} - e^{(h+\rho\eta_1 d_1)(t-T)}) \right\} \right. \\ & \quad \left. + \frac{\sigma_1^2 d_1^3 (\rho^2-1)}{2(h+\rho\eta_1 d_1)^2} \left\{ + \frac{1}{\rho\eta_1 d_1} (e^{h(t-T)} - 1) \right\} \right. \\ & \quad \left. + \frac{2e^{h(t-T)}}{\rho\eta_1 d_1} (1 - e^{\rho\eta_1 d_1(t-T)}) \right\} \\ & + 2h\varpi \int_t^T (f_2(\tau) - g_2(\tau)) d\tau \end{aligned} \right\} \right)$$

$$Var_{t,w,l}[\mathcal{W}^{\ell*}(T)] = \frac{1}{\gamma^2} I(t)$$

Where

$$I(t) = \left(2l \left\{ \begin{aligned} & \left(\frac{d_1^2}{h+\rho\eta_1 d_1} + \frac{\eta_1^2 d_1^3 (\rho^2-1)}{2(h+\rho\eta_1 d_1)^2} + \frac{e^{h(t-T)}}{h+2\rho\eta_1 d_1} \right) (1 - e^{(h+\rho\eta_1 d_1)(t-T)}) \\ & + \frac{d_1^2}{2h} (1 - e^{h(t-T)}) \\ & - \left[\frac{\rho\eta_1 d_1^3}{h+\rho\eta_1 d_1} \left\{ + \frac{1}{\rho\eta_1 d_1} (e^{h(t-T)} - e^{(h+\rho\eta_1 d_1)(t-T)}) \right\} \right. \\ & \quad \left. + \frac{\sigma_1^2 d_1^3 (\rho^2-1)}{2(h+\rho\eta_1 d_1)^2} \left\{ + \frac{1}{\rho\eta_1 d_1} (e^{h(t-T)} - 1) \right\} \right. \\ & \quad \left. + \frac{2e^{h(t-T)}}{\rho\eta_1 d_1} (1 - e^{\rho\eta_1 d_1(t-T)}) \right\} \\ & + 2h\varpi \int_t^T (f_2(\tau) - g_2(\tau)) d\tau \end{aligned} \right\} \right)$$

$$\frac{1}{\gamma} = \frac{\sqrt{Var_{t,w,l}[\mathcal{W}^{\ell*}(T)]}}{\sqrt{I(t)}} \quad (4.15)$$

$$E_{t,w,l}[\mathcal{W}^{\ell*}(T)] = \mathcal{B}(t, w, l)$$

$$E_{t,w,l}[\mathcal{W}^{\ell*}(T)] = we^{\gamma_1(T-t)} + \frac{1}{\gamma} \left\{ \begin{aligned} & \frac{ld_1^2}{h+\rho\eta_1 d_1} (1 - e^{(h+\rho\eta_1 d_1)(t-T)}) \\ & - h\varpi \int_t^T g_2(\tau) d\tau \\ & + m\gamma \int_t^T \frac{\tau e^{\gamma_1(T-\tau)}}{\beta - \beta_0 - \tau} d\tau \\ & + \frac{m\gamma}{\gamma_1} (e^{\gamma_1(t-T)} - 1) \\ & + \frac{d_2^2}{2(\eta_2^2 + \eta_3^2)} (T - t) \end{aligned} \right\} \quad (4.16)$$

Substituting (4.15) i (4.16), we have

$$E_{t,w,l}[\mathcal{W}^{\ell*}(T)] = we^{\gamma_1(T-t)} + \sqrt{\frac{Var_{t,w,l}[\mathcal{W}^{\ell*}(T)]}{I(t)}} \left\{ \begin{aligned} & \frac{ld_1^2}{h+\rho\eta_1 d_1} (1 - e^{(h+\rho\eta_1 d_1)(t-T)}) \\ & - h\varpi \int_t^T g_2(\tau) d\tau \\ & + m\gamma \int_t^T \frac{\tau e^{\gamma_1(T-\tau)}}{\beta - \beta_0 - \tau} d\tau \\ & + \frac{m\gamma}{\gamma_1} (e^{\gamma_1(t-T)} - 1) \\ & + \frac{d_2^2}{2(\eta_2^2 + \eta_3^2)} (T - t) \end{aligned} \right\} \quad (4.17)$$

Proposition 4.3 The optimal fund size $\mathcal{W}^{\ell^*}(t)$ corresponding to the optimal allocation strategy ℓ^* is given as

$$\begin{aligned}
\mathcal{W}(t) = & \frac{\bar{w}hd_1}{\gamma} + \left(\left(\frac{t}{h} - \frac{1}{h^2}\right)e^{ht} + \frac{1}{h^2}\right)\left(1 - \frac{\rho\eta_1d_1}{h + \rho\eta_1d_1}\right)e^{\gamma_1(t-T)} \\
& + e^{ht}\left(\frac{t}{2h + \rho\eta_1d_1} - \frac{1}{(2h + \rho\eta_1d_1)^2}\right)e^{(\gamma_1 + h + \rho\eta_1d_1)((t-T))} \\
& + \left(\frac{e^{\gamma_1(t-T)} - (h + \rho\eta_1d_1)T}{(2h + \rho\eta_1d_1)^2}\right)e^{\gamma_1(t-T)} \\
& + \frac{d_1^2}{h\gamma}[\exp(\gamma_1(t-T))\left(1 - \frac{\rho\eta_1d_1}{h(h + \rho\eta_1d_1)}\right) \\
& + \frac{1}{2h + \rho\eta_1d_1}\exp(\gamma_1 + h + \rho\eta_1d_1(t-T))\exp(ht-1) \\
& + \frac{td_2^2\exp(\gamma_1(t-T))}{\gamma(\eta_2^2 + \eta_3^2)} + w_0\exp(\gamma_1t) \\
& + \frac{m}{h}(\exp(\gamma_1t-1)) - h\gamma\exp(\gamma_1t)\int_t^T \left(\frac{\tau e^{-(\gamma_1t)}}{\beta - \beta_0 - \tau}\right)d\tau.
\end{aligned} \tag{4.18}$$

Proof From equation (2.9) we have

$$d\mathcal{W}(t) = \left(\begin{array}{c} \left(\mathcal{W}(t) \left(\ell_2 d_1 L(t) + \ell_3 d_2 + \gamma_1 \right) + m \left(\frac{\beta - \beta_0 - 2t}{\beta - \beta_0 - t} \right) \right) dt + \\ \mathcal{W}(t) \left(\ell_3 (\eta_2 d_3(t) + \eta_3 d\dot{B}_4(t)) + \ell_2 \sqrt{L(t)} d\dot{B}_1 \right) \end{array} \right) \mathcal{W}(0) = w_0$$

Where $L(t) = (h\varpi t + 1)e^{mt}$

Solving the linear differential equation, we have the result above

5 Sensitivity Analysis

Proposition 5.1 Suppose $w > 0, \gamma > 0, \gamma_1 > 0, t \in [0, T], h > 0, d_1 > 0, \eta_1 > 0, \gamma_1 > 0$ and $\rho \in [-1, 1]$ then (i) $\frac{\partial \ell_2^*}{\partial w} < 0$ (ii) $\frac{\partial \ell_2^*}{\partial \gamma} < 0$ (iii) $\frac{\partial \ell_2^*}{\partial \gamma_1} < 0$ (iv) $\frac{\partial \ell_2^*}{\partial t} > 0$

Proof

(i) Recall that $\ell_2^* = \frac{d_1 e^{\gamma_1(t-T)}}{\gamma w (h + \rho\eta_1 d_1)} \left[h + \rho\eta_1 d_1 e^{(h + \rho\eta_1 d_1)(t-T)} \right]$

$$\frac{\partial \ell_2^*}{\partial w} = -\frac{d_1 e^{\gamma_1(t-T)}}{\gamma w^2 (h + \rho\eta_1 d_1)} \left[h + \rho\eta_1 d_1 e^{(h + \rho\eta_1 d_1)(t-T)} \right]$$

For $\rho = 0$

$$\frac{\left[h + \rho\eta_1 d_1 e^{(h + \rho\eta_1 d_1)(t-T)} \right]}{h + \rho\eta_1 d_1} = \frac{h}{h} = 1 > 0$$

For $\rho < 0$,

$\frac{\left[h + \rho\eta_1 d_1 e^{(h + \rho\eta_1 d_1)(t-T)} \right]}{h + \rho\eta_1 d_1} > 0$ since either $\rho\eta_1 d_1 e^{(h + \rho\eta_1 d_1)(t-T)} < 0$ when $\rho\eta_1 d_1 < 0$ and $\rho\eta_1 d_1 e^{(h + \rho\eta_1 d_1)(t-T)} > 0$ when $\rho\eta_1 d_1 > 0$

For $\rho > 0$,

$$\frac{\left[h + \rho\eta_1 d_1 e^{(h + \rho\eta_1 d_1)(t-T)} \right]}{h + \rho\eta_1 d_1} > 0$$

Therefore for all $\rho \in [-1, 1]$, $\frac{\left[h + \rho\eta_1 d_1 e^{(h + \rho\eta_1 d_1)(t-T)} \right]}{h + \rho\eta_1 d_1} > 0$

Since $\frac{d_1 e^{\gamma_1(t-T)}}{\gamma w^2} > 0$ and $\frac{\left[h + \rho\eta_1 d_1 e^{(h + \rho\eta_1 d_1)(t-T)} \right]}{h + \rho\eta_1 d_1} > 0$. Hence

$$\frac{\partial \ell_2^*}{\partial w} = -\frac{d_1 e^{\gamma_1(t-T)}}{\gamma w^2 (h + \rho\eta_1 d_1)} \left[h + \rho\eta_1 d_1 e^{(h + \rho\eta_1 d_1)(t-T)} \right] < 0$$

$$(ii) \quad \frac{\partial \ell_2^*}{\partial \gamma} = -\frac{d_1 e^{\gamma_1(t-T)}}{\gamma^2 w(h + \rho \eta_1 d_1)} \left[h + \rho \eta_1 d_1 e^{(h + \rho \eta_1 d_1)(t-T)} \right]$$

Since $\frac{d_1 e^{\gamma_1(t-T)}}{\gamma^2 w} > 0$ and $\frac{[h + \rho \eta_1 d_1 e^{(h + \rho \eta_1 d_1)(t-T)}]}{h + \rho \eta_1 d_1} > 0$. Hence

$$\frac{\partial \ell_2^*}{\partial \gamma} < 0$$

$$(iii) \quad \frac{\partial \ell_2^*}{\partial \gamma_1} = \frac{(t-T)d_1 e^{\gamma_1(t-T)}}{\gamma w(h + \rho \eta_1 d_1)} [h + \rho \eta_1 d_1 e^{(h + \rho \eta_1 d_1)(t-T)}]$$

Since $\frac{d_1 e^{\gamma_1(t-T)}}{\gamma w} > 0$, $\frac{h + \rho \eta_1 d_1 e^{(h + \rho \eta_1 d_1)(t-T)}}{h + \rho \eta_1 d_1} > 0$ and $(t-T) < 0$. Hence $\frac{\partial \ell_2^*}{\partial \gamma_1} < 0$

$$(iv) \quad \ell_2^* = \frac{d_1 \exp(\gamma_1(t-T))}{\gamma w(h + \rho \eta_1 d_1)} \left[h + \rho \eta_1 d_1 e^{(h + \rho \eta_1 d_1)(t-T)} \right] = \frac{h d_1 e^{\gamma_1(t-T)}}{\gamma w(h + \rho \eta_1 d_1)} + \frac{\rho \eta_1 d_1^2 e^{(\gamma_1 + h + \rho \eta_1 d_1)(t-T)}}{\gamma w(h + \rho \eta_1 d_1)}$$

$$\frac{\partial \ell_2^*}{\partial t} = \frac{\gamma_1 h d_1 e^{\gamma_1(t-T)}}{\gamma w(h + \rho \eta_1 d_1)} + \frac{(\gamma_1 + h + \rho \eta_1 d_1) \rho \eta_1 d_1^2 e^{(\gamma_1 + h + \rho \eta_1 d_1)(t-T)}}{\gamma w(h + \rho \eta_1 d_1)}$$

Since $\gamma_1 h d_1 e^{\gamma_1(t-T)} > 0$ and $(\gamma_1 + h + \rho \eta_1 d_1) \rho \eta_1 d_1^2 e^{(\gamma_1 + h + \rho \eta_1 d_1)(t-T)} > 0$, then $\frac{\partial \ell_2^*}{\partial t} > 0$

Proposition 5.2

Suppose $w > 0$, $\gamma > 0$, $d_2 > 0$, $\eta_2 > 0$, $\eta_3 > 0$, $\gamma_1 > 0$, $t \in [0, T]$ then (i) $\frac{\partial \ell_3^*}{\partial w} < 0$ (ii) $\frac{\partial \ell_3^*}{\partial \gamma} < 0$ (iii) $\frac{\partial \ell_3^*}{\partial \gamma_1} < 0$ (iv) $\frac{\partial \ell_3^*}{\partial t} > 0$

Proof

(i) Recall that $\ell_3^* = \frac{d_2 e^{\gamma_1(t-T)}}{\gamma w(\eta_2^2 + \eta_3^2)}$

$$\frac{\partial \ell_3^*}{\partial w} = -\frac{d_2 e^{\gamma_1(t-T)}}{\gamma w^2(\eta_2^2 + \eta_3^2)}$$

Since $e^{\gamma_1(t-T)} > 0$ and $\frac{d_2}{\gamma w^2(\eta_2^2 + \eta_3^2)} > 0$ then

$$\frac{\partial \ell_3^*}{\partial w} = -\frac{d_2 e^{\gamma_1(t-T)}}{\gamma w^2(\eta_2^2 + \eta_3^2)} < 0$$

$$(ii) \quad \frac{\partial \ell_3^*}{\partial \gamma} = -\frac{d_2 e^{\gamma_1(t-T)}}{\gamma^2 w(\eta_2^2 + \eta_3^2)}$$

Since $e^{\gamma_1(t-T)} > 0$ and $\frac{d_2}{\gamma^2 w(\eta_2^2 + \eta_3^2)} > 0$. Hence

$$\frac{\partial \ell_3^*}{\partial \gamma} = -e^{\gamma_1(t-T)} \cdot \frac{d_2}{\gamma^2 w(\eta_2^2 + \eta_3^2)} < 0$$

$$(iii) \quad \frac{\partial \ell_3^*}{\partial \gamma_1} = (t-T) \frac{d_2 e^{\gamma_1(t-T)}}{\gamma w(\eta_2^2 + \eta_3^2)}$$

Since $\frac{d_2 e^{\gamma_1(t-T)}}{\gamma w(\eta_2^2 + \eta_3^2)} > 0$ and $t < T$, implies that $t - T < 0$. Hence $\frac{\partial \ell_3^*}{\partial \gamma_1} < 0$ (iv) $\ell_3^* = \frac{d_2 e^{\gamma_1(t-T)}}{\gamma w(\eta_2^2 + \eta_3^2)}$

$$\frac{\partial \ell_3^*}{\partial t} = \frac{\gamma_1 d_2 e^{\gamma_1(t-T)}}{\gamma w(\eta_2^2 + \eta_3^2)}$$

Since $e^{\gamma_1(t-T)} > 0$ and $\frac{\gamma_1 d_2}{\gamma w(\eta_2^2 + \eta_3^2)} > 0$. Hence

$$\frac{\partial \ell_3^*}{\partial t} > 0$$

Proposition 5.3 Suppose $h > 0$, $w > 0$, $\gamma > 0$, $\gamma_1 > 0$, $t \in [0, T]$, $d_1 > 0$, $d_2 > 0$, $\eta_1 > 0$, $\eta_2 > 0$, $\eta_3 > 0$, and $\rho \in [-1, 1]$ then (i) $\frac{\partial \ell_1^*}{\partial w} > 0$ (ii) $\frac{\partial \ell_1^*}{\partial \gamma} > 0$ (iii) $\frac{\partial \ell_1^*}{\partial \gamma_1} > 0$ (iv) $\frac{\partial \ell_1^*}{\partial t} < 0$

$$\ell_1^* = 1 - \frac{d_1 e^{\gamma_1(t-T)}}{\gamma w(h + \rho \eta_1 d_1)} \left[h + \rho \eta_1 d_1 e^{(h + \rho \eta_1 d_1)(t-T)} \right] - \frac{d_2 e^{\gamma_1(t-T)}}{\gamma w(\eta_2^2 + \eta_3^2)}$$

$$(i) \ell_1^* = 1 - (\ell_2^* - \ell_3^*)$$

$$\frac{\partial \ell_1^*}{\partial w} = -\frac{\partial \ell_2^*}{\partial w} - \frac{\partial \ell_3^*}{\partial w} = -\left(\frac{\partial \ell_2^*}{\partial w} + \frac{\partial \ell_3^*}{\partial w}\right)$$

But $\frac{\partial \ell_2^*}{\partial w} < 0$ and $\frac{\partial \ell_3^*}{\partial w} < 0$, therefore $\frac{\partial \ell_2^*}{\partial w} + \frac{\partial \ell_3^*}{\partial w} < 0$. hence

$$\frac{\partial \ell_1^*}{\partial w} = -\left(\frac{\partial \ell_2^*}{\partial w} + \frac{\partial \ell_3^*}{\partial w}\right) > 0$$

$$(ii) \frac{\partial \ell_1^*}{\partial \gamma} = -\frac{\partial \ell_2^*}{\partial \gamma} - \frac{\partial \ell_3^*}{\partial \gamma} = -\left(\frac{\partial \ell_2^*}{\partial \gamma} + \frac{\partial \ell_3^*}{\partial \gamma}\right)$$

Since $\frac{\partial \ell_2^*}{\partial \gamma} < 0$ and $\frac{\partial \ell_3^*}{\partial \gamma} < 0$, such that $\frac{\partial \ell_2^*}{\partial \gamma} + \frac{\partial \ell_3^*}{\partial \gamma} < 0$. Therefore

$$\frac{\partial \ell_1^*}{\partial \gamma} > 0$$

$$(iii) \frac{\partial \ell_1^*}{\partial \gamma_1} = -\frac{\partial \ell_2^*}{\partial \gamma_1} - \frac{\partial \ell_3^*}{\partial \gamma_1} = -\left(\frac{\partial \ell_2^*}{\partial \gamma_1} + \frac{\partial \ell_3^*}{\partial \gamma_1}\right)$$

Since $\frac{\partial \ell_2^*}{\partial \gamma_1} < 0$ and $\frac{\partial \ell_3^*}{\partial \gamma_1} < 0$, such that $\frac{\partial \ell_2^*}{\partial \gamma_1} + \frac{\partial \ell_3^*}{\partial \gamma_1} < 0$. Therefore

$$\frac{\partial \ell_1^*}{\partial \gamma_1} > 0$$

$$(iii) \frac{\partial \ell_1^*}{\partial t} = -\frac{\partial \ell_2^*}{\partial t} - \frac{\partial \ell_3^*}{\partial t} = -\left(\frac{\partial \ell_2^*}{\partial t} + \frac{\partial \ell_3^*}{\partial t}\right)$$

Since $\frac{\partial \ell_2^*}{\partial t} > 0$ and $\frac{\partial \ell_3^*}{\partial t} > 0$, such that $\frac{\partial \ell_2^*}{\partial t} + \frac{\partial \ell_3^*}{\partial t} > 0$. Therefore

$$\frac{\partial \ell_1^*}{\partial t} < 0$$

6 Discussion

Based on our observation, proposition 5.1 and 5.2 shows that the optimal portfolio policies of the two risky assets are inversely proportional to the risk aversion coefficient, initial fund size and the predetermined interest rate and directly proportional to time. Also, proposition 5.3 shows that the optimal portfolio policy of the risk free asset is directly proportional to the risk aversion coefficient, initial fund size and the predetermined interest rate and inversely proportional to time.

The implications of the above propositions is that members who are scared to take risk should invest more in risk free asset while members who love taking risk can invest in more in risky asset to increase their expectations. Secondly, if the initial fund size at the time of investment is high, members may decide to reduce the amount of risk to be taken, thereby reducing the proportion of their funds to be invested in risky assets. Thirdly, when the risk free investment offers high interest rate, members will be attracted to such investment since human being naturally does not like much risk but prefers good returns, they will prefer to invest more in risk free asset at the expense of the risky assets. Finally, we observed that as retirement age of the remaining members draw closer, there will be anxiety in the mind of these members on what their final accumulations will be like considering the effect of the return clause which has taken some good amount out of the system, these members will like to increase their investments in risky assets in order to boost their investment return since DC pension members' benefit depend on the investment returns.

From equation (4.15), we observed that the risk aversion coefficient is inversely proportional to the variance; this shows that members of the scheme with high risk aversion coefficient will prefer to invest more in risk free asset and invest less in risky assets and vice versa. Also, equation (4.17), we observed that the expectation is directly dependent on the variance; this means that when members take more risk, their expectation from such investment will be higher compared to the members who invest less in the risky asset.

7. Conclusion

We investigated optimal investment policy in a pension fund system (DC) with return of premium clauses under Heston's Volatility model using mean-variance utility function. Investments in cash, equity and loan were considered

to help increase the accumulated funds of the remaining members to meet their retirement needs. We established the extended Hamilton Jacobi Bellman equations using and obtained the optimal investment policies for the three assets and also the efficient frontier using game theoretic approach and variable separable method. A theoretical analysis of the effects of some parameters on the optimal investment policies was given. We conclude that the optimal investment policies of the risky assets decrease with the initial fund size, predetermined interest rate, risk aversion coefficient and increases as retirement age approaches.

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