Regular Solution for the Generalized Relativistic Boltzmann Equation in Yang-Mills Field

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Abstract. We consider in this work the generalized relativistic Boltzmann equation in the presence of a Yang-Mills field in temporal gauge on a Bianchi type 1 space-time. Such an equation governs the evolution which collisions of plasmas, for instance of quarks and gluons (quagma), where non abelian Yang-Mills field replace the usual electromagnetic field. A local in time existence and uniqueness result for the regular solution is established, using a suitable combination of Faedo Galerkin method and fixed point theorem.

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1 Introduction

Statistical mechanics may be naturally divided into two branches, one dealing with equilibrium systems, the other with nonequilibrium systems. Non equilibrium phenomena are much less understood, a notable exception is offered by the case of dilute gases: here a basic equation was established by Ludwig Boltzmann in 1872. The Boltzmann equation is one of the basic equations of the relativistic Kinetic theory and has proved fruitful, not only for the study of the classical gases that Boltzmann had in mind, but also properly generalized for studying electron transport in solids and plasmas, neutron transport in nuclear reactors, phonon transport in superfluids and radiative transfer in planetary and stellar atmospheres.

A plasma is a collection of particles moving at a very high speed and under the effect of the forces they create. In the abelian case, forces created by charged particles are electromagnetic forces and when we consider particles evolving with collisions, the self-mained phenomenon is governed by the Maxwell-Boltzmann system. Many authors have studied the Boltzmann equation alone or coupled with other type of equations for instance, we can refer to [4, 13, 14, 15, 16, 17, 18, 19].

In this work, we consider a more general plasma with particles having non-abelian charges. For example, the field of quarks and gluons that one meets in chromodynamics, the electromagnetic field is replaced by the Yang-Mills

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field. The plasma obtained here, called "plasma quarks-gluons", is supposed to exist at a very high temperature. In this case, unlike in the abelian's one, the unknown of the Boltzmann equation depends, not only on the position $(x^\alpha) = (x^0, x^i)$ and the 4-momentum of particles denoted by $p = (p^\alpha) = (p^0, \bar{p})$, but also on the non-abelian charge of particles denoted by $q = (q^a)$, $a = 1, 2, \dots, N$ (where N is the dimension of the Lie algebra $\mathcal{G}$ of a Lie group G). In the collisionless case, the Boltzmann equation is replaced by the Vlasov equation, many authors have already studied this kind of phenomenon: Y. Choquet-Bruhat and N. Noutchegueme in [7] studied the Yang-Mills-Vlasov system using the characteristic method and obtained a local in time existence result. They also studied in [8] the Yang-Mills-Vlasov system only for the zero mass particles case and obtained a global existence theorem in Minkowski space-time for small initial data; N. Noutchegueme and P. Noudjeu in [19] proved a local in time existence theorem of solutions of the Cauchy problem for the Yang-Mills system in temporal gauge with current generated by a distribution function that satisfies a Vlasov equation; R. D. Ayissi and al in [3] obtained the viscosity solutions for the one-Body Liouville equation in Yang-Mills charged Bianchi models with non-zero mass.

In the instantaneous, binary and elastic scheme due to Licherowicz and Chernikov [10], we consider that at a given position $x^\alpha = (t, x)$, two particles of momenta $p = (p^0, \bar{p})$, $p_* = (p_*^0, \bar{p}_*)$, and charges $q = (\tilde{q}, q^N)$, $q_* = (\tilde{q}_*, q_*^N)$ respectively, collide without destroying each other. The collision affects their momenta and charges that change after the collision to become $p' = (p'^0, \bar{p}')$ and $p'_* = (p'^0_*, \bar{p}'_*)$ for momenta, $q' = (\tilde{q}', q'^N)$ and $q'_* = (\tilde{q}'_*, q'^N_*)$ for charges respectively. They satisfy $p + p_* = p' + p'_*$ and $q + q_* = q' + q'_*$, which expresses the momentum conservation law and the charges conservation law respectively. Using the line element (1), the "collision operator" $\mathcal{L} = \mathcal{L}(f, g)$ is defined by :

$$\mathcal{L}(f, g) = \int_{\mathbb{R}^3 \times \theta} \frac{hr^2}{p_*^0} \int_{S^2 \times S^{N-2}} \left[f(\bar{p}', \tilde{q}') g(\bar{p}'_*, \tilde{q}'_*) - f(\bar{p}, \tilde{q}) g(\bar{p}_*, \tilde{q}_*) \right] \sigma dw d\theta,$$

where w is an element on the unit sphere S^2 of $\mathbb{R}^3$, θ an element on the unit sphere S^{N-2} of $\mathbb{R}^{N-1}$ and σ the differential cross section of collisions defined in section 2.

Rather than the usual Boltzmann equation, we consider in this work the generalized Boltzmann equation writing in the form:

$$\frac{\partial f}{\partial t} + P^i \frac{\partial f}{\partial p^i} + Q^I \frac{\partial f}{\partial q^I} = \frac{\mathcal{L}(f, f)}{p^0} + \frac{a_0}{p^0} f^2,$$

where P^i and Q^I are giving in the next section by (12) and a_0 is a strictly positive constant.

We prove here a local in time existence theorem of regular solution for the above equation in the presence of a given Yang-Mills field in Bianchi type 1 space-time, following the method used in [4] and [15]. To our knowledge, there is not such result in the literature.

The paper is organized as follows:

In section 2, we describe the generalized relativistic Boltzmann equation in Yang-Mills field.

In section 3, we introduce the function spaces and we give the energy estimates.

In section 4, we state and prove the existence and uniqueness theorem.

2 Description of the generalized relativistic Boltzmann equation in the presence of a Yang-Mills fields

2.1 The generalized relativistic Boltzmann equation in Yang-Mills fields

- In all what follows, unless otherwise specified, Greek indices range from 0 to 3 and latin ones from 1 to 3. We use the Einstein summation convention i.e., $A_\alpha B^\alpha = \sum_\alpha A_\alpha B^\alpha$.

- We consider the collision evolution of a kind of fast moving massive and charged particles in the time-oriented Bianchi type 1 space-time with locally rotationally symmetric in the form:

$$g = -dt^2 + h^2(t)dx_1^2 + r^2(t)(dx_2^2 + dx_3^2), \quad (1)$$

where $h > 0$, $r > 0$ are two continuously differentiable functions of time t . We assume that $\frac{\dot{h}}{h}$ and $\frac{\dot{r}}{r}$ are bounded. Hence there exists a constant $C > 0$ such that : $|\frac{\dot{h}}{h}| \leq C$, $|\frac{\dot{r}}{r}| \leq C$. As a direct consequence, we have for $t \in \mathbb{R}^+$:

$$0 < h(t) \leq h_0 e^{ct}, \quad 0 < r(t) \leq r_0 e^{ct}, \quad \frac{1}{h(t)} \leq \frac{1}{h_0} e^{ct} \quad \text{and} \quad \frac{1}{r(t)} \leq \frac{1}{r_0} e^{ct}, \quad (2)$$

where $h_0 = h(0)$, $r_0 = r(0)$.

- The massive particles have the same rest mass $m > 0$, normalized to the unity i.e., $m = 1$. We denote by $T(\mathbb{R}^4)$, the tangent bundle of $\mathbb{R}^4$ with coordinates (x^α, p^β) , where $p = (p^\beta) = (p^0, \bar{p})$ stands for the momentum of each particle and $\bar{p} = (p^i)$, $i = 1, 2, 3$. The charged particles move on the mass hyperboloid $\mathbb{P}(\mathbb{R}^4) \subset T(\mathbb{R}^4)$, whose equation is $\mathbb{P}_{t,x}(p) = g_{\alpha\beta}p^\alpha p^\beta = -1$ or equivalently, using (1), we have:

$$p^0 = [1 + h^2(t)(p^1)^2 + r^2(t)((p^2)^2 + (p^3)^2)]^{\frac{1}{2}}, \quad (3)$$

where the choice of $p^0 > 0$ symbolizes the fact that, naturally, the particles eject towards the future.

- We denote by A a Yang-Mills potential represented by a 1-form on $\mathbb{R}^4$ which takes its values in an N -dimensional real Lie algebra $\mathcal{G}$ of a Lie group G , endowed with an Ad-invariant scalar product denoted by a dot “.”, which satisfies the following property:

$$f.[k, l] = [f, k].l, \quad \forall f, k, l \in \mathcal{G}, \quad (4)$$

where $[\cdot, \cdot]$ stands for the Lie brackets of the Lie algebra $\mathcal{G}$. We consider that $\mathcal{G}$ is a vector space on $\mathbb{R}$ with dimension $N \geq 2$ and $(\varepsilon_I), I = 1, \dots, N$ an orthonormal basis of $\mathcal{G}$. Then the Yang-Mills potential is locally defined as follows:

$$A = A_\mu dx^\mu, \quad \text{with} \quad A_\mu = A_\mu^I \varepsilon_I, \quad I = 1, 2, \dots, N. \quad (5)$$

- The Yang-Mills field is the curvature of the Yang-Mills potential. It is represented by a $\mathcal{G}$ -valued antisymmetric 2-form $F = (F_{\lambda\mu}^I)$, linked to the potential A by:

$$F_{\lambda\mu}^I = \nabla_\lambda A_\mu^I - \nabla_\mu A_\lambda^I + [A_\lambda, A_\mu]^I, \quad \text{with} \quad [A_\lambda, A_\mu]^I = C_{bc}^I A_\lambda^b A_\mu^c, \quad (6)$$

where C_{bc}^I are the structure constants of $\mathcal{G}$. We require that A satisfies the temporal gauge condition, which means that $A_0 = 0$.

- We also suppose that the non-abelian charge q of Yang-Mills particles is a C^∞ class function from $\mathbb{R}^4$ to $\mathcal{G}$ whose given norm is $e > 0$. To clarify this idea, q takes its values in an orbit of $\mathcal{G}$, which is a sphere ϑ whose equation is:

$$(\vartheta) : \quad q.q = |q|^2 = e^2, \quad (7)$$

where $|\cdot|$ stands for the norm deduced from the scalar product of $\mathcal{G}$. The relation (7) allows to express the component q^N of q as a function $\tilde{q} = (q^I)$, $I = 1, 2, \dots, N-1$. We have:

$$q^N = \left[e^2 - \sum_{I=1}^{N-1} (q^I)^2 \right]^{\frac{1}{2}}. \quad (8)$$

- We denote by f the unknown distribution function which measures the probability density of the presence of particles in a given domain. f is a function defined on $T(\mathbb{R}^4) \times \mathcal{G}$ and will be subject to the generalized relativistic Boltzmann equation. Using relations (3), (8) and the fact that we are studying an homogeneous phenomenon, we obtain that the distribution function of Yang-Mills particles is definitely a function of independant variables $(t, p^i, q^I) = (t, \bar{p}, \tilde{q})$. Then, $f = f(t, \bar{p}, \tilde{q})$, $t \in \mathbb{R}$, $\bar{p} \in \mathbb{R}^3$, $\tilde{q} \in \mathbb{R}^{N-1}$.

- The trajectories of particles with momentum $p = (p^\alpha) = (p^0, \bar{p})$ and charge $q = (\tilde{q}, q^N)$ in a Yang-Mills field F , are no longer geodesics of space-time $(\mathbb{R}^4, g)$, but satisfy the following differential system:

$$\frac{dx^\alpha}{ds} = p^\alpha, \quad \frac{dp^\alpha}{ds} = -\Gamma_{\lambda\mu}^\alpha p^\lambda p^\mu + p^\beta q.F_\beta^\alpha, \quad \frac{dq^I}{ds} = -C_{bc}^I p^\alpha A_\alpha^b q^c, \quad (9)$$

where $\Gamma_{\lambda\mu}^\alpha$ are the Christoffel symbols of the Levi-Civita connection associated to g . The last equation in (9), called Wong's equation, expresses the fact that the covariant derivative of gauge of q along a trajectory is null. The system (9) expresses the fact that, tangent vector Y to the trajectories of particles is:

$$Y = (p^\alpha, -\Gamma_{\lambda\mu}^\alpha p^\lambda p^\mu + p^\beta q.F_\beta^\alpha, -C_{bc}^I p^\alpha A_\alpha^b q^c). \quad (10)$$

According to relations (3) and (8), the phase space of such Yang-Mills particles is in fact the subset $\mathbb{P}_{t,x} \times \vartheta$ of $T(\mathbb{R}^4) \times \mathcal{G}$.

- The generalized relativistic Boltzmann equation in f for the Yang-Mills charged particles in the Bianchi type 1 space-time can be written:

$$\frac{\partial f}{\partial t} + P^i \frac{\partial f}{\partial p^i} + Q^I \frac{\partial f}{\partial q^I} = \frac{\mathcal{L}(f, f)}{p^0} + \frac{a_0}{p^0} f^2, \quad (11)$$

where

$$P^i = -2\Gamma_{i0}^i p^i - q \cdot F^{i0} - \frac{p^j g^{ij} q \cdot F_{ij}}{p^0} \quad \text{and} \quad Q^I = -\frac{p^i}{p^0} C_{bc}^I A_i^b q^c. \quad (12)$$

The left hand side is obtained from the Lie derivative of f with respect to the vectors field Y defined by (10). In the right hand side, $\mathcal{L}$ is the "collision operator" and a_0 a positive constant.

- From the conservation law of momentum and charges, we have:

$$p + p_* = p' + p'_* \quad (\text{a}) \quad \text{and} \quad q + q_* = q' + q'_* \quad (\text{b}). \quad (13)$$

Given two functions f and g on $\mathbb{P}_{t,x} \times \vartheta$, the "collision operator" is often formally written as the difference between the gain term $\mathcal{L}^+$ and the loss term $\mathcal{L}^-$ (see [5]):

$$\mathcal{L}(f, g)(t, \bar{p}, \bar{q}) = \mathcal{L}^+(t, \bar{p}, \bar{q}) - \mathcal{L}^-(t, \bar{p}, \bar{q}), \quad (14)$$

where

$$\begin{aligned} \mathcal{L}^+(f, g) &= \int_{\mathbb{R}^3 \times \vartheta} \frac{hr^2}{p_*^0} \int_{S^2 \times S^{N-2}} f(\bar{p}', \bar{q}') g(\bar{p}_*, \bar{q}_*) \sigma dw d\theta \\ \mathcal{L}^-(f, g) &= \int_{\mathbb{R}^3 \times \vartheta} \frac{hr^2}{p_*^0} \int_{S^2 \times S^{N-2}} f(\bar{p}, \bar{q}) g(\bar{p}_*, \bar{q}_*) \sigma dw d\theta \end{aligned}$$

with:

- S^2 is the unit sphere of $\mathbb{R}^3$, whose element is denoted dw
- S^{N-2} is the unit sphere of $\mathbb{R}^{N-1}$, whose element is denoted θ , and $d\theta$ his volume element
- $\sigma = \sigma(t, \bar{p}, \bar{q}, \bar{p}_*, \bar{q}_*, \bar{p}', \bar{q}', \bar{p}_*, \bar{q}_*)$ is a positive regular function called the collision kernel or the cross-section of the collisions, on which we require that

$$(H_1) : \begin{cases} \exists C > 0, 0 \leq \sigma \leq C \\ (1 + |\bar{p}|)^l \|\partial_{(\bar{p}, \bar{q})} \sigma\|_{L^1(\Omega \times \mathfrak{S})} \in L^\infty(\Omega), 0 \leq |\beta| \leq m+3, 0 \leq l \leq m+3 \\ (1 + |\bar{p}|)^{|\beta|-1} \partial_{(\bar{p}, \bar{q})}^\beta \sigma \in L^\infty(\Omega \times \Omega \times \mathfrak{S}), 1 \leq |\beta| \leq m+3, \end{cases}$$

where $\beta \in \mathbb{N}^{m+3}$, $\Omega = \mathbb{R}^3 \times \mathbb{R}^{N-1}$ and $\mathfrak{S} = S^2 \times S^{N-2}$. σ measures interactions effects between particles and determines their nature.

2.2 Parametrization of the post-collisional momenta and post-collisional charges

Suppose that the pre-collisional momenta p and p_* and the pre-collisional charges q and q_* are given. Then:

a) The conservation law of momenta (13-a) splits into:

$$p^0 + p_*^0 = p'^0 + p_*'^0 \quad (\text{a}) \quad \text{and} \quad \bar{p} + \bar{p}_* = \bar{p}' + \bar{p}'_* \quad (\text{b}). \quad (15)$$

(15-a) expresses the conservation of the quantity $\tilde{e} = \sqrt{1 + |\bar{p}|_g^2} + \sqrt{1 + |\bar{p}_*|_g^2}$, called the elementary energy of the unit rest mass of particles. We parametrize (15-b) by setting, following N. Noutchegueme and al in [14], $\bar{p}' = \bar{p} + d(\bar{p}, \bar{p}_*, \omega)w$ and $\bar{p}'_* = \bar{p}_* - d(\bar{p}, \bar{p}_*, \omega)w$ with $w \in S^2$ in which, d is a regular function given by:

$$d(\bar{p}, \bar{p}_*, \omega) = \frac{2\tilde{e}p^0 p_*^0 [\omega \cdot (\hat{p}_* - \hat{p})]}{\tilde{e}^2 - [\omega \cdot (\bar{p} + \bar{p}_*)]^2}, \quad (16)$$

where $\hat{p} = \frac{\bar{p}}{p^0}$, $\hat{p}_* = \frac{\bar{p}_*}{p_*^0}$. The scalar product in (16) which is defined by $(\bar{p}, \bar{p}_*) = h^2 p^1 p_*^1 + r^2 (p^2 p_*^2 + p^3 p_*^3)$ gives for $(\bar{p} = \bar{p}_*) : (\bar{p}, \bar{p}_*) \equiv |\bar{p}|_g^2 = h^2 (p^1)^2 + r^2 [(p^2)^2 + (p^3)^2]$.

The jacobian of the transformation $(\bar{p}, \bar{p}_*) \longrightarrow (\bar{p}', \bar{p}'_*)$ is given by

$$\frac{\partial(\bar{p}', \bar{p}'_*)}{\partial(\bar{p}, \bar{p}_*)} = -\frac{p'^0 p_*'^0}{p^0 p_*^0}. \quad (17)$$

b) The conservation law of charges (13-b) can also be split into

$$q^N + q_*^N = q'^N + q_*'^N \quad (\text{a}) \quad \text{and} \quad \tilde{q} + \tilde{q}_* = \tilde{q}' + \tilde{q}'_* \quad (\text{b}). \quad (18)$$

(18-a) express the conservation of the quantity $\hat{\mathbf{c}} = \left[e^2 - \sum_{I=1}^{N-1} (q^I)^2 \right]^{\frac{1}{2}} + \left[e^2 - \sum_{I=1}^{N-1} (q_*^I)^2 \right]^{\frac{1}{2}}$. We also parametrized (18-b) by setting, following the sketch given in [14] for the parametrization of the post-collisional momenta:

$$\tilde{q}' = \tilde{q} + \eta(\tilde{q}, \tilde{q}_*, \theta_1)\theta_1, \quad \tilde{q}'_* = \tilde{q}_* - \eta(\tilde{q}, \tilde{q}_*, \theta_1)\theta_1; \quad \theta_1 \in S^{N-2}. \quad (19)$$

By raising each member of (18-a) to the square and by (19), we get

$$\eta^2 + \eta(\tilde{q} - \tilde{q}_* \cdot \theta_1) + q^N q_*^N - \sqrt{B} = 0, \quad (20)$$

where

$$B = \eta^4 + 2\eta^3(\tilde{q} - \tilde{q}_* \cdot \theta_1) + \eta^2 \left[-2e^2 + (\tilde{q}_*)^2 + (\tilde{q})^2 - 4(\tilde{q} \cdot \theta_1)(\tilde{q}_* \cdot \theta_1) \right] + 2\eta \left[(\tilde{q}_* - \tilde{q} \cdot \theta_1)(q_*^N)^2 + \{(q^N)^2 - (q_*^N)^2\}(\tilde{q}_* \cdot \theta_1) \right] + e^4 - e^2 \left[(\tilde{q}_*)^2 + (\tilde{q})^2 \right] + (\tilde{q} \cdot \tilde{q}_*)^2.$$

By multiplying (20) by $\eta^2 + \eta(\tilde{q} - \tilde{q}_* \cdot \theta_1) + q^N q_*^N + \sqrt{B}$, we obtain

$$\eta^2 \left[(q^N + q_*^N)^2 + (\tilde{q}_* + \tilde{q} \cdot \theta_1)^2 \right] + 2\eta(q^N + q_*^N) \left[(\tilde{q} \cdot \theta_1)q_*^N - (\tilde{q}_* \cdot \theta_1)q^N \right] = 0,$$

since $\eta \neq 0$, we finally have:

$$\eta(\tilde{q}, \tilde{q}_*, \theta_1) = \frac{2(q^N + q_*^N) \left[(\tilde{q}_* \cdot \theta_1)q^N - (\tilde{q} \cdot \theta_1)q_*^N \right]}{[(q^N + q_*^N)^2 + \{(\tilde{q}_* + \tilde{q}) \cdot \theta_1\}^2]}. \quad (21)$$

The jacobian of the transformation $(\tilde{q}, \tilde{q}_*) \longmapsto (\tilde{q}', \tilde{q}'_*)$ is given by

$$J = \frac{\partial(\tilde{q}', \tilde{q}'_*)}{\partial(\tilde{q}, \tilde{q}_*)} = -\frac{q'^N q_*'^N}{q^N q_*^N}. \quad (22)$$

3 Function spaces and energy estimates

In all what follows, set $\Omega = \mathbb{R}^3 \times \mathbb{R}^{N-1}$ and $\mathfrak{S} = S^2 \times S^{N-2}$.

Definition 1. Let $T > 0$, $l \in \mathbb{N}$ and $d \in \mathbb{R}^+$.

1. $\mathbb{K}_d^l(\Omega) = \{u : \Omega \longrightarrow \mathbb{R}, (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta u \in L^2(\Omega), |\beta| \leq l\}$, endowed with the norm

$$\|u\|_{\mathbb{K}_d^l(\Omega)} = \max_{0 \leq |\beta| \leq l} \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta u \right\|_{L^2(\Omega)}.$$

2. $\mathbb{K}_d^l(0, T; \Omega) = \{u : [0, T] \times \Omega \longrightarrow \mathbb{R}, u \text{ continuous}, u(t, \cdot) \in \mathbb{K}_d^l(\Omega), \forall t \in [0, T]\}$, endowed with the norm

$$\|u\|_{\mathbb{K}_d^l(0, T; \Omega)} = \sup_{0 \leq t \leq T} \max_{0 \leq |\beta| \leq l} \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta u(t, \cdot) \right\|_{L^2(\Omega)}.$$

For $\delta > 0$, we also define

3. $\mathbb{K}_{d, \delta}^l(0, T; \Omega) = \{u \in \mathbb{K}_d^l(0, T; \Omega), \|u\|_{\mathbb{K}_d^l(0, T; \Omega)} \leq \delta\}$

4. $\mathbb{E}_d^l(\Omega) = \{u : \Omega \longrightarrow \mathbb{R}, (1 + |\bar{p}|)^{d+|\beta|} D_{(\bar{p}, \bar{q})}^\beta u \in L^2(\Omega), |\beta| \leq l\}$, endowed with the norm

$$\|u\|_{\mathbb{E}_d^l(\Omega)} = \max_{0 \leq |\beta| \leq l} \left\| (1 + |\bar{p}|)^{d+|\beta|} D_{(\bar{p}, \bar{q})}^\beta u \right\|_{L^2(\Omega)}.$$

5. $\mathbb{E}_d^l(0, T; \Omega) = \{u : [0, T] \times \Omega \rightarrow \mathbb{R}, u \text{ continuous}, u(t, \cdot) \in \mathbb{E}_d^l(\Omega), \forall t \in [0, T]\}$, endowed with the norm

$$\|u\|_{\mathbb{E}_d^l(0, T; \Omega)} = \sup_{0 \leq t \leq T} \max_{0 \leq |\beta| \leq l} \left\| (1 + |\bar{p}|)^{d+|\beta|} D_{(\bar{p}, \bar{q})}^\beta u(t, \cdot) \right\|_{L^2(\Omega)}.$$

6. $\mathbb{E}_{d, \delta}^l(0, T; \Omega) = \{u \in \mathbb{E}_d^l(0, T; \Omega), \|u\|_{\mathbb{E}_d^l(0, T; \Omega)} \leq \delta\}$.

$D_{(\bar{p}, \bar{q})}^\beta$ denotes the derivative in the sens of distributions.

Lemma 2 ([15]). Let $T > 0, l \in \mathbb{N}$ and $d \in \mathbb{R}$;

i) $\mathbb{E}_d^l(0, T; \Omega)$ is a Banach space.

2i) $\mathbb{E}_d^l(0, T; \Omega)$ is dense in $\mathbb{E}_d^l(0, T, \Omega)$ for the norm $\|\cdot\|_{\mathbb{E}_d^l(0, T, \Omega)}$.

3i) $\mathbb{E}_d^l(0, T; \Omega)$ is a separable Hilbert space.

4i) The space $\mathbb{E}_{d, \delta}^l(0, T; \Omega)$ is a completed metric subspace of $\mathbb{E}_d^l(0, T; \Omega)$.

Theorem 3. Let U be and open subset of $\mathbb{R}^n$ satisfying the cone property; $l, l_1, l_2 \in \mathbb{N}$; $d, d_1, d_2 \in \mathbb{R}^+$ such that $l_1, l_2 \geq l$, $l_1 + l_2 - \frac{n}{2} \geq l$ and $d_1 + d_2 + \frac{n}{2} \geq d$. Then, the application $(f, g) \mapsto fg$ is continuous from $\mathbb{E}_{d_1}^{l_1} \times \mathbb{E}_{d_2}^{l_2}$ to $\mathbb{E}_d^l$.

Before establishing the proof of this theorem, we first state the following lemma.

Lemma 4 ([9]). let $0 < \varepsilon \leq 1$.

i) There is an automorphism T_ε of the ring of functions on $\mathbb{R}^n$,

2i) T_ε is an isomorphism from $\mathbb{E}_d^l(\mathbb{R}^n)$ to $\mathbb{E}_{(d+\frac{n}{2})/\varepsilon-n/2}^l(\mathbb{R}^n)$

Proof of theorem 1. Let's recall that, if $l \geq l_1$, $d \geq d_1$, we obviously have $\mathbb{E}_d^l(\mathbb{R}^n) \subset \mathbb{E}_{d_1}^{l_1}(\mathbb{R}^n)$.

Now since $f \in \mathbb{E}_d^l(\mathbb{R}^n)$ is equivalent to $(1 + |\bar{p}|)^{d+|\beta|} D^\beta f \in \mathbb{E}^{l-\beta}(\mathbb{R}^n)$ for $0 \leq |\beta| \leq l$, it is clear that we have the following continuous multiplication property

$$\mathbb{E}_{d_1}^{l_1}(\mathbb{R}^n) \times \mathbb{E}_{d_2}^{l_2}(\mathbb{R}^n) \longrightarrow \mathbb{E}_{d_1+d_2}^l(\mathbb{R}^n) \quad \text{if } l_1 + l_2 - \frac{n}{2} \geq l. \quad (23)$$

Let $f \in \mathbb{E}_{d_1}^{l_1}(\mathbb{R}^n)$ and $g \in \mathbb{E}_{d_2}^{l_2}(\mathbb{R}^n)$. According to lemma 2(2i), we have :

$$T_\varepsilon(f) \in \mathbb{E}_{(d_1+\frac{n}{2})/\varepsilon-n/2}^l(\mathbb{R}^n) \quad \text{and} \quad T_\varepsilon(g) \in \mathbb{E}_{(d_2+\frac{n}{2})/\varepsilon-n/2}^l(\mathbb{R}^n).$$

Therefore, the relation (23) implies $T_\varepsilon(f)T_\varepsilon(g) = T_\varepsilon(fg) \in \mathbb{E}_{(d_1+d_2+n)/\varepsilon-n}^l(\mathbb{R}^n)$.

We also have $fg \in \mathbb{E}_{d_1+d_2+\frac{n}{2}-\varepsilon n/2}^l(\mathbb{R}^n)$.

Since this holds for every $0 < \varepsilon \leq 1$, we conclude that $fg \in \mathbb{E}_d^l(\mathbb{R}^n)$ for every $d \leq d_1 + d_2 + \frac{n}{2}$. The continuity comes from the fact that T_ε is an isomorphism. $\square$

$\mathbb{E}_d^l(\mathbb{R}^n)$ is an algebra if $l > \frac{n}{2}$ and $d > -\frac{n}{2}$.

Proof. We have $l - \frac{n}{2} > 0$ and $d + \frac{n}{2} > 0$. If we set $l_1 = l_2 = l$ and $d_1 = d_2 = d$ in theorem 1, it follows that $l + l - \frac{n}{2} > l$ and $d + d + \frac{n}{2} > d$. Thus, the bilinear application $(f, g) \mapsto fg$ is continuous from $\mathbb{E}_d^l(\mathbb{R}^n) \times \mathbb{E}_d^l(\mathbb{R}^n)$ to $\mathbb{E}_d^l(\mathbb{R}^n)$. $\square$

Remark 5. i) $f = f(t)$ is a function of $(\bar{p}, \bar{q}) \in \mathbb{R}^{N+2}$. We have from Sobolev injection theorem

$$W_2^l \hookrightarrow C_b^1(\mathbb{R}^n) \quad \text{if} \quad l > 1 + \frac{N+2}{2} = \frac{N+4}{2}.$$

Then, the smallest interger which satisfied the above property is $l = E(\frac{N+4}{2}) + 1$.

2i) If $N = 2m$ or $N = 2m + 1$ then, $l = m + 3$. From now and on, we will use $l = m + 3$.

3i) We have $l = m + 3$, $n = N + 2$ and $d \in [\frac{N+4}{2}, +\infty[$ Therefore, $\mathbb{E}_d^l$ is an algebra.

Lemma 6. *i) Let $T > 0$. If h_0 and r_0 satisfy $(h_0^2 + 2r_0^2)e^{2cT} \leq 1$. Then we have:*

$$(\tilde{e})^2 - [w \cdot (\bar{p} + \bar{p}_*)]^2 > 2. \quad (24)$$

2i) The function $(\bar{p}, \bar{p}_, w) \mapsto \partial_{\bar{p}}^\beta d(\bar{p}, \bar{p}_*, w)$, is bounded for $\beta \in \mathbb{N}^3$, $1 \leq |\beta| \leq m+3$.*

3i) The function $(\tilde{q}, \tilde{q}_, \theta) \mapsto \partial_{\tilde{q}}^\beta \eta(\tilde{q}, \tilde{q}_*, \theta)$, is bounded for $\beta \in \mathbb{N}^{N-1}$, $1 \leq |\beta| \leq m+3$.*

Proof. i) We proceed as in [4] by choosing h_0 and r_0 such as, $(h_0^2 + 2r_0^2)e^{2cT} \leq 1$ to conclude.

2i) For the second case, let $\beta \in \mathbb{N}^3$ such that $1 \leq |\beta| \leq m+3$. If $|\beta| = 1$, see [15].

If $|\beta| \geq 2$, we can look at $\partial_{\bar{p}}^\beta d(\bar{p}, \bar{p}_*, w)$, with d given by (16), as a rational function whose the degree of the denominator is greater or equal to the degree of the numerator. As the order of derivation increases, the degree of the denominator gains more power compared to that of the numerator. So, $\partial_{\bar{p}}^\beta d(\bar{p}, \bar{p}_*, w) \rightarrow 0$ when $|\bar{p}| \rightarrow +\infty$, hence for $\varepsilon = 1$, there exists $\delta > 0$ such that $|\bar{p}| > \delta \Rightarrow \left| \partial_{\bar{p}}^\beta d(\bar{p}, \bar{p}_*, w) \right| < 1$.

In other hand, according to (24), the denominator of the function $\partial_{\bar{p}}^\beta d(\bar{p}, \bar{p}_*, w)$ never vanishes. So it's of class C^∞ and therefore bounded on the compact $|\bar{p}| \leq \delta$. Hence 2i).

3i) is similar to 2i) □

Lemma 7. *The following estimates hold: i) $(1 + |\bar{p}|)^2 \leq C(T)(p^0)^2$,*

2i) $(1 + |\bar{p}|) \leq C(T)(1 + |\bar{p}'|)(1 + |\bar{p}_'|)$, 3i) $(1 + |\bar{p}|)^2 \leq C(T)p^0 p_*^0 p'^0 p_*'^0$.*

Proof. i) $(1 + |\bar{p}|)^2 = \left(1 + \sqrt{\sum_{i=1}^3 (p^i)^2}\right)^2 = \left(1 + \sqrt{\sum_{i=1}^3 \frac{1}{g_{ii}} g_{ii} (p^i)^2}\right)^2 \leq \left(1 + \sqrt{\left(\sum_{i=1}^3 \frac{1}{g_{ii}}\right) g_{ii} p^i p^i}\right)^2 \leq 2 \left(1 + \sum_{i=1}^3 \frac{1}{g_{ii}} g_{ii} p^i p^i\right) \leq 2 \left(1 + \sum_{i=1}^3 \frac{1}{g_{ii}}\right) (1 + g_{ii} p^i p^i) \leq C(T)(p^0)^2$.

2i) It follows from i) that, $(1 + |\bar{p}|) \leq C(T)p^0 \leq C(T)(p^0 + p_*^0) = C(T)(p'^0 + p_*'^0)$. Besides, $p'^0 \leq C(T)(1 + |\bar{p}'|)$ and $p_*'^0 \leq C(T)(1 + |\bar{p}_*'|)$. From where $(1 + |\bar{p}|) \leq C(T)(1 + |\bar{p}'|)(1 + |\bar{p}_*'|)$.

3i) Thereafter, combining 2i) and i), we then have:

$$(1 + |\bar{p}|)^2 \leq C(T)(1 + |\bar{p}'|)(1 + |\bar{p}_*'|)(1 + |\bar{p}'|)(1 + |\bar{p}_*'|) \leq C(T)p^0 p_*^0 p'^0 p_*'^0. \quad \square$$

Remark 8. *Let $(a_i)_{i=1}^n \in \mathbb{R}^n$. We have:*

$$\left(\sum_{i=1}^n a_i\right)^2 \leq n \sum_{i=1}^n a_i^2. \quad (25)$$

Proposition 9. *Let $d \in]\frac{N+4}{2}, +\infty[$. If $f, g \in \mathbb{K}_d^{m+3}(\Omega)$ then,*

$$\left\| (1 + |\bar{p}|)^d \frac{1}{p^0} \mathcal{L}^+(f, g) \right\|_{L^2(\Omega)} \leq C(T) \|f\|_{\mathbb{K}_d^{m+3}(\Omega)} \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}.$$

Proof. We have:

$$\begin{aligned} \left\| (1 + |\bar{p}|)^d \frac{1}{p^0} \mathcal{L}^+(f, g) \right\|_{L^2(\Omega)}^2 &= \int_{\Omega} \left[\frac{(1 + |\bar{p}|)^d}{p^0} \int_{\Omega} \frac{hr^2}{p_*^0} d\bar{p}_* w_{\tilde{q}_*} \int_{S^2 \times S^{N-2}} f(\bar{p}', \tilde{q}') g(\bar{p}_*, \tilde{q}_*) \sigma dw d\theta \right]^2 d\bar{p} w_{\tilde{q}} \\ &\leq C(T) \int_{\Omega} \frac{(1 + |\bar{p}|)^{2d}}{(p^0)^2} \left[\int_{\Omega \times \mathfrak{S}} \frac{f(\bar{p}', \tilde{q}') g(\bar{p}_*, \tilde{q}_*)}{p_*^0} \sigma^{\frac{1}{2}} \sigma^{\frac{1}{2}} dw d\theta d\bar{p}_* w_{\tilde{q}_*} \right]^2 d\bar{p} w_{\tilde{q}}. \end{aligned}$$

We apply the Cauchy-Schwartz inequality to the integral over $\Omega \times \mathfrak{S}$ to obtain:

$$\begin{aligned} \left\| (1 + |\bar{p}|)^d \frac{1}{p^0} \mathcal{L}^+(f, g) \right\|_{L^2(\Omega)}^2 &\leq \\ C(T) \int_{\mathfrak{S}} dw d\theta \int_{\Omega \times \Omega} \frac{(1 + |\bar{p}|)^{2d}}{p^0 p_*^0} |f(\bar{p}', \tilde{q}')|^2 |g(\bar{p}_*, \tilde{q}_*)|^2 \frac{d\bar{p} d\bar{p}_*}{p^0 p_*^0} w_{\tilde{q}} w_{\tilde{q}_*}, \end{aligned} \quad (26)$$

since $\|\sigma\|_{L^1(\Omega \times \mathfrak{S})} \in L^\infty(\Omega)$ and σ is bounded. According to (17) and (22), we also have that:

$$\frac{d\bar{p}d\bar{p}_*}{p^0p_*^0} = -\frac{d\bar{p}'d\bar{p}'_*}{p'^0p'^0_*} \text{ and } w_{\bar{q}}w_{\bar{q}_*} = w_{\bar{q}'}w_{\bar{q}'_*}. \quad (27)$$

Noting that $\frac{1}{p^0p_*^0} < 1$, (26) gives using 2i) and 3i) in lemma 4:

$$\left\| (1 + |\bar{p}|)^d \frac{1}{p^0} \mathcal{L}^+(f, g) \right\|_{L^2(\Omega)}^2 \leq C(T) \int_{\Omega} (1 + |\bar{p}'|)^{2d} |f(\bar{p}, \bar{q}')|^2 d\bar{p}' w_{\bar{q}'} \times \int_{\Omega} (1 + |\bar{p}'_*|)^{2d} |g(\bar{p}'_*, \bar{q}'_*)|^2 d\bar{p}'_* w_{\bar{q}'_*}.$$

Then, $\left\| (1 + |\bar{p}|)^d \frac{1}{p^0} \mathcal{L}^+(f, g) \right\|_{L^2(\Omega)} \leq C(T) \|f\|_{\mathbb{K}_d^{m+3}(\Omega)} \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}.$ $\square$

Proposition 10. Let $d \in]\frac{N+4}{2}, +\infty[$ and $\beta \in \mathbb{N}^{N+2}$, $|\beta| = 1$. If $f, g \in \mathbb{K}_d^{m+3}(\Omega)$ then:

$$\left\| (1 + |\bar{p}|)^{d+1} \frac{\partial^\beta}{\partial(\bar{p}, \bar{q})} \left[\frac{1}{p^0} \mathcal{L}^+(f, g) \right] \right\|_{L^2(\Omega)} \leq C(T) \|f\|_{\mathbb{K}_d^{m+3}(\Omega)} \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}.$$

Proof. The proof of this proposition will be given by lemmas 5 and 6. $\square$

Lemma 11. Let $d \in]\frac{N+4}{2}, +\infty[$ and $i_1 = 1, 2, 3$. If $f, g \in \mathbb{K}_d^{m+3}(\Omega)$, then:

$$\left\| (1 + |\bar{p}|)^{d+1} \frac{\partial}{\partial p^{i_1}} \left[\frac{1}{p^0} \mathcal{L}^+(f, g) \right] \right\|_{L^2(\Omega)} \leq C(T) \|f\|_{\mathbb{K}_d^{m+3}(\Omega)} \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}.$$

Proof. Since $(1 + |\bar{p}|) \leq C(T)p^0$ and $|g_{i_1 i_1} p^{i_1}| \leq C(T)p^0$, after computation, we have:

$$\left\| (1 + |\bar{p}|)^{d+1} \frac{\partial}{\partial p^{i_1}} \left[\frac{1}{p^0} \mathcal{L}^+(f, g) \right] \right\|_{L^2(\Omega)} \leq A_0 + C(T)B_0, \quad (28)$$

where $A_0 = \left\| \frac{(1+|\bar{p}|)^{d+1}}{p^0} \frac{\partial}{\partial p^{i_1}} [\mathcal{L}^+(f, g)] \right\|_{L^2(\Omega)}$ and $B_0 = \left\| (1 + |\bar{p}|)^d \frac{1}{p^0} \mathcal{L}^+(f, g) \right\|_{L^2(\Omega)}.$

The term B_0 in (28) is estimated in the proposition 1. In the sequel, for the term A_0 , the lemma 3 gives that $|\partial_{p^{i_1}} \bar{p}'|$ and $|\partial_{p^{i_1}} \bar{p}'_*|$ are bounded and using (25), it comes that:

$$\begin{aligned} A_0 &= \int_{\Omega} \left[\frac{(1+|\bar{p}|)^{d+1}}{p^0} \int_{\Omega} \frac{1}{p_*^0} d\bar{p}_* w_{\bar{q}_*} \int_{\mathfrak{S}} \frac{\partial}{\partial p^{i_1}} \{f(\bar{p}', \bar{q}') g(\bar{p}'_*, \bar{q}'_*) \sigma\} dw d\theta \right]^2 d\bar{p} w_{\bar{q}} \\ &\leq C(T) (A_1 + B_1 + C_1), \end{aligned} \quad (29)$$

$$\text{where } \begin{cases} A_1 = \int_{\Omega} d\bar{p} w_{\bar{q}} \frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2} \left[\int_{\Omega \times \mathfrak{S}} \frac{|g \sigma \partial_{p^{i_1}} f|}{p_*^0} d\bar{p}_* d\theta w_{\bar{q}_*} dw \right]^2 \\ B_1 = \int_{\Omega} d\bar{p} w_{\bar{q}} \frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2} \left[\int_{\Omega \times \mathfrak{S}} \frac{|f \sigma \partial_{p^{i_1}} g|}{p_*^0} d\bar{p}_* d\theta w_{\bar{q}_*} dw \right]^2 \\ C_1 = \int_{\Omega} d\bar{p} w_{\bar{q}} \frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2} \left[\int_{\Omega \times \mathfrak{S}} \frac{|f g \partial_{p^{i_1}} \sigma|}{p_*^0} d\bar{p}_* d\theta w_{\bar{q}_*} dw \right]^2 \end{cases}.$$

For the term A_1 , we use the Cauchy Schwartz inequality and the assumptions (H_1) , to obtain:

$$A_1 \leq C(T) \int_{\Omega} \frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2} \left(\int_{\Omega \times \mathfrak{S}} \frac{|\partial_{p^{i_1}} f|^2 |g|^2 \sigma}{(p_*^0)^2} d\bar{p}_* d\theta w_{\bar{q}_*} dw \right) d\bar{p} w_{\bar{q}}.$$

Considering (27) and lemma 4, yields to

$$\begin{aligned}
A_1 &\leq C(T) \int_{\mathfrak{S}} dw d\theta \int_{\Omega \times \Omega} \frac{(1 + |\bar{p}|)^{2d+2}}{p^0 p_*^0} |\partial_{p'^{i_1}} f|^2 |g|^2 \frac{d\bar{p} d\bar{p}_*}{p^0 p_*^0} w_{\bar{q}} w_{\bar{q}_*} \\
&\leq C(T) \int_{\Omega \times \Omega} \frac{(1 + |\bar{p}|)^2 (1 + |\bar{p}|)^{2d}}{p^0 p_*^0} |\partial_{p'^{i_1}} f|^2 |g|^2 \frac{d\bar{p} d\bar{p}_*}{p^0 p_*^0} w_{\bar{q}} w_{\bar{q}_*} \\
&\leq C(T) \int_{\Omega \times \Omega} \frac{(1 + |\bar{p}|)^2 (1 + |\bar{p}'|)^{2d} (1 + |\bar{p}'_*|)^{2d}}{p^0 p_*^0} |\partial_{p'^{i_1}} f|^2 |g|^2 \frac{d\bar{p}' d\bar{p}'_*}{p'^0 p'^0_*} w_{\bar{q}'} w_{\bar{q}'_*} \\
&\leq C(T) \int_{\Omega} (1 + |\bar{p}'|)^{2d+2} |\partial_{p'^{i_1}} f|^2 d\bar{p}' w_{\bar{q}'} \int_{\Omega} (1 + |\bar{p}'_*|)^{2d} |g|^2 d\bar{p}'_* w_{\bar{q}'_*}.
\end{aligned}$$

Which means that:

$$A_1 \leq C(T) \|f\|_{\mathbb{K}_d^{m+3}(\Omega)}^2 \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}^2. \quad (30)$$

For the term B_1 , interchanging f and g in A_1 , we obtain an inequality similar to (30) for B_1 .

Now we have for the last term C_1 , using cauchy-Schwartz inequality and (H_1) :

$$\begin{aligned}
C_1 &\leq C(T) \int_{\Omega} d\bar{p} w_{\bar{q}} \frac{(1 + |\bar{p}|)^{2d+2}}{(p^0)^2} \int_{\Omega \times \mathfrak{S}} \frac{|f|^2 |g|^2 |\partial_{p'^{i_1}} \sigma|}{(p_*^0)^2} d\bar{p}_* w_{\bar{q}_*} dw d\theta \int_{\Omega \times \mathfrak{S}} |\partial_{p'^{i_1}} \sigma| d\bar{p}_* w_{\bar{q}_*} dw d\theta \\
&\leq C(T) \int_{\Omega \times \Omega} \frac{(1 + |\bar{p}|)^{2d+2}}{p^0 p_*^0} |f|^2 |g|^2 \frac{d\bar{p} d\bar{p}_*}{p^0 p_*^0} w_{\bar{q}} w_{\bar{q}_*}.
\end{aligned}$$

We use (30) in which we replace $\partial_{p'^{i_1}} f$ by f to obtain:

$$\begin{aligned}
C_1 &\leq C(T) \int_{\Omega} (1 + |\bar{p}'|)^{2d} |f|^2 d\bar{p}' w_{\bar{q}'} \times \int_{\Omega} (1 + |\bar{p}'_*|)^{2d} |g|^2 d\bar{p}'_* w_{\bar{q}'_*} \\
&\leq C(T) \|f\|_{\mathbb{K}_d^{m+3}(\Omega)}^2 \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}^2. \quad (31)
\end{aligned}$$

From (30), (31) and proposition 1, we have the proof of lemma 5. $\square$

Lemma 12. let $d \in]\frac{N+4}{2}, +\infty[$ and $a_1 = 1, \dots, N-1$. If $f, g \in \mathbb{K}_d^{m+3}(\Omega)$, then:

$$\left\| (1 + |\bar{p}|)^{d+1} \frac{\partial}{\partial q^{a_1}} \left[\frac{1}{p^0} \mathcal{L}^+(f, g) \right] \right\|_{L^2(\Omega)} \leq C(T) \|f\|_{\mathbb{K}_d^{m+3}(\Omega)}^2 \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}^2.$$

Proof. Given lemma 3, $|\partial_{q^{a_1}} \tilde{q}'|$ and $|\partial_{q^{a_1}} \tilde{q}'_*|$ are bounded and using (25), we obtain after computation:

$$\begin{aligned}
&\left\| (1 + |\bar{p}|)^{d+1} \frac{\partial}{\partial q^{a_1}} \left(\frac{1}{p^0} \mathcal{L}^+(f, g) \right) \right\|_{L^2(\Omega)}^2 = \\
&\int_{\Omega} d\bar{p} w_{\bar{q}} \left[\frac{(1 + |\bar{p}|)^{d+1}}{p^0} \int_{\Omega} \frac{h^3}{p_*^0} d\bar{p}_* w_{\bar{q}_*} \int_{\mathfrak{S}} \frac{\partial}{\partial q^{a_1}} \{f(\bar{p}', \tilde{q}') g(\bar{p}'_*, \tilde{q}'_*) \sigma\} dw d\theta \right]^2 \leq A_2 + B_2 + C_2, \\
\text{where } \begin{cases} A_2 = C(T) \int_{\Omega} d\bar{p} w_{\bar{q}} \frac{(1 + |\bar{p}|)^{2d+2}}{(p^0)^2} \left[\int_{\Omega \times \mathfrak{S}} \frac{|\sigma g \partial_{q^{a_1}} f|}{p_*^0} dw d\theta d\bar{p}_* w_{\bar{q}_*} \right]^2 \\ B_2 = C(T) \int_{\Omega} d\bar{p} w_{\bar{q}} \frac{(1 + |\bar{p}|)^{2d+2}}{(p^0)^2} \left[\int_{\Omega \times \mathfrak{S}} \frac{|\sigma f \partial_{q^{a_1}} g|}{p_*^0} dw d\theta d\bar{p}_* w_{\bar{q}_*} \right]^2 \\ C_2 = C(T) \int_{\Omega} d\bar{p} w_{\bar{q}} \frac{(1 + |\bar{p}|)^{2d+2}}{(p^0)^2} \left[\int_{\Omega \times \mathfrak{S}} \frac{|f g \partial_{q^{a_1}} \sigma|}{p_*^0} dw d\theta d\bar{p}_* w_{\bar{q}_*} \right]^2 \end{cases}.
\end{aligned}$$

We proceed like in the proof of lemma 5 to ends the proof of lemma 6. $\square$

Lemma 13. There exists a constant $C = C(T) > 0$ such that, $\forall \beta \in \mathbb{N}^3$ with $0 \leq |\beta| \leq m+3$, we have:

$$\left| (1 + |\bar{p}|)^{|\beta|} \partial_{\bar{p}}^{\beta} \left(\frac{1}{p^0} \right) \right| \leq C \frac{1}{p^0}.$$

Proof. For the sake of clarity, we start the proof of this lemma by a conjecture on the module of the multi-index β , before concluding by a recursion reasoning. Let $\beta \in \mathbb{N}^3$ with $0 \leq |\beta| \leq m+3$ and $q(|\beta|)$ be the proposition defined by:

" $\exists C > 0$, such that $0 \leq |\beta| \leq m+3$, we have $\left| (1 + |\bar{p}|)^{|\beta|} \partial_{\bar{p}}^{\beta} \left(\frac{1}{p^0} \right) \right| \leq C \frac{1}{p^0}$ ".

For $|\beta| = 0$, the result is obvious, and therefore, $q(0)$ is true.

For $|\beta| = 1$, let $i_1 = 1, 2, 3$, we have $(1 + |\bar{p}|) \partial_{p^{i_1}} \left(\frac{1}{p^0} \right) = -(1 + |\bar{p}|) \frac{p^{i_1}}{(p^0)^3}$. Considering the fact that: $(1 + |\bar{p}|) \leq C(T)p^0$, $|g_{i_1 i_1} p^{i_1}| \leq C(T)p^0$ and $\frac{1}{p^0} < 1$, we obtain:

$$\left| (1 + |\bar{p}|) \partial_{p^{i_1}} \left(\frac{1}{p^0} \right) \right| \leq C \frac{1}{p^0}, \text{ which implies that } q(1) \text{ is true.}$$

For $|\beta| = 2$, let $i_1, i_2 = 1, 2, 3$, we have $\partial_{p^{i_2}} \left[\partial_{p^{i_1}} \left(\frac{1}{p^0} \right) \right] = -g_{i_1 i_2} \left[\frac{\delta_{i_1 i_2} (p^0)^3 - 3g_{1_2 i_2} p^{i_2} p^{i_1} p^0}{(p^0)^6} \right]$. Since $(1 + |\bar{p}|)^2 \leq C(T)(p^0)^2$ and $|g_{ii} p^i| \leq C(T)p^0$, we obtain:

$$\left| (1 + |\bar{p}|)^2 \partial_{p^{i_2} p^{i_1}}^2 \left(\frac{1}{p^0} \right) \right| \leq C \frac{1}{p^0}, \text{ which implies that } q(2) \text{ is true.}$$

For $|\beta| = 3$, let $i_1, i_2, i_3 = 1, 2, 3$, we have:

$$\partial_{p^{i_3}} \left[\partial_{p^{i_2} p^{i_1}}^2 \left(\frac{1}{p^0} \right) \right] = \frac{3\delta_{i_1 i_2} g_{i_1 i_1} g_{i_3 i_3} p^{i_3}}{(p^0)^5} + \frac{3g_{i_1 i_1} g_{i_2 i_2} (\delta_{i_1 i_3} p^{i_2} + \delta_{i_2 i_3} g_{i_1 i_1} g_{i_2 i_2} g_{i_3 i_3} p^{i_1})}{(p^0)^5} - \frac{3p^{i_1} p^{i_2} p^{i_3}}{(p^0)^7}.$$

Likewise, we get:

$$\left| (1 + |\bar{p}|)^3 \partial_{p^{i_3} p^{i_2} p^{i_1}}^3 \left(\frac{1}{p^0} \right) \right| \leq C \frac{1}{p^0}, \text{ which implies that } q(3) \text{ is true.}$$

We now suppose that $\forall \beta \in \mathbb{N}^3$ such that $3 \leq |\beta| \leq m+2$, the proposition $q(|\beta|)$ is true and show that $q(|\beta| + 1)$ is also true.

Let $\beta \in \mathbb{N}^3$ such that $3 \leq |\beta| \leq m+2$, in view of the foregoing, we can write:

$$\partial_{\bar{p}}^{\beta} \left(\frac{1}{p^0} \right) = \frac{\chi}{(p^0)^{2|\beta|+1}}, \quad (1A)$$

where χ can be regarded as a polynomial in $\bar{p}$ with bounded coefficients of degree strictly less than $2|\beta| + 1$.

Let $j = 1, 2, 3$. Using (1A), we have:

$$\partial_{p^j} \left[\partial_{\bar{p}}^{\beta} \left(\frac{1}{p^0} \right) \right] = \frac{(\partial_{p^j} \chi)(p^0)^{2|\beta|} - 2(|\beta| + 1)p^j (p^0)^{2|\beta|-2} \chi}{(p^0)^{4|\beta|+1}}, \quad (2A)$$

and taking into account $(1 + |\bar{p}|)^{|\beta|} \leq C(T)(p^0)^{|\beta|}$ in the recurrence hypothesis, we established that:

$$|\chi| \leq C(p^0)^{|\beta|}. \quad (3A)$$

Therefore,

$$|\partial_{p^j} \chi| \leq C(p^0)^{|\beta|-1}. \quad (4A)$$

Combining (3A), (4A), $(1 + |\bar{p}|)^{|\beta|+1} \leq C(T)(p^0)^{|\beta|+1}$ and $\frac{1}{p^0} < 1$ in (2A), we obtain:

$$\left| (1 + |\bar{p}|)^{|\beta|+1} \partial_{\bar{p}}^{\beta+\beta'} \left(\frac{1}{p^0} \right) \right| \leq C \frac{1}{p^0}, \quad \forall \beta' \in \mathbb{N}^3 \text{ such that } |\beta'| = 1.$$

Hence $q(|\beta| + 1)$ is true, which completes the proof of lemma 7. $\square$

Remark 14. $\forall k \in \mathbb{N}^{N+2}$ with $0 \leq |k| \leq m+3$, there is a unique pair of multi index $(k_1, k_2) \in \mathbb{N}^3 \times \mathbb{N}^{N-1}$ such that $k = (k_1, k_2)$. We have

$$\partial_{(\bar{p}, \bar{q})}^k \left(\frac{1}{p^0} \right) = \begin{cases} \partial_{\bar{p}}^k \left(\frac{1}{p^0} \right) & \text{if } k_2 = 0_{\mathbb{N}^{N-1}} \\ 0 & \text{else} \end{cases}$$

Proposition 15. let $d \in]\frac{N+4}{2}, \infty[$; $f, g \in \mathbb{K}_d^{m+3}(\Omega)$ and $\beta \in \mathbb{N}^{N+2}$ such that: $2 < |\beta| \leq m+2$. If

$$\left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta \left(\frac{1}{p^0} \mathcal{L}^+(f, g) \right) \right\|_{L^2(\Omega)} \leq C(T) \|f\|_{\mathbb{K}_d^{m+3}(\Omega)} \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}$$

then, $\forall \beta' \in \mathbb{N}^{N+2}$ such that $|\beta'| = 1$, we have:

$$\left\| (1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} \left(\frac{1}{p^0} \mathcal{L}^+(f, g) \right) \right\|_{L^2(\Omega)} \leq C \|f\|_{\mathbb{K}_d^{m+3}(\Omega)} \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}.$$

Proof. Setting

$$\alpha = \beta + \beta', \quad \text{we have} \quad |\alpha| \leq |\beta| + |\beta'| \leq m+3,$$

and using the Leibnitz formula, we obtain:

$$\partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} \left(\frac{1}{p^0} \mathcal{L}^+(f, g) \right) = \frac{1}{p^0} \partial_{(\bar{p}, \bar{q})}^\alpha (\mathcal{L}^+) + \sum_{k \leq \alpha, |k| \geq 1} C_\alpha^k \partial_{(\bar{p}, \bar{q})}^k \left(\frac{1}{p^0} \right) \partial_{(\bar{p}, \bar{q})}^{\alpha-k} (\mathcal{L}^+).$$

Which implies

$$\left\| (1 + |\bar{p}|)^{d+|\alpha|} \partial_{(\bar{p}, \bar{q})}^\alpha \left(\frac{1}{p^0} \mathcal{L}^+(f, g) \right) \right\|_{L^2(\Omega)}^2 \leq A_1 + A_2, \quad (33)$$

where

$$\begin{cases} A_1 = \left\| (1 + |\bar{p}|)^{d+|\alpha|} \frac{hr^2}{p^0} \partial_{(\bar{p}, \bar{q})}^\alpha \mathcal{L}^+ \right\|_{L^2(\Omega)}^2 \\ A_2 = \left\| (1 + |\bar{p}|)^{d+|\alpha|} \sum_{k \leq \alpha, |k| \geq 1} C_\alpha^k \partial_{(\bar{p}, \bar{q})}^k \left(\frac{1}{p^0} \right) \partial_{(\bar{p}, \bar{q})}^{\alpha-k} (\mathcal{L}^+) \right\|_{L^2(\Omega)}^2 \end{cases} \quad (34)$$

In A_2 , perform the following factorization

$$A_2 = \left\| (1 + |\bar{p}|)^{d+|\alpha|} \sum_{k \leq \alpha, |k| \geq 1} C_\alpha^k (1 + |\bar{p}|)^{|k|} \partial_{(\bar{p}, \bar{q})}^k \left(\frac{1}{p^0} \right) \partial_{(\bar{p}, \bar{q})}^{\alpha-k} (\mathcal{L}^+) \frac{1}{(1 + |\bar{p}|)^{|k|}} \right\|_{L^2(\Omega)}^2.$$

Using lemma 7 and (25), we have:

$$A_2 \leq C(T) \|f\|_{\mathbb{K}_d^{m+3}(\Omega)}^2 \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}^2, \quad (35)$$

since $|\alpha| - |k| \leq m+2$ because $(|k| \geq 1)$ and $|\alpha| \leq m+3$, we conclude from the recurrence hypothesis.

We then have to study A_1 given by (34). One has

$$\partial_{(\bar{p}, \bar{q})}^\alpha \mathcal{L}^+(f, g) = \int_{\Omega} \frac{hr^2}{p_*^0} d\bar{p}_* w_{\bar{q}_*} \int_{\mathfrak{S}} \partial_{(\bar{p}, \bar{q})}^\alpha (fg\sigma) dw d\theta.$$

Applying twice successively the Leibniz formula, we obtain:

$$\partial_{(\bar{p}, \bar{q})}^\alpha (fg\sigma) = \sum_{k \leq \alpha} C_\alpha^k \partial_{(\bar{p}, \bar{q})}^{\alpha-k} \sigma \sum_{\lambda \leq k} C_k^\lambda \partial_{(\bar{p}, \bar{q})}^\lambda f \partial_{(\bar{p}, \bar{q})}^{k-\lambda} g.$$

Which yields

$$A_1 = \int_{\Omega} d\bar{p}w_{\bar{q}} \left[\frac{(1+|\bar{p}|)^{d+|\alpha|}}{p^0} \int_{\Omega} \frac{hr^2}{p_*^0} d\bar{p}_*w_{\bar{q}_*} \int_{\mathfrak{S}} \sum_{k \leq \alpha} C_{\alpha}^k \partial_{(\bar{p}, \bar{q})}^{\alpha-k} \sigma \sum_{\lambda \leq k} C_k^{\lambda} \partial_{(\bar{p}, \bar{q})}^{\lambda} f \partial_{(\bar{p}, \bar{q})}^{k-\lambda} g dw d\theta \right]^2,$$

Using (25), we get:

$$A_1 \leq C(T) \sum_{k \leq \alpha} J_k, \quad (36)$$

$$\text{where } J_k = \int_{\Omega} d\bar{p}w_{\bar{q}} \left[\frac{(1+|\bar{p}|)^{d+|\alpha|}}{p^0} \int_{\Omega} \frac{hr^2}{p_*^0} d\bar{p}_*w_{\bar{q}_*} \int_{\mathfrak{S}} \partial_{(\bar{p}, \bar{q})}^{\alpha-k} \sigma \sum_{\lambda \leq k} C_k^{\lambda} \partial_{(\bar{p}, \bar{q})}^{\lambda} f \partial_{(\bar{p}, \bar{q})}^{k-\lambda} g dw d\theta \right]^2.$$

Let $s_1 \in \mathbb{N}^{m+3}$, arbitrarily chosen and satisfying $s_1 \leq \alpha$. For $k = s_1$, we have according to (25):

$$J_{s_1} \leq C(T) \sum_{\lambda \leq s_1} I_{\lambda},$$

$$\text{where, } I_{\lambda} = \int_{\Omega} d\bar{p}w_{\bar{q}} \left[\frac{(1+|\bar{p}|)^{d+|\alpha|}}{p^0} \int_{\Omega} \frac{hr^2}{p_*^0} d\bar{p}_*w_{\bar{q}_*} \int_{\mathfrak{S}} \partial_{(\bar{p}, \bar{q})}^{\alpha-s_1} \sigma \partial_{(\bar{p}, \bar{q})}^{\lambda} f \partial_{(\bar{p}, \bar{q})}^{s_1-\lambda} g dw d\theta \right]^2.$$

let $s_2 \in \mathbb{N}^{N+2}$, arbitrarily chosen and satisfying $s_2 \leq s_1$.

Estimation of I_{λ} for $\lambda = s_2$

Applying the Cauchy-Schwartz inequality for the integral on $\Omega \times \mathfrak{S}$, we get by considering the assumptions on the collision kernel (H_1):

$$I_{s_2} \leq C(T) \int_{\Omega} d\bar{p}w_{\bar{q}} \frac{(1+|\bar{p}|)^{2d+2|\alpha|}}{(p^0)^2} \int_{\Omega \times \mathfrak{S}} \frac{|\partial_{(\bar{p}, \bar{q})}^{s_2} f|^2 |\partial_{(\bar{p}, \bar{q})}^{s_1-s_2} g|^2}{(p_*^0)^2} d\bar{p}_*w_{\bar{q}_*} dw d\theta,$$

Using lemma 4 and the inequality $|\alpha| \leq |\beta| + 1$ we have:

$$\begin{aligned} I_{s_2} &\leq C(T) \int_{\Omega \times \Omega} \frac{(1+|\bar{p}|)^{2d+2|\alpha|} |\partial^{s_2} f|^2 |\partial^{s_1-s_2} g|^2}{(p_*^0)^2 (p^0)^2} d\bar{p}_*w_{\bar{q}_*} d\bar{p}w_{\bar{q}} \\ &\leq C \int_{\Omega \times \Omega} (1+|\bar{p}'|)^{2d+2|\beta|} |\partial^{s_2} f|^2 (1+|\bar{p}'_*|)^{2d+2|\beta|} |\partial^{s_1-s_2} g|^2 \frac{(1+|\bar{p}|)^2}{p_*^0 p^0} \frac{d\bar{p}d\bar{p}_*}{p_*^0 p^0} w_{\bar{q}_*} w_{\bar{q}} \\ &\leq C \int_{\Omega \times \Omega} (1+|\bar{p}'|)^{2d+2|\beta|} |\partial^{s_2} f|^2 (1+|\bar{p}'_*|)^{2d+2|\beta|} |\partial^{s_1-s_2} g|^2 d\bar{p}' d\bar{p}'_* w_{\bar{q}'_*} w_{\bar{q}} \\ &\leq C \int_{\Omega} (1+|\bar{p}'|)^{2d+2|\beta|} |\partial^{s_2} f|^2 d\bar{p}' w_{\bar{q}} \int_{\Omega} (1+|\bar{p}'_*|)^{2d+2|\beta|} |\partial^{s_1-s_2} g|^2 d\bar{p}'_* w_{\bar{q}'_*}, \end{aligned}$$

which means :

$$I_{s_2} \leq C(T) \|f\|_{\mathbb{K}_d^{m+3}(\Omega)}^2 \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}^2. \quad (37)$$

As s_2 is arbitrarily chosen, we have:

$$J_{s_1} \leq C(T) \sum_{\lambda=0}^{s_1} I_{\lambda} \leq C \|f\|_{\mathbb{K}_d^{m+3}(\Omega)}^2 \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}^2. \quad (38)$$

Likewise, as s_1 is arbitrarily chosen in (38)

$$A_1 \leq C(T) \sum_{k \leq \alpha} J_k \leq C \|f\|_{\mathbb{K}_d^{m+3}(\Omega)}^2 \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}^2. \quad (39)$$

(35) and (39) end the proof of proposition 3. $\square$

Theorem 16. Let $d \in]\frac{N+4}{2}, +\infty[$ and $f, g \in \mathbb{K}_d^{m+3}(\Omega)$. We have:

i)

$$\frac{1}{p^0} \mathcal{L}^+(f, g) \in \mathbb{K}_d^{m+3}(\Omega) \quad \text{and} \quad \frac{1}{p^0} \mathcal{L}^-(f, g) \in \mathbb{K}_d^{m+3}(\Omega).$$

2i) There exists $C = C(T) > 0$ such that: $\left\| \frac{1}{p^0} \mathcal{L}^+(f, g) \right\|_{\mathbb{K}_d^{m+3}(\Omega)} \leq C \|f\|_{\mathbb{K}_d^{m+3}(\Omega)} \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}$ and $\left\| \frac{1}{p^0} \mathcal{L}^-(f, g) \right\|_{\mathbb{K}_d^{m+3}(\Omega)} \leq C \|f\|_{\mathbb{K}_d^{m+3}(\Omega)} \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}.$

Proof. We show that, $\forall \beta \in \mathbb{N}^{N+2}$ such that $0 \leq |\beta| \leq m+3$ we have:

$$\left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta \left(\frac{1}{p^0} \mathcal{L}^+(f, g) \right) \right\|_{L^2(\Omega)} \leq C \|f\|_{\mathbb{K}_d^{m+3}(\Omega)} \|g\|_{\mathbb{K}_d^{m+3}(\Omega)}.$$

For that, we reason by recurrence on the module of the multi index β . The conclusion is obtained using propositions 1, 2 and 3. For the term $\mathcal{L}^-(f, g)$, we proceed in the same way. $\square$

Lemma 17. Let $d \in]\frac{N+4}{2}, +\infty[$ and $f, g \in \mathbb{E}_d^{m+3}(\Omega)$. We have:

i) $\left\| \frac{1}{p^0} \mathcal{L}^+(f, f) - \frac{1}{p^0} \mathcal{L}^+(g, g) \right\|_{\mathbb{K}_d^{m+3}(\Omega)} \leq C \left(\|f\|_{\mathbb{K}_d^{m+3}(\Omega)} + \|g\|_{\mathbb{K}_d^{m+3}(\Omega)} \right) \|f - g\|_{\mathbb{K}_d^{m+3}(\Omega)}$

2i) $\left\| \frac{1}{p^0} \mathcal{L}^-(f, f) - \frac{1}{p^0} \mathcal{L}^-(g, g) \right\|_{\mathbb{K}_d^{m+3}(\Omega)} \leq C \left(\|f\|_{\mathbb{K}_d^{m+3}(\Omega)} + \|g\|_{\mathbb{K}_d^{m+3}(\Omega)} \right) \|f - g\|_{\mathbb{K}_d^{m+3}(\Omega)}.$

Proof. For the proof of this lemma, we use the bilinearity of the collision operator. $\square$

Theorem 18. Let $d \in]\frac{N+4}{2}, +\infty[$, $a_0 > 0$ and $f, g \in \mathbb{E}_d^{m+3}(\Omega)$. then, we have:

i)

$$\frac{1}{p^0} \mathcal{L}(f, f) + a_0 f^2 \in \mathbb{E}_d^{m+3}(\Omega).$$

2i) There exists $C = C(T) > 0$ such that :

$$\left\| \frac{1}{p^0} \mathcal{L}(f, f) + a_0 f^2 \right\|_{\mathbb{E}_d^{m+3}(\Omega)} \leq C \|f\|_{\mathbb{E}_d^{m+3}(\Omega)}^2,$$

3i) There exists $C = C(T) > 0$ such that:

$$\left\| \frac{1}{p^0} \mathcal{L}(f, f) - \frac{1}{p^0} \mathcal{L}(g, g) + a_0(f^2 - g^2) \right\|_{\mathbb{E}_d^{m+3}(\Omega)} \leq 2C \left(\|f\|_{\mathbb{E}_d^{m+3}(\Omega)} + \|g\|_{\mathbb{E}_d^{m+3}(\Omega)} \right) \|f - g\|_{\mathbb{E}_d^{m+3}(\Omega)}.$$

Proof. For i) and 2i), the proof will be divided in two steps:

a) We use the density of $\mathbb{K}_d^{m+3}(\Omega)$ in $\mathbb{E}_d^{m+3}(\Omega)$ and therefore, reason for $f \in \mathbb{K}_d^{m+3}(\Omega)$ to conclude classically that $\frac{1}{p^0} \mathcal{L}(f, f) \in \mathbb{E}_d^{m+3}(\Omega)$.

Let $f \in \mathbb{K}_d^{m+3}(\Omega)$ and $\beta \in \mathbb{N}^{m+3}$, $0 \leq |\beta| \leq m+3$, we have:

$$\frac{1}{p^0} \mathcal{L}(f, f) = \frac{1}{p^0} \mathcal{L}^+(f, f) - \frac{1}{p^0} \mathcal{L}^-(f, f).$$

We conclude using theorem 2, that $\frac{1}{p^0} \mathcal{L}(f, f) \in \mathbb{K}_d^{m+3}(\Omega)$, and there exists $C = C(T) > 0$ such that $\left\| \frac{1}{p^0} \mathcal{L}(f, f) \right\|_{\mathbb{K}_d^{m+3}(\Omega)} \leq C \|f\|_{\mathbb{K}_d^{m+3}(\Omega)}^2.$

b) We conclude using remark 1(3i) that $f^2 \in \mathbb{E}_d^{m+3}(\Omega)$ and there exists $C = C(T) > 0$ such that $\|f^2\|_{\mathbb{E}_d^{m+3}(\Omega)} \leq C \|f\|_{\mathbb{E}_d^{m+3}(\Omega)}^2.$

For 3i), we use the bilinearity of the collision operator and lemma 8. $\square$

4 Existence and uniqueness theorem

We will prove that the generalized relativistic Boltzmann equation that we write in the form:

$$\frac{\partial f}{\partial t} + P^i \frac{\partial f}{\partial p^i} + Q^I \frac{\partial f}{\partial q^I} = \frac{1}{p^0} \mathcal{L}(f, f) + \frac{a_0}{p^0} f^2, \quad (40)$$

has a unique solution in $\mathbb{E}_{d,\delta}^{m+3}(\Omega)$ weak star, where

$$P^i = - \left(2\Gamma_{i0}^i p^i + q \cdot F^{i0} + \frac{p^j q \cdot F_{ij}}{p^0} \right), \quad Q^I = - \frac{p^i}{p^0} C_{bc}^I A_i^b q^c \quad \text{and} \quad f(0, \bar{p}, \tilde{q}) = f_0 \in \mathbb{E}_{d,\delta}^{m+3}(\Omega).$$

Since we are studying a homogeneous phenomenon, we will suppose that the Yang-Mills potential A and the Yang-Mills field F are two given continuous and bounded functions of time. So there exists two positive constants C_A and C_F such as:

$$|A| = \max_{1 \leq i \leq 3, 1 \leq a \leq N} |A_i^a(t)| < C_A \quad \text{and} \quad |F| = \max_{1 \leq i, j \leq 3, 1 \leq a \leq N} (|F^{i0 \ a}(t)|, |F_{ij}^a(t)|) < C_F.$$

Lemma 19. *i) Let $t \in [0, T[$. The application $t \rightarrow \bar{p}(t)$ is uniformly bounded.*

2i) P^i and Q^I are bounded.

3i) $(1 + |\bar{p}|)^{|\beta|-1} \partial_{(\bar{p}, \tilde{q})}^\beta P^i$ is bounded, for all $\beta \in \mathbb{N}^{N+2}$ with $1 \leq |\beta| \leq m+3$,

4i) $(1 + |\bar{p}|)^{|\beta|-1} \partial_{(\bar{p}, \tilde{q})}^\beta Q^I$ is bounded, for all $\beta \in \mathbb{N}^{N+2}$ with $1 \leq |\beta| \leq m+3$.

Proof. For i), see [3].

2i) is immediate using i) and the fact that the Yang-Mills potential A and the Yang-Mills field F are with compact support.

3i) For all $\beta \in \mathbb{N}^{N+2}$ such that $1 \leq |\beta| \leq m+3$, we have:

$$\partial_{(\bar{p}, \tilde{q})}^\beta P^i = -2\Gamma_{i0}^i \partial_{(\bar{p}, \tilde{q})}^\beta (p^i) - \partial_{(\bar{p}, \tilde{q})}^\beta (q \cdot F^{i0}) - \partial_{(\bar{p}, \tilde{q})}^\beta \left(\frac{p^j q \cdot F_{ij}}{p^0} \right).$$

Knowing that the application $t \rightarrow \bar{p}(t)$ is uniformly bounded on $[0, T[$, immediatly

$$(1 + |\bar{p}|)^{|\beta|-1} \left| 2\Gamma_{i0}^i \partial_{(\bar{p}, \tilde{q})}^\beta (p^i) + \partial_{(\bar{p}, \tilde{q})}^\beta (q \cdot F^{i0}) \right| \leq C(T, e, C_F). \quad 1B$$

Furthermore, the Leibniz formula applied twice successively gives

$$\begin{aligned} \partial_{(\bar{p}, \tilde{q})}^\beta \left(\frac{p^j q \cdot F_{ij}}{p^0} \right) &= \sum_{k \leq \beta} C_{\beta-k}^k \partial_{(\bar{p}, \tilde{q})}^k \left(\frac{1}{p^0} \right) \sum_{\lambda \leq \beta-k} C_{\beta-k-\lambda}^\lambda \partial_{(\bar{p}, \tilde{q})}^\lambda p^j \partial_{(\bar{p}, \tilde{q})}^{\beta-k-\lambda} (q \cdot F_{ij}) \\ &= \sum_{k \leq \beta} C_{\beta-k}^k (1 + |\bar{p}|)^{|k|} \partial_{(\bar{p}, \tilde{q})}^k \left(\frac{1}{p^0} \right) \sum_{\lambda \leq \beta-k} C_{\beta-k-\lambda}^\lambda \partial_{(\bar{p}, \tilde{q})}^\lambda p^j \partial_{(\bar{p}, \tilde{q})}^{\beta-k-\lambda} (q \cdot F_{ij}) \times \frac{1}{(1 + |\bar{p}|)^{|k|}}. \end{aligned}$$

By lemma 7 and the fact that $\frac{1}{(1+|\bar{p}|)^{|k|}} < 1$, we get:

$$|(1 + |\bar{p}|)^{|\beta|-1} \partial_{(\bar{p}, \tilde{q})}^\beta (\tilde{P}^i)| \leq C(1 + |\bar{p}|)^{|\beta|-1} \sum_{k \leq \beta} C_{\beta-k}^k \frac{1}{p^0} \sum_{\lambda \leq \beta-k} C_{\beta-k-\lambda}^\lambda |\partial_{(\bar{p}, \tilde{q})}^\lambda p^j| |\partial_{(\bar{p}, \tilde{q})}^{\beta-k-\lambda} (q \cdot F_{ij})|,$$

and we conclude that:

$$\left| (1 + |\bar{p}|)^{|\beta|-1} \partial_{(\bar{p}, \tilde{q})}^\beta P^i \right| \leq C(T, e, C_F). \quad (2B)$$

(1B) and (2B) complete the proof of 3i). The proof for 4i) is similar to 3i) □

Let $\tilde{f} \in \mathbb{E}_{d,\delta}^{m+3}(\Omega)$. The linearized generalized relativistic Boltzmann equation in f is the equation

$$\frac{\partial \tilde{f}}{\partial t} + P^i \frac{\partial \tilde{f}}{\partial p^i} + Q^I \frac{\partial \tilde{f}}{\partial q^I} = \frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0(\tilde{f})^2, \quad (41)$$

where $f(0, \bar{p}, \tilde{q}) = f_0$, $P^i = - \left(2\Gamma_{i0}^i p^i + q \cdot F^{i0} + \frac{p^j q \cdot F_{ij}}{p^0} \right)$ and $Q^I = - \frac{p^i}{p^0} C_{bc}^I A_i^b q^c$.

We will prove that (41) has a unique solution in $\mathbb{E}_d^{m+3}(\Omega)$ weak star. To proceed, we use the Faedo-Galerkin scheme, which consists in finding the approximate solutions of the problem, estimating them uniformly and passing to the limit in a suitable weak sense, to get the expected solution [11].

Construction of the sequence

Let $\tilde{f}$ be given in $\mathbb{E}_d^{m+3}$. Let $(v_k)_{k \in \mathbb{N}^*}$ be a hilbertian basis of $\mathbb{E}_d^{m+3}$ (since it is a separable Hilbert space). We follow the method used in [4] for the construction of the sequence (f^M) which will converge to the solution f of (41). We then write

$$f^M = \sum_{k=1}^M \lambda_k v_k, \quad M \in \mathbb{N}^* \quad (42)$$

where coefficients λ_k are derivables functions of t and are given as solutions of the following system of first order linear differential equations:

$$(\partial_t f^M / v_j) + (P^i \partial_{p^i} f^M / v_j) + (Q^I \partial_{q^I} f^M / v_j) = \left(\frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0(\tilde{f})^2 / v_j \right), \quad (43)$$

with the initial data:

$$\lambda_j(0) = (f_0 / v_j), \quad (44)$$

where $(./.)$ stands for the scalar product in $\mathbb{E}_d^{m+3}$. Thus we obtain that:

$$\partial_t f^M + P^i \partial_{p^i} f^M + Q^I \partial_{q^I} f^M = \frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0(\tilde{f})^2. \quad (45)$$

So f^M is solution of (41), with the initial data:

$$f^M(0) = \sum_{k=1}^M \lambda_k(0) v_k = \sum_{k=1}^M (f_0 / v_k) v_k.$$

Proposition 20. Let $d \in]\frac{N+4}{2}, \infty[$, $f^M \in \mathbb{E}_d^{m+3}(\Omega)$, $\alpha, \beta \in \mathbb{N}^{N+2}$ such that $|\alpha| \leq |\beta| \leq m+3$. Then

$$\left| \left((1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta [P^i \partial_{p^i} f^M] / (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})} f^M \right) \right|_{L^2} \leq C \left(\sum_{|\alpha| \leq |\beta|} \left\| (1 + |\bar{p}|)^{d+|\alpha|} \partial_{(\bar{p}, \bar{q})}^\alpha f^M \right\|_{L^2} \right) \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2},$$

where : $C = C(T, e, C_F)$ is a positive constant and $(/)$ stands for the scalar product in $L^2(\Omega)$.

Proof. The proof of this proposition is given by lemmas 9, 10 and 11. □

Lemma 21. Let $d \in]\frac{N+4}{2}, \infty[$ and $f^M \in \mathbb{E}_d^{m+3}(\Omega)$, then :

$$\left| \left((1 + |\bar{p}|)^d \left[P^i \frac{\partial f^M}{\partial p^i} \right] / (1 + |\bar{p}|)^d f^M \right) \right|_{L^2(\Omega)} \leq C \left\| (1 + |\bar{p}|)^d f^M \right\|_{L^2}^2,$$

where $C = C(T, e, C_F)$ is a positive constant.

Proof. We have $\partial_{p^i} [(1 + |\bar{p}|)^d f^M] = \partial_{p^i} [(1 + |\bar{p}|)^d] f^M + (1 + |\bar{p}|)^d \partial_{p^i} f^M$, which implies

$$\left| \left((1 + |\bar{p}|)^d P^i \left[\frac{\partial f^M}{\partial p^i} \right] / (1 + |\bar{p}|)^d f^M \right) \right|_{L^2} \leq |B_1| + |B_2|, \quad (46)$$

where $\begin{cases} B_1 = (P^i \partial_{p^i} [(1 + |\bar{p}|)^d f^M] / (1 + |\bar{p}|)^d f^M)_{L^2} \\ B_2 = (P^i \partial_{p^i} [(1 + |\bar{p}|)^d] f^M / (1 + |\bar{p}|)^d f^M)_{L^2} \end{cases}$.

For the term B_1 , we also have :

$$\begin{aligned} B_1 &= (\partial_{p^i} [(1 + |\bar{p}|)^d f^M] / P^i (1 + |\bar{p}|)^d f^M)_{L^2} \\ &= -((1 + |\bar{p}|)^d f^M / \partial_{p^i} (P^i) (1 + |\bar{p}|)^d f^M + P^i \partial_{p^i} [(1 + |\bar{p}|)^d f^M])_{L^2}. \end{aligned}$$

Using the symmetry and bilinearity of the scalar product on the second member of the previous inequality, one has :

$$B_1 = -\frac{1}{2} (\partial_{p^i} P^i [(1 + |\bar{p}|)^d f^M] / (1 + |\bar{p}|)^d f^M)_{L^2}.$$

Accoding to lemma 9, $\partial_{p^i} (P^i)$ is bounded. Hence,

$$|B_1| \leq \frac{1}{2} |\partial_{p^i} P^i| \|(1 + |\bar{p}|)^d f^M\|_{L^2}^2 \leq C(T, e, C_F) \|(1 + |\bar{p}|)^d f^M\|_{L^2}^2. \quad (47)$$

For the last term B_2 , we have $\partial_{p^i} [(1 + |\bar{p}|)^d] = \frac{p^i d(1 + |\bar{p}|)^{d-1}}{|\bar{p}|(1 + |\bar{p}|)}$, which implies that:

$$|B_2| \leq \left| \frac{P^i p^i d(1 + |\bar{p}|)^{d-1}}{|\bar{p}|(1 + |\bar{p}|)} \right| \|(1 + |\bar{p}|)^d f^M\|_{L^2}^2 \leq C(T, e, C_F) \|(1 + |\bar{p}|)^d f^M\|_{L^2}^2 \quad (48)$$

(46), (47) and (48) complete the proof of lemma 10 □

Lemma 22. Let $d \in [\frac{N+4}{2}, \infty[$, $f^M \in \mathbb{E}_d^{m+3}(\Omega)$, $i_1 = 1, 2, 3$ and $a_1 = 1, \dots, N-1$. Then :

$$\begin{aligned} i) \quad & \left| \left((1 + |\bar{p}|)^{d+1} \partial_{p^{i_1}} \left[P^i \frac{\partial f^M}{\partial p^i} \right] / (1 + |\bar{p}|)^d \partial_{p^{i_1}} f^M \right)_{L^2} \right| \leq \\ & C(T, e, C_F) \left\{ \sum_{i=1}^3 \|(1 + |\bar{p}|)^{d+1} \partial_{p^i} f^M\|_{L^2(\Omega)} \right\} \times \|(1 + |\bar{p}|)^{d+1} \partial_{p^{i_1}} f^M\|_{L^2(\Omega)} ; \\ 2i) \quad & \left| \left((1 + |\bar{p}|)^{d+1} \partial_{q^{a_1}} \left[P^i \frac{\partial f^M}{\partial p^i} \right] / (1 + |\bar{p}|)^d \partial_{q^{a_1}} f^M \right)_{L^2} \right| \leq \\ & C(T, e, C_F) \left\{ \sum_{i=1}^3 \|(1 + |\bar{p}|)^{d+1} \partial_{p^i} f^M\|_{L^2(\Omega)} \right\} \times \|(1 + |\bar{p}|)^{d+1} \partial_{q^{a_1}} f^M\|_{L^2(\Omega)}. \end{aligned}$$

Proof. 1) We have $\partial_{p^{i_1}} [P^i \partial_{p^i} f^M] = \partial_{p^{i_1}} P^i \partial_{p^i} f^M + P^i \partial_{p^{i_1} p^i}^2 f^M$, which implies that

$$|((1 + |\bar{p}|)^{d+1} \partial_{p^{i_1}} [P^i \partial_{p^i} f^M] / (1 + |\bar{p}|)^d \partial_{p^{i_1}} f^M)_{L^2}| \leq |B_3| + |B_4|, \quad (49)$$

where $\begin{cases} B_3 = ((1 + |\bar{p}|)^{d+1} P^i \partial_{p^{i_1} p^i}^2 f^M / \partial_{p^{i_1}} f^M)_{L^2} \\ B_4 = ((1 + |\bar{p}|)^{d+1} \partial_{p^{i_1}} [P^i] \partial_{p^i} f^M / (1 + |\bar{p}|)^d \partial_{p^{i_1}} f^M)_{L^2} \end{cases}$.

For the term B_4 , since $\partial_{p^{i_1}} P^i$ is bounded (lemma 9), we get:

$$|B_4| \leq C(T, e, C_F) \sum_{i=1}^3 \|(1 + |\bar{p}|)^{d+1} \partial_{p^i} f^M\|_{L^2} \times \|(1 + |\bar{p}|)^{d+1} \partial_{p^{i_1}} f^M\|_{L^2}^2. \quad (50)$$

Now, for the term B_3 , we have

$$\partial_{p^i} [(1 + |\bar{p}|)^{d+1} \partial_{p^{i_1}} f^M] = (1 + |\bar{p}|)^{d+1} \partial_{p^{i_1} p^i}^2 f^M + \partial_{p^i} [(1 + |\bar{p}|)^{d+1}] \partial_{p^{i_1}} f^M.$$

Thus, $|B_3| \leq |B'_3| + |B''_3|$, with $B'_3 = (P^i \partial_{p^i} [(1 + |\bar{p}|)^{d+1} \partial_{p^{i_1}} f^M] / (1 + |\bar{p}|)^d \partial_{p^{i_1}} f^M)_{L^2}$ and $B''_3 = (P^i \partial_{p^i} [(1 + |\bar{p}|)^{d+1}] \partial_{p^{i_1}} f^M / (1 + |\bar{p}|)^d \partial_{p^{i_1}} f^M)_{L^2}$. On the one hand, we have

$$|B'_3| \leq C(T, e, C_F) \|(1 + |\bar{p}|)^{d+1} \partial_{p^{i_1}} f^M\|_{L^2}^2, \quad (51)$$

and on the other hand $\partial_{p^i}[(1 + |\bar{p}|)^{d+1}] = \frac{(d+1)p^i(1+|\bar{p}|)^{d+1}}{|\bar{p}|(1+|\bar{p}|)}$ which implies that:

$$\begin{aligned} |B_3''| &\leq \left| \frac{(d+1)P^i p^i (1+|\bar{p}|)^{d+1}}{|\bar{p}|(1+|\bar{p}|)} \right| \|(1+|\bar{p}|)^{d+1} \partial_{p^i} f^M\|_{L^2}^2 \\ &\leq C(T, e, C_F) \|(1+|\bar{p}|)^{d+1} \partial_{p^i} f^M\|_{L^2}^2. \end{aligned} \quad (52)$$

The inequality (50), (51) and (52) end the proof for (i).

2) For the case 2i) We have $\partial_{q^{a_1}} [P^i \partial_{p^i} f^M] = \partial_{q^{a_1}} P^i \partial_{p^i} f^M + P^i \partial_{q^{a_1} p^i}^2 f^M$, and then

$$\left| \left((1+|\bar{p}|)^{d+1} \partial_{q^{a_1}} \left[P^i \frac{\partial f^M}{\partial p^i} \right] / (1+|\bar{p}|)^d \partial_{q^{a_1}} f^M \right) \right|_{L^2} \leq |B_5| + |B_6|, \quad (53)$$

$$\text{where } \begin{cases} B_5 = \left((1+|\bar{p}|)^{d+1} \partial_{q^{a_1}} P^i \partial_{p^i} f^M / (1+|\bar{p}|)^d \partial_{q^{a_1}} f^M \right)_{L^2} \\ B_6 = \left((1+|\bar{p}|)^{d+1} P^i \partial_{q^{a_1} p^i}^2 f^M / (1+|\bar{p}|)^d \partial_{q^{a_1}} f^M \right)_{L^2} \end{cases}.$$

In the term B_5 , since $\partial_{q^{a_1}} P^i$ is boubed (lemme 9), we get:

$$|B_5| \leq C(T, e, C_F) \|(1+|\bar{p}|)^{d+1} \partial_{p^i} f^M\|_{L^2} \|(1+|\bar{p}|)^{d+1} \partial_{q^{a_1}} f^M\|_{L^2}. \quad (54)$$

For the term B_6 , we have:

$$\partial_{p^i} [(1+|\bar{p}|)^{d+1} \partial_{q^{a_1}} f^M] = (1+|\bar{p}|)^{d+1} \partial_{p^i q^{a_1}}^2 f^M + \partial_{p^i} (1+|\bar{p}|)^{d+1} \partial_{q^{a_1}} f^M.$$

Thus $|B_6| \leq |B_6'| + |B_6''|$, with $B_6' = (P^i \partial_{p^i} \{ (1+|\bar{p}|)^{d+1} \partial_{q^{a_1}} f^M \} / (1+|\bar{p}|)^{d+1} \partial_{q^{a_1}} f^M)_{L^2}$ and $B_6'' = (P^i \partial_{p^i} (1+|\bar{p}|)^{d+1} \partial_{q^{a_1}} f^M / (1+|\bar{p}|)^{d+1} \partial_{q^{a_1}} f^M)_{L^2}$.

On the one hand, we have

$$|B_6'| \leq C(T, e, C_F) \|(1+|\bar{p}|)^{d+1} \partial_{q^{a_1}} f^M\|_{L^2}^2, \quad (55)$$

and the other hand,

$$\begin{aligned} |B_6''| &\leq \left| \frac{(d+1)p^i(1+|\bar{p}|)^{d+1}}{|\bar{p}|(1+|\bar{p}|)} \right| \|(1+|\bar{p}|)^{d+1} \partial_{q^{a_1}} f^M\|_{L^2} \\ &\leq C(T, e, C_F) \|(1+|\bar{p}|)^{d+1} \partial_{q^{a_1}} f^M\|_{L^2}. \end{aligned} \quad (56)$$

The inequality (54), (55) and (56) end the proof for 2i) □

Lemma 23. Let $d \in]\frac{N+4}{2}, \infty[$, $f^M \in \mathbb{E}_d^{m+3}(\Omega)$ and $\beta \in \mathbb{N}^{N+2}$, $|\beta| \leq m+2$ if

$$\begin{aligned} &\left((1+|\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta \left[P^i \frac{\partial f^M}{\partial p^i} \right] / (1+|\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right)_{L^2} \leq \\ &C(T, e, C_F) \left(\sum_{|\alpha| \leq |\beta|} \left\| (1+|\bar{p}|)^{d+|\alpha|} \partial_{(\bar{p}, \bar{q})}^\alpha f^M \right\|_{L^2} \right) \left\| (1+|\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2} \end{aligned}$$

then, $\forall \beta' \in \mathbb{N}^{N+3}$, $|\beta'| = 1$, we have:

$$\begin{aligned} &\left((1+|\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} \left[P^i \frac{\partial f^M}{\partial p^i} \right] / (1+|\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right)_{L^2} \leq \\ &C(T, e, C_F) \left(\sum_{|\alpha| \leq |\beta+\beta'|} \left\| (1+|\bar{p}|)^{d+|\alpha|} \partial_{(\bar{p}, \bar{q})}^\alpha f^M \right\|_{L^2} \right) \left\| (1+|\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right\|_{L^2} \end{aligned}$$

Proof. We set $\lambda = \beta + \beta'$. According to the Leibniz formula, we have:

$$\partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} \left[P^i \frac{\partial f^M}{\partial p^i} \right] = \sum_{k \leq \lambda} C_\lambda^k \partial_{(\bar{p}, \bar{q})}^k P^i \partial_{(\bar{p}, \bar{q})}^{\lambda-k} \left(\frac{\partial f^M}{\partial p^i} \right),$$

which implies

$$\left| \left((1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} \left[P^i \frac{\partial f^M}{\partial p^i} \right] / (1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right) \right|_{L^2} \leq |K_1| + |K_2|, \quad (57)$$

$$\text{where } \begin{cases} K_1 = \left((1 + |\bar{p}|)^{d+|\beta|+1} P^i \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} \left(\frac{\partial f^M}{\partial p^i} \right) / (1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right)_{L^2} \\ K_2 = \left((1 + |\bar{p}|)^{d+|\beta|+1} \sum_{k \leq \lambda, |k| \geq 1} C_\lambda^k \partial_{(\bar{p}, \bar{q})}^k P^i \partial_{(\bar{p}, \bar{q})}^{\lambda-k} \left(\frac{\partial f^M}{\partial p^i} \right) / (1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right)_{L^2} \end{cases} \quad \text{Estimation of } K_2:$$

According to lemma 9, $\partial_{(\bar{p}, \bar{q})}^k \tilde{P}^i$ is bounded, thus,

$$\begin{aligned} |K_2| &\leq C \sum_{k \leq \lambda, |k| \geq 1} C_\lambda^k \left| \left((1 + |\bar{p}|)^{d+|\beta|-|k|+2} \partial_{(\bar{p}, \bar{q})}^{\lambda-k} \left(\frac{\partial f^M}{\partial p^i} \right) / (1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right) \right|_{L^2} \\ &= C \sum_{k \leq \lambda, |k| \geq 1} C_\lambda^k \left| \left((1 + |\bar{p}|)^{d+|\beta|-|k|+2} \partial_{(\bar{p}, \bar{q})}^{\lambda+\gamma-k} (f^M) / (1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right) \right|_{L^2}, \end{aligned}$$

with $\gamma \in \mathbb{N}^3$ such that $|\gamma| = 1$. Hence,

$$|K_2| \leq C \sum_{k \leq \lambda, |k| \geq 1} \left\| (1 + |\bar{p}|)^{d+|\beta|-|k|+2} \partial_{(\bar{p}, \bar{q})}^{\lambda+\gamma-k} f^{MN} \right\|_{L^2} \left\| (1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right\|_{L^2}. \quad (58)$$

Estimation of K_1 given by (57). We have:

$$\partial_{p^i} \left[(1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right] = \partial_{p^i} \left[(1 + |\bar{p}|)^{d+|\beta|+1} \right] \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M + (1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} \left(\frac{\partial f^M}{\partial p^i} \right),$$

which implies that

$$|K_1| \leq |K'_1| + |K''_1|, \quad (59)$$

$$\text{where } \begin{cases} K'_1 = \left(P^i \partial_{p^i} \left[(1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right] / (1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right)_{L^2} \\ K''_1 = \left(P^i \partial_{p^i} \left[(1 + |\bar{p}|)^{d+|\beta|+1} \right] \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M / (1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right)_{L^2} \end{cases}.$$

Since $\tilde{P}^i$ is bounded, one has:

$$K'_1 \leq C(T, e, C_F) \left\| (1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right\|_{L^2}^2. \quad (60)$$

Furhtermore, $\partial_{p^i} \left[(1 + |\bar{p}|)^{d+|\beta|+1} \right] = \frac{(d+|\beta|+1) p^i (1+|\bar{p}|)^{d+|\beta|+1}}{|\bar{p}|(1+|\bar{p}|)}$, and then,

$$K''_1 \leq C(T, e, C_F) \left\| (1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right\|_{L^2}^2. \quad (61)$$

Combining (59), (60) and (61), we get:

$$K_1 \leq C(T, e, C_F) \left\| (1 + |\bar{p}|)^{d+|\beta|+1} \partial_{(\bar{p}, \bar{q})}^{\beta+\beta'} f^M \right\|_{L^2}^2. \quad (62)$$

The inequalities (57), (58) and (62) complete the proof of lemma 12. $\square$

Proposition 24. Let $d \in]\frac{N+4}{2}, \infty[$, $f^M \in \mathbb{E}_d^{m+3}(\Omega)$, $\alpha, \beta \in \mathbb{N}^{N+2}$ such that $|\alpha| \leq |\beta| \leq m+3$. Then

$$\begin{aligned} &\left| \left((1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta [Q^I \partial_{q^I} f^M] / (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right) \right|_{L^2} \leq \\ &C \left(\sum_{|\alpha| \leq |\beta|} \left\| (1 + |\bar{p}|)^{d+|\alpha|} \partial_{(\bar{p}, \bar{q})}^\alpha f^M \right\|_{L^2} \right) \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2}, \end{aligned}$$

where: $C = C(T, e, C_F)$ is a positive constant and $(/)$ stands for the scalar product on $L^2(\Omega)$.

Proof. The proof of this proposition is similar to that of proposition 4. $\square$

Proposition 25. Let $d \in]\frac{N+4}{2}, \infty[$, $f^M \in \mathbb{E}_d^{m+3}(\Omega)$, $\tilde{f} \in \mathbb{E}_{d,\delta}^{m+3}(\Omega)$, $\beta \in \mathbb{N}^{N+2}$, $|\beta| \leq m+3$ and $T > 0$. We have :

$$\left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right\|_{L^2} \leq C,$$

where $C = C(T, e, C_F)$ is a positive constant.

Proof. Since $\tilde{f} \in \mathbb{E}_{d,\delta}^{m+3}(0, T, \Omega)$, $f^M \in \mathbb{E}_d^{m+3}(\Omega)$ and using theorem 3, we have

$\frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0 (\tilde{f})^2 \in \mathbb{E}_d^{m+3}(\Omega)$ and $(1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \in L^2(\Omega)$, for $\beta \in \mathbb{N}^{m+3}$ such that $|\beta| \leq m+3$. From (45), using the bilinearity of the scalar product, we have:

$$\begin{aligned} & \left((1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta [\partial_t f^M + P^i \partial_{p^i} f^M + Q^I \partial_{q^I} f^M] / (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right)_{L^2} = \\ & \left((1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta \left[\frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0 (\tilde{f})^2 \right] / (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right)_{L^2}. \end{aligned} \quad (63)$$

Given that $(1 + |\bar{p}|)^{d+|\beta|}$ does not depend on t , we have:

$$(1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta [\partial_t f^M] = \partial_t (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta [f^M].$$

It comes that:

$$\begin{aligned} & \left((1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta [\partial_t f^M] / (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right)_{L^2} = \\ & \left(\partial_t (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M / (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right)_{L^2} = \frac{1}{2} \frac{d}{dt} \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right\|_{L^2}^2. \end{aligned}$$

From (63), we then have:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta [f^M] \right\|_{L^2}^2 \\ & + \left((1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta [P^i \partial_{p^i} f^M] / (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right)_{L^2} \\ & + \left((1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta [Q^I \partial_{q^I} f^M] / (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right)_{L^2} \\ & = \left((1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta \left[\frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0 (\tilde{f})^2 \right] / (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right)_{L^2}. \end{aligned} \quad (64)$$

Furthermore, according to the Cauchy-Schwartz inequality, we have:

$$\begin{aligned} & \left| \left((1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta \left[\frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0 (\tilde{f})^2 \right] / (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right)_{L^2} \right| \\ & \leq \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta \left[\frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0 (\tilde{f})^2 \right] \right\|_{L^2} \times \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right\|_{L^2} \\ & \leq \left\| \frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0 (\tilde{f})^2 \right\|_{E_d^{m+3}} \times \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right\|_{L^2}. \end{aligned}$$

So, from (64) we have:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta [f^M] \right\|_{L^2}^2 \leq \\ & - \left((1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta [P^i \partial_{p^i} f^M] / (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right)_{L^2} \\ & - \left((1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta [Q^I \partial_{q^I} f^M] / (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right)_{L^2} + \\ & \left\| \frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0 (\tilde{f})^2 \right\|_{E_d^{m+3}} \times \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p},\bar{q})}^\beta f^M \right\|_{L^2}. \end{aligned} \quad (65)$$

Introducing propositions 4 and 5 in (65), we get:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\| (1 + |\bar{p}|)^{d+\beta} \partial_{(\bar{p}, \bar{q})}^\beta [f^M] \right\|_{L^2}^2 \\ & \leq C(T, e, C_F) \left(\sum_{|\alpha| \leq |\beta|} \left\| (1 + |\bar{p}|)^{d+|\alpha|} \partial_{(\bar{p}, \bar{q})}^\alpha f^M \right\|_{L^2} \right) \left\| (1 + |\bar{p}|)^{d+\beta} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2} \\ & \quad + \left\| \frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0 (\tilde{f})^2 \right\|_{\mathbb{E}_d^{m+3}} \times \left\| (1 + |\bar{p}|)^{d+\beta} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2}. \end{aligned} \quad (66)$$

Knowing that:

$$\frac{1}{2} \frac{d}{dt} \left\| (1 + |\bar{p}|)^{d+\beta} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2}^2 = \left\| (1 + |\bar{p}|)^{d+\beta} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2} \frac{d}{dt} \left\| (1 + |\bar{p}|)^{d+\beta} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2},$$

dividing (66) by $\left\| (1 + |\bar{p}|)^{d+\beta} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2}$, we have :

$$\begin{aligned} & \frac{d}{dt} \left\| (1 + |\bar{p}|)^{d+\beta} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2} \\ & \leq C(T, e, C_F) \left(\sum_{|\alpha| \leq |\beta|} \left\| (1 + |\bar{p}|)^{d+|\alpha|} \partial_{(\bar{p}, \bar{q})}^\alpha f^M \right\|_{L^2} \right) + \left\| \frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0 (\tilde{f})^2 \right\|_{\mathbb{E}_d^{m+3}}. \end{aligned} \quad (67)$$

Tanking the sum of (67) over $|\beta| = 0, 1, 2, \dots, m+3$, we get:

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{|\beta| \leq m+3} \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2} \right) \leq \\ & C(T, e, C_F) \left(\sum_{|\beta| \leq m+3} \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\alpha f^M \right\|_{L^2} \right) + C \left\| \frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0 (\tilde{f})^2 \right\|_{\mathbb{E}_d^{m+3}}. \end{aligned} \quad (68)$$

Integrating (68) on $[0, t]$ for $t \in [0, T[$ and applying Gronwall lemma, we obtain:

$$\begin{aligned} & \sum_{|\beta| \leq m+3} \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2} \leq \\ & e^{CT} \left[\sum_{|\beta| \leq m+3} \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta f^M(0) \right\|_{L^2} + C \int_0^t \left\| \frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) \right\|_{\mathbb{E}_d^{m+3}} ds \right]. \end{aligned} \quad (69)$$

Using theorem 3, one has:

$$\sum_{|\beta| \leq m+3} \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2} \leq e^{CT} \left[\|f_0\|_{\mathbb{E}_d^{m+3}(\Omega)} + CT \left\| \tilde{f} \right\|_{\mathbb{E}_d^{m+3}(\Omega)}^2 \right]. \quad (70)$$

Since $\tilde{f}, f_0 \in \mathbb{E}_{d,\delta}^{m+3}$, we conclude that $\left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2} \leq C(T, e, C_F)$. $\square$

Theorem 26. Let $f_0 \in E_{d,\delta}^{m+3}$, $\tilde{f} \in E_{d,\delta}^{m+3}$ be given. Then the linearized generalized relativistic Boltzmann equation (41) has a local unique and bounded weak star solution in $\mathbb{E}_d^{m+3}(\Omega)$.

Proof. The proof of this theorem will be done into two steps.

Existence: According to proposition 6, the sequence (f^M) is bounded in $\mathbb{E}_d^{m+3}$ which is a Hilbert space. So (f^M) admits a subsequence (f^{M_k}) which converges weakly to $f \in \mathbb{E}_d^{m+3}$, the solution of the linearized generalized relativistic Boltzmann equation (41). From (43), we have:

$$(\partial_t f^{M_k} / v_k) + (P^i \partial_{p^i} f^{M_k} / v_k) + (Q^I \partial_{q^I} f^{M_k} / v_k) = \left(\frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0 (\tilde{f})^2 / v_k \right).$$

At the limit when $k \rightarrow +\infty$, we obtain $\partial_t f + P^i \partial_{p^i} f + Q^I \partial_{q^I} f = \frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0(\tilde{f})^2$. Hence f is a solution of the linearized generalized relativistic Boltzmann equation (41) satisfying the condition $f(0) = f_0$.

Uniqueness: We assume that, there is another solution f' of (41) with the same initial data f_0 . By setting $g = f - f'$, g is the solution of the following cauchy problem

$$\begin{cases} \frac{\partial g}{\partial t} + P^i \frac{\partial g}{\partial p^i} + Q^I \frac{\partial g}{\partial q^I} = 0 \\ g(0) = 0 \end{cases} \quad (71)$$

Next, proceeding as in the proof of the proposition 6, we show that (71) admits $g = 0$ as a unique solution. Thus $f = f'$ $\square$

Theorem 27. *Let $f_0 \in E_{d,\delta}^{m+3}$ be given. Then the generalized relativistic Boltzmann equation (40) has in $E_{d,\delta}^{m+3}$ a local unique * weak solution f such that $f(0, \bar{p}, \bar{q}) = f_0(\bar{p}, \bar{q})$.*

Proof. We consider the application

$$\phi : \mathbb{E}_d^{m+3} \rightarrow \mathbb{E}_d^{m+3}; \tilde{f} \mapsto \phi(\tilde{f}) = f,$$

where f is the unique solution of (40). The proof of this theorem will be divided into two steps.

1) Firstly, let us show that one can choose $f_0 \in \mathbb{E}_{d,\delta}^{m+3}$ and $T > 0$ such that ϕ sends $\mathbb{E}_{d,\delta}^{m+3}$ in itself.

In fact, this is to show that one can choose $f_0 \in \mathbb{E}_{d,\delta}^{m+3}$ and $T > 0$ such that

$$\|\tilde{f}\|_{\mathbb{E}_d^{m+3}} \leq \delta \quad \Rightarrow \quad \|f\|_{\mathbb{E}_d^{m+3}} \leq \delta.$$

Introducing (70) in (68), one has:

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{|\beta| \leq m+3} \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2} \right) \leq \\ & C(T, e, C_F) \left(\|f_0\|_{\mathbb{E}_d^{m+3}(\Omega)} + CT \left\| \tilde{f} \right\|_{\mathbb{E}_d^{m+3}(\Omega)}^2 + \left\| \frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0(\tilde{f})^2 \right\|_{\mathbb{E}_d^{m+3}} \right). \end{aligned}$$

Integrating this inequality on $[0, t]$, $0 \leq t < T$, we obtain using $\left\| \frac{1}{p^0} \mathcal{L}(\tilde{f}, \tilde{f}) + a_0(\tilde{f})^2 \right\|_{\mathbb{E}_d^{m+3}} \leq \delta^2$, that:

$$\begin{aligned} \sum_{|\beta| \leq m+3} \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta f^M \right\|_{L^2} & \leq \sum_{|\beta| \leq m+3} \left\| (1 + |\bar{p}|)^{d+|\beta|} \partial_{(\bar{p}, \bar{q})}^\beta f^M(0) \right\|_{L^2} + \\ & C(T, e, C_F) \left(\|f_0\|_{\mathbb{E}_d^{m+3}(\Omega)} T + CT^2 \delta^2 + \delta^2 T \right). \end{aligned}$$

But, we have :

$$\|f^M\|_{\mathbb{E}_d^{m+3}} \leq \|f_0\|_{\mathbb{E}_d^{m+3}} + C(T, e, C_F) \left(\|f_0\|_{\mathbb{E}_d^{m+3}(\Omega)} T + CT^2 \delta^2 + \delta^2 T \right) \quad (72)$$

Since $\delta > 0$ is given, if we take $f_0 \in \mathbb{E}_{d,\delta}^{m+3}(\Omega)$ and $T > 0$ such that:

$$\|f_0\|_{\mathbb{E}_d^{m+3}(\Omega)} \leq \frac{\delta}{2} \quad \text{and} \quad C(T, e, C_F) \left(\|f_0\|_{\mathbb{E}_d^{m+3}(\Omega)} T + CT^2 \delta^2 + \delta^2 T \right) \leq \frac{\delta}{2}$$

(72) implies $\|f^M\|_{\mathbb{E}_d^{m+3}(\Omega)} \leq \delta$. Since the sequence (f^M) converges weak star to f in $\mathbb{E}_d^{m+3}(\Omega)$, we have $\|f\|_{\mathbb{E}_d^{m+3}(\Omega)} \leq \delta$. This shows that $\left\| \tilde{f} \right\|_{\mathbb{E}_d^{m+3}} \leq \delta \Rightarrow \|f\|_{\mathbb{E}_d^{m+3}} \leq \delta$.

2) Let us show that ϕ is a contraction in the space $\mathbb{E}_{d,\delta}^{m+3}$.

Suppose that $\phi(\tilde{f}_j) = f_j$, $j = 1, 2$, where

$$\tilde{f}_j(0) = f_0 \quad \text{and} \quad \left\| \tilde{f}_j \right\|_{\mathbb{E}_d^{m+3}} \leq \delta, \quad j = 1, 2$$

and show that

$$\|f_1 - f_2\|_{\mathbb{E}_d^{m+3}} \leq TC(\tilde{F}, T) \left\| \tilde{f}_1 - \tilde{f}_2 \right\|_{\mathbb{E}_d^{m+3}}.$$

Setting $G = f_1 - f_2$ and $\tilde{G} = \tilde{f}_1 - \tilde{f}_2$ then, $G \in \mathbb{E}_d^{m+3}$ is a solution of the following Cauchy problem

$$\begin{cases} \frac{\partial G}{\partial t} + P^i \partial_{p^i} G + Q^I \partial_{q^I} G = \frac{1}{p^0} \mathcal{L}(\tilde{f}_1, \tilde{G}) + \frac{1}{p^0} \mathcal{L}(\tilde{G}, \tilde{f}_2) + a_0 \tilde{G} (\tilde{f}_1 + \tilde{f}_2) \\ G(0) = 0 \end{cases} \quad (73)$$

(68) gives

$$\begin{aligned} & \frac{d}{dt} \left\| (1 + |\bar{p}|)^{d+|\beta|} D_{(\bar{p}, \bar{q})}^\beta G(t) \right\|_{L^2} \leq \\ & C \left\| (1 + |\bar{p}|)^{d+|\beta|} D_{(\bar{p}, \bar{q})}^\beta G(t) \right\|_{L^2} + \left\| (1 + |\bar{p}|)^{d+|\beta|} D_{(\bar{p}, \bar{q})}^\beta \left(\frac{1}{p^0} \mathcal{L}(\tilde{f}_1, \tilde{G}) + \frac{1}{p^0} \mathcal{L}(\tilde{G}, \tilde{f}_1) \right) + a_0 \tilde{G} (\tilde{f}_1 + \tilde{f}_2) \right\|_{L^2}. \end{aligned}$$

Integrating on $[0, t]$ for $t \in [0, T[$ and applying Gronwall lemma, knowing that $G(0) = 0$, we obtain :

$$\begin{aligned} & \left\| (1 + |\bar{p}|)^{d+|\beta|} D_{(\bar{p}, \bar{q})}^\beta G(t) \right\|_{L^2} \leq \\ & C(T, e, C_F) \int_0^t \left\| (1 + |\bar{p}|)^{d+|\beta|} D_{(\bar{p}, \bar{q})}^\beta \left(\frac{1}{p^0} \mathcal{L}(\tilde{f}_1, \tilde{G}) + \frac{1}{p^0} \mathcal{L}(\tilde{G}, \tilde{f}_1) \right) + a_0 \tilde{G} (\tilde{f}_1 + \tilde{f}_2) \right\|_{L^2} d\tau. \end{aligned} \quad (74)$$

Then, taking the supremum in (74), for t , we get :

$$\|G(t)\|_{\mathbb{E}_d^{m+3}} \leq C(T, e, C_F) T \|\tilde{G}(t)\|_{\mathbb{E}_d^{m+3}}. \quad (75)$$

According to (75), ϕ is a contracting map if $T > 0$ is chosen such that $T \times C(T, e, C_F) < 1$. Thus, ϕ has a unique fixed point f solution of the generalized relativistic Boltzmann equation (40). $\square$

Conclusion. We have studied in this work the regular solution of the generalized relativistic Boltzmann equation in the presence of a given Yang-Mills field; taking as background the Bianchi type 1 space-time. We have obtained a local in time solution. We know that, there are results for the Yang-Mills-Vlasov system, and also results for the global in time regular solution for the Boltzmann equation in the abelian case. Interesting open questions are on one hand the global in time regular solution for the Boltzmann equation in the non-abelian case and on the other hand the Boltzmann-Yang-Mills system.

References

- [1] B. V. Alexeev. The generalized Boltzmann equation, generalized hydrodynamic equations and their applications, Phil. Trans. R. Soc. Lond. A (1994), 417-443. <https://doi.org/10.1098/rsta.1994.0140>
- [2] L. Arlotti, N. Bellomo and E. De Angeli. Generalized kinetic (Boltzmann) models: Mathematical structures and applications, Math. Models Methods Appl. (2001), 567-591. <https://doi.org/10.1142/s0218202502001799>
- [3] R. D. Ayissi, N. Noutchegueme, R. M. Etoua, and H. P. Mbeutchagna. Viscosity solutions for the one-body Liouville equation in Yang-Mills charged Bianchi Models with Non-Zero mass, Lett. Math. Phys., 105 (2015), 1289-1299. <https://doi.org/10.1007/s11005-015-0777-7>
- [4] R. D. Ayissi, N. Noutchegueme. The Faedo-Galerkin method for the relativistic Boltzmann equation in Bianchi type 1 space-time, Comm. Math. Appl., 4(2) (2013), 93-118.

- [5] D. Bancel and Y. Choquet-Bruhat. Existence, uniqueness and local stability for the Einstein-Maxwell-Boltzmann system, *Commun. Math. Phys.*, 33 (1973), 83-96.
<https://doi.org/10.1007/bf01645621>
- [6] C. Cercignani. *Rarefied gas dynamics, From basic concepts to actual calculations*, Cambridge University Press, Cambridge 2000.
- [7] Y. Choquet-Bruhat, N. Noutchegueme. Systeme de Yang-Mills-Vlasov en jauge temporelle, *Ann. Inst. Henri Poincaré*, 55(3) (1991), 759-787.
- [8] Y. Choquet-Bruhat, N. Noutchegueme. Solution globale des equations de Yang-Mills-Vlasov (masse nulle), *C. R. Acad. Sc. Paris*, 311 (1990), no. 1, 785-788.
- [9] Y. Choquet-Bruhat, D. Christodoulou. Elliptic systems in $H_{s,\delta}$ spaces on manifolds which are euclidean at infinity, *Acta Mathematica*, 146, 129-150 (1981). <https://doi.org/10.1007/bf02392460>
- [10] A. Lichnerowicz. *Theorie relativiste de la Gravitation et de l'Électromagnétisme*, Masson et cie (1955).
- [11] J. L. Lions, E. Magenes. *Problèmes aux limites non homogènes et applications*, Vol. 1, Paris, Dunod, 1968.
- [12] P. B. Mucha. Global existence of solutions of the Einstein-Boltzmann equation in the flat Robertson-Walker space-time, *Comm. Math. Phys.*, 203 (1999), 107-118. <https://doi.org/10.1007/s002200050607>
- [13] N. Noutchegueme, D. Dongo and F. E. Djiofack. Global regular solution for the Einstein-Maxwell-Boltzmann-Scalar field system in Bianchi type I space-time, *J. A. M.*, 13 (2017), 7087-7118 .
<https://doi.org/10.24297/jam.v13i1.5982>
- [14] N. Noutchegueme, E. Takou and E. K. Tchuengue. The relativistic Boltzmann equation on Bianchi type I space-time for hard potentials, *reports on Mathematical Physics*, 80(2017), 87-114. [https://doi.org/10.1016/s0034-4877\(17\)30063-0](https://doi.org/10.1016/s0034-4877(17)30063-0)
- [15] N. Noutchegueme, M. Kenmogne. Regular solutions to the Boltzmann equation on a Robertson-Walker space-time, *J. P. M. S.*, 5(4)(2015), 66-104.
- [16] N. Noutchegueme, R. Ayissi. Bianchi type 1 magnetized cosmological models for the Einstein-Boltzmann equation with the cosmological constant, *J. Math. Phys.*, 56 (2015), 012501 (35pp).
<https://doi.org/10.1063/1.4905648>
- [17] N. Noutchegueme, D. Dongo. Global existence of solutions for the Einstein-Boltzmann system in Bianchi type 1 space-time, *Class. Quantum Grav.*, 23 (2006), 2979-3003. <https://doi.org/10.1088/0264-9381/23/9/013>
- [18] N. Noutchegueme, D. Dongo and E. Takou. Global existence of solutions for the relativistic Boltzmann equation with arbitrarily large initial data on a Bianchi type 1 space-time, *Gen. Relat. Grav.*, 37(12) (2005), 2047-2062.
<https://doi.org/10.1007/s10714-005-0179-8>
- [19] N. Noutchegueme, P. Noundjeu. Systeme de Yang-Mills-Vlasov pour des particules avec densité de charge de jauge non-abélienne sur un espace-temps courbe, *Ann. Inst. Henri Poincaré*, 1 (2000), 385-404.
<https://doi.org/10.1007/s000230050008>
- [20] E. Takou and F. L. Ciake. The relativistic Boltzmann equation on a spherically symmetric gravitational field, *Class. Quantum Grav.*, 34 (2017), 1-23. <https://doi.org/10.1088/1361-6382/aa85d1>