Application Of Aboodh Transform For Solving First Order Constant Coefficients Complex Equation

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Abstract. In this work , we investigate the Aboodh transformation method to solve first order constant coefficients complex equations. This method provides an effective and efficient way of solving a wide range of linear operator equations.

Keywords: *Aboodh Transform, complex equation, Partial derivatives.*

2010 MSC No: 45E05, 30G20, 30E20.

1 Introduction

Many Complex differential equations can be transformed into a system of partial differential equations that has two system of unknowns, two independent variants. separating real and imaginary parties[1]. Aboodh transformation method that is used in several areas of mathematics is an integral transform. We can solve linear differential equations, integral equations, integro-differential equations with the Aboodh transform [2, 3, 4]. This method can not be used for solving nonlinear differential equations because of nonlinear terms . But nonlinear differential equations can be solved using the Aboodh transformation aid with differential transform method. In this work, we study the solutions of first order constant coefficients complex equations. These equations were solved by Laplace transformation [1]. The organization of this paper is as follows. In Section 2, definitions and basic theorem are given. In Section 3, contains numerical illustrations.

2 Basic definitions and theorems

The Aboodh transform is a new integral transform similar to the Laplace transform and other integral transforms that are defined in the time domain $t \geq 0$, such as the Sumudu transform, the Natural transform and the Elzaki transform, respectively.

Aboodh transform is defined for functions of exponential order. We consider functions in the set F defined by;

$$F = \{f(t) : |f(t)| < Me^{-vt}, \text{ if } t \in [0; \infty[, M, k_1, k_2 > 0; k_1 \leq v \leq k_2\} \quad (1)$$

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For a given function in the set F , the constant M must be finite number and k_1, k_2 may be finite or infinite with variable v define as $k_1 \leq v \leq k_2$. Then, the Aboodh integral transform denoted by the operator $A(\cdot)$ is defined by the integral equation:

$$A[f(t)] = \frac{1}{v} \int_0^{\infty} f(t) e^{-vt} dt, \quad t \geq 0, \quad k_1 \leq v \leq k_2. \quad (2)$$

Standard Aboodh transform for some special functions found are given below in Table 1.

$f(x)$	$A[f(x)]$
1	$\frac{1}{v^2}$
x	$\frac{1}{v^3}$
$x^n, n \geq 1$	$\frac{n!}{v^{n+2}}$
e^{at}	$\frac{1}{v^2 - av}$
$\sin(ax)$	$\frac{1}{v(v^2 + a^2)}$
$\cos(ax)$	$\frac{1}{v^2 + a^2}$
$\sinh(ax)$	$\frac{1}{v(v^2 - a^2)}$
$x \cosh(ax)$	$\frac{1}{v^2 - a^2}$

TABLE1: Aboodh transform of some functions.

Aboodh transform of some partial derivatives in the form :

$$A[u(x; t)] = K(x; v)$$

$$(i) - A\left[\frac{\partial u(x; t)}{\partial t}\right] = vK(x; v) - \frac{u(x; 0)}{v}.$$

$$(ii) - A\left[\frac{\partial^2 u(x; t)}{\partial t^2}\right] = v^2 K(x; v) - \frac{1}{v} \frac{\partial u(x; v)}{\partial t} - u(x; 0).$$

$$(iii) - A\left[\frac{\partial u(x; t)}{\partial x}\right] = K'(x; v) = \frac{dK(x; v)}{dx}.$$

(iv) - $A\left[\frac{\partial^2 u(x; t)}{\partial x^2}\right] = K''(x; v) = \frac{d^2 K(x; v)}{dx^2}$. The Aboodh transform is linear, i.e., if α and β are any constants and $f(t)$ and $g(t)$ are functions defined over the set F above, then

$$A[\alpha f(x)] + A[\beta g(x)] = \alpha A[f(x)] + \beta A[g(x)].$$

2.1 Complex derivatives

Let $w = w(z, \bar{z})$ be a complex function. Here $z = x + iy$, and

$w(z, \bar{z}) = p(x, y) + i.q(x, y)$. First order derivatives according to z and $\bar{z}$ of $w(z, \bar{z})$ are defined as following:

$$\frac{\partial w}{\partial z} = \frac{1}{2} \left(\frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right). \quad (3)$$

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right). \quad (4)$$

3 Solution of complex differential equations from first order which is constant coefficients

Theorem 3.1. Let a, b, c are real constants, $F(z; \bar{z})$ is a polynomial of $z; \bar{z}$ and $w = p + iq$ is a complex function. Then the real and imaginary parts of solution of

$$a \frac{\partial w}{\partial z} + b \frac{\partial w}{\partial \bar{z}} + cw = F(z; \bar{z}) \quad (5)$$

$$w(x, 0) = 0$$

are

$$\begin{aligned} p &= A^{-1} \left[\frac{(a+b) \frac{\partial}{\partial x} (2A_3 + \frac{(a-b)}{v} q(x, 0))}{\Delta} \right. \\ &\quad \left. + \frac{2c \left((2A_3 + \frac{(a-b)}{v} q(x, 0)) \right) - (a-b)v \left(2A_4 + \frac{(a-b)}{v} p(x, 0) \right)}{\Delta} \right], \\ q &= A^{-1} \left[\frac{(a+b) \frac{\partial}{\partial x} (2A_4 + \frac{(a-b)}{v} p(x, 0))}{\Delta} \right. \\ &\quad \left. + \frac{2c \left((2A_4 + \frac{(a-b)}{v} p(x, 0)) \right) - (a-b)v \left(2A_3 + \frac{(a-b)}{v} q(x, 0) \right)}{\Delta} \right]. \end{aligned}$$

Where

$$\Delta = \begin{vmatrix} (a+b)D + 2c & v(a-b) \\ v(b-a) & (a+b)D + 2c \end{vmatrix} = ((a+b)D + 2c)^2 + (v(b-a))^2.$$

and A_1, A_2, A_3, A_4 are Aboodh transforms of p, q, F_1, F_2 respectively.

Proof. We use equalities (3), (4) in equality (5), we have

$$\frac{a}{2} \left(\frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right) + \frac{b}{2} \left(\frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right) + cw = F_1(x, y) + iF_2(x, y) \quad (6)$$

If we chose $w = p + iq$ in (6), following equality is obtained

$$\begin{aligned} a \left(\frac{\partial p}{\partial x} + i \frac{\partial p}{\partial x} - i \frac{\partial p}{\partial y} + \frac{\partial q}{\partial y} \right) + b \left(\frac{\partial p}{\partial x} + i \frac{\partial p}{\partial x} + i \frac{\partial p}{\partial y} - \frac{\partial q}{\partial y} \right) + \\ + 2cw = 2F_1(x, y) + 2iF_2(x, y) \end{aligned} \quad (7)$$

If (7) is separated to real and imagine parts, then following equation system is obtained

$$(a+b) \frac{\partial p}{\partial x} + (a-b) \frac{\partial q}{\partial y} + 2cp = 2F_1(x, y) \quad (8)$$

$$(a+b) \frac{\partial q}{\partial x} + (b-a) \frac{\partial p}{\partial y} + 2cq = 2F_2(x, y) \quad (9)$$

If we use Aboodh transform for above (8), (9) equalities, then we get following equalities

$$(a+b) \frac{\partial A_1}{\partial x} + (a-b) \left(vA_2 - \frac{q(x, 0)}{v} \right) + 2cA_1 = 2A_3 \quad (10)$$

$$(a+b) \frac{\partial A_2}{\partial x} + (a-b) \left(v A_1 - \frac{p(x,0)}{v} \right) + 2c A_2 = 2A_4 \quad (11)$$

If (10), (11) is regulate and is used Cramer rule, then equalities are obtained.

$$\begin{aligned} (a+b) \frac{\partial A_1}{\partial x} + (a-b) v A_2 + 2c A_1 &= 2A_3 + \frac{(a-b)}{v} q(x,0) \\ (b-a) v A_1 + (a+b) \frac{\partial A_2}{\partial x} + 2c A_2 &= 2A_4 + \frac{(b-a)}{v} p(x,0) \end{aligned}$$

$$A_1 = \frac{\begin{vmatrix} 2A_3 + \frac{(a-b)}{v} q(x,0) & (a-b) v \\ 2A_4 + \frac{(b-a)}{v} p(x,0) & (a+b) D + 2c \end{vmatrix}}{\Delta}$$

$$A_1 = \frac{(a+b) \frac{\partial}{\partial x} \left[2A_3 + \frac{(a-b)}{v} q(x,0) \right]}{\Delta} +$$

$$\frac{2c \left(2A_3 + \frac{(a-b)}{v} q(x,0) \right) - (a-b) v \left(2A_4 + \frac{(b-a)}{v} p(x,0) \right)}{\Delta} \quad (12)$$

and

$$A_2 = \frac{\begin{vmatrix} (a+b) D + 2c & 2A_3 + \frac{(a-b)}{v} q(x,0) \\ (b-a) v & 2A_4 + \frac{(b-a)}{v} p(x,0) \end{vmatrix}}{\Delta}$$

$$A_2 = \frac{(a+b) \frac{\partial}{\partial x} \left[2A_4 + \frac{(b-a)}{v} p(x,0) \right]}{\Delta} +$$

$$\frac{2c \left(2A_4 + \frac{(b-a)}{v} p(x,0) \right) - (b-a) v \left(2A_3 + \frac{(a-b)}{v} q(x,0) \right)}{\Delta} \quad (13)$$

Followings are obtained from inverse Aboodh of (12), (13)

$$p(x, y) = A^{-1} \left[\frac{(a+b) \frac{\partial}{\partial x} \left[2A_3 + \frac{(a-b)}{v} q(x,0) \right]}{\Delta} + \right.$$

$$\left. \frac{2c \left(2A_3 + \frac{(a-b)}{v} q(x,0) \right) - (a-b) v \left(2A_4 + \frac{(b-a)}{v} p(x,0) \right)}{\Delta} \right] \quad (14)$$

and

$$q(x, y) = A^{-1} \left[\frac{(a+b) \frac{\partial}{\partial x} \left[2A_4 + \frac{(b-a)}{v} p(x,0) \right]}{\Delta} + \right.$$

$$\left. \frac{2c \left(2A_4 + \frac{(b-a)}{v} p(x,0) \right) - (b-a) v \left(2A_3 + \frac{(a-b)}{v} q(x,0) \right)}{\Delta} \right]. \quad (15)$$

□

Example 3.2. Consider the following differential equation,

$$3\frac{\partial w}{\partial z} + \frac{\partial w}{\partial \bar{z}} = 0$$

with the condition

$$w(x, 0) = x^2$$

Coefficients of equation are $a = 3$, $b = 1$, $c = 0$, and $F(x; y) = 0$. From the theorem (3.1), we have

$$\Delta = ((a + b)D + 2c)^2 + (v(b - a))^2 = 16D^2 + 4v^2,$$

and

$$\begin{aligned} p(x, y) &= A^{-1} \left[\frac{-2v \left(\frac{-2}{v} x^2 \right)}{16D^2 + 4v^2} \right] = 4x^2 A^{-1} \left[\frac{1}{16D^2 + 4v^2} \right] \\ &= x^2 A^{-1} \left[\frac{1}{v^2} \cdot \frac{1}{\left(\frac{16D^2}{4v^2} + 1 \right)} \right] = x^2 A^{-1} \left[\frac{1}{v^2} \cdot \frac{1}{\left(\left(\frac{2D}{v} \right)^2 + 1 \right)} \right] \\ &= x^2 A^{-1} \left[\frac{1}{v^2} \cdot \left(1 - \left(\frac{2D}{v} \right)^2 \right) \right] = A^{-1} \left[\frac{x^2}{v^2} \cdot \left(1 - \left(\frac{2}{v} \right)^2 \right) \right] \\ &= A^{-1} \left(\frac{x^2}{v^2} - \frac{8}{v^4} \right) = x^2 \left(1 - \frac{y^2}{4} \right) = x^2 - \frac{x^2 y^2}{4}. \end{aligned}$$

And on the other hand, we have

$$\begin{aligned} q(x; y) &= A^{-1} \left[\frac{4\partial}{\partial x} \left[\frac{\frac{-2}{v} x^2}{16D^2 + 4v^2} \right] \right] = -16x A^{-1} \left[\frac{1}{v(16D^2 + 4v^2)} \right] \\ &= -4x A^{-1} \left[\frac{1}{v^3 \left(1 + \frac{16D^2}{4v^2} \right)} \right] = -4x A^{-1} \left[\frac{1}{v^3 \left(1 + \left(\frac{2D}{v} \right)^2 \right)} \right] \\ &= -4x A^{-1} \left[\frac{1}{v^3} \cdot \left(1 - \left(\frac{2D}{v} \right)^2 \right) \right] = -4A^{-1} \left[\frac{x}{v^3} \cdot (1) \right] = -4xy \end{aligned}$$

So, we obtain

$$w = x^2 - \frac{x^2 y^2}{4} - 4ixy.$$

Example 3.3. Consider the following differential equation,

$$\frac{\partial w}{\partial z} + 2\frac{\partial w}{\partial \bar{z}} = z$$

with the condition

$$w(x; 0) = x$$

Coefficients of equation are $a = 1$, $b = 2$, $c = 0$, and $F(x; y) = x + iy$. From the theorem (3.1), we have

$$\Delta = ((a + b)D + 2c)^2 + (v(b - a))^2 = 9D^2 + v^2,$$

and

$$\begin{aligned}
p(x, y) &= A^{-1} \left[\frac{3 \frac{\partial}{\partial x} \left[\frac{2x}{v^2} \right] + v \left(\frac{2}{v^3} + \frac{1}{v} \cdot x \right)}{9D^2 + v^2} \right] = A^{-1} \left[\frac{\frac{6}{v^2} + v \left(\frac{2}{v^3} + \frac{1}{v} \cdot x \right)}{9D^2 + v^2} \right] \\
&= A^{-1} \left[\frac{\frac{8}{v^2} + x}{v^2 (9D^2 + 1)} \right] = A^{-1} \left[\frac{\left(\frac{8}{v^2} + x \right)}{v^2 \left(1 + \left(\frac{3D}{v} \right)^2 \right)} \right] \\
&= A^{-1} \left[\frac{\left(\frac{8}{v^2} + x \right)}{v^2} \left(1 - \left(\frac{3D}{v} \right)^2 \right) \right] = A^{-1} \left[\frac{1}{v^2} \left(\frac{8}{v^2} + x \right) \right]
\end{aligned}$$

$$= A^{-1} \left[\frac{8}{v^4} + \frac{x}{v^2} \right] = 4y^2 + x.$$

$$p(x, y) = A^{-1} \left[\frac{3 \frac{\partial}{\partial x} \left[\frac{2x}{v^2} \right] + v \left(\frac{2}{v^3} + \frac{1}{v} \cdot x \right)}{9D^2 + v^2} \right] = A^{-1} \left[\frac{\frac{6}{v^2} + v \left(\frac{2}{v^3} + \frac{1}{v} \cdot x \right)}{9D^2 + v^2} \right]$$

And on the other hand, we have

$$q(x, y) = A^{-1} \left[\frac{3 \frac{\partial}{\partial x} \left[\frac{2}{v^3} + \frac{1}{v} \cdot x \right] - v \left(2 \frac{x}{v^2} \right)}{9D^2 + v^2} \right] = A^{-1} \left[\frac{\frac{3}{v} - \frac{2x}{v}}{9D^2 + v^2} \right]$$

$$= A^{-1} \left[\frac{(3-2x)}{v^3 \left(1 + \left(\frac{3D}{v} \right)^2 \right)} \right] = A^{-1} \left[\frac{(3-2x)}{v^3} \left(1 - \left(\frac{3D}{v} \right)^2 \right) \right]$$

$$= A^{-1} \left[\frac{1}{v^3} (3 - 2x) \right] = (3 - 2x) y.$$

Finally, we have

$$w = 4y^2 + x + i(3 - 2x)y.$$

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