Topological Group Via Operations defined on α -Open Sets

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Abstract. In this paper, we begin the study of β - α -continuous multiplication map. Here we begin by stating some properties of β - α -continuous multiplication and define the β - α -continuity of the inversion map as well. We then introduce the concept of β - α -topological groups and we go over some important theorems and examples of such spaces.

Keywords: *operations, α -open set, groups, β - α -topological group.*

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1 Introduction

The most natural way for such a study is to require algebraic operations to be continuous. Topology is an umbrella term that includes several fields of study including point set topology, algebraic topology, and differential topology. Because of this it is difficult to credit a single mathematician with introducing topology.

Topological groups have the algebraic structure of a group and the topological structure of a topological space and they are linked by the requirement that multiplication and inversion are continuous functions.

In the literature the most papers are related to the case of topological groups where group operations are continuous mappings. However, there are also many nice papers and interesting, deep results on other classes of topologized groups. In all these cases continuity play the crucial role. We refer the reader to the monograph [1] and the survey paper [12] where detail information about mentioned classes of topologized groups can be found, as well as references related to them.

In 2013, Ibrahim [9] introduced a strong form of α -open sets called α_γ -open via operation and studied some of its properties. Khalaf and Ibrahim [2, 3, 4, 5, 6, 7, 8] continued studying the properties of operations defined on the family of α -open sets introduced by Ibrahim [9], they defined and discussed several properties of α_γ -regular, α - β -compact and α_γ -connected spaces also the studied α - β -continuous and α - (γ, β) -continuous functions.

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2 Preliminaries

Let A be a subset of a topological space (G, τ) . We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$ respectively. A subset A of a topological space (G, τ) is called α -open [11] if $A \subseteq Int(Cl(Int(A)))$. The complement of an α -open set is called α -closed. The intersection of all α -closed sets containing A is called the α -closure of A and is denoted by $\alpha Cl(A)$. By $\alpha O(G, \tau)$, we denote the family of all α -open sets of G . An operation $\beta : \alpha O(G, \tau) \rightarrow P(G)$ [9] is a mapping satisfying the condition, $V \subseteq V^\beta$ for each $V \in \alpha O(G, \tau)$. We call the mapping β an operation on $\alpha O(G, \tau)$. A subset A of G is called an α_β -open set [9] if for each point $x \in A$, there exists an α -open set U of G containing x such that $U^\beta \subseteq A$. The complement of an α_β -open set is said to be α_β -closed. We denote the set of all α_β -open sets of (G, τ) by $\alpha O(G, \tau)_\beta$. The α_β -closure [9] of a subset A of G with an operation β on $\alpha O(G)$ is denoted by $\alpha_\beta Cl(A)$ and is defined to be the intersection of all α_β -closed sets containing A . A point $x \in G$ is in αCl_β -closure [9] of a set $A \subseteq G$, if $U^\beta \cap A \neq \emptyset$ for each α -open set U containing x . The αCl_β -closure of A is denoted by $\alpha Cl_\beta(A)$. The union of all α_β -open sets contained in A is called the α_β -interior of A and denoted by $\alpha_\beta Int(A)$ [2]. An operation β on $\alpha O(G, \tau)$ is said to be α -open [9] if for every α -open set U of $x \in G$, there exists an α_β -open set V of G such that $x \in V$ and $V \subseteq U^\beta$. The operation $id : \alpha O(G, \tau) \rightarrow P(G)$ is defined by $id(V) = V$ for any set $V \in \alpha O(G, \tau)$ this operation is called the identity operation on $\alpha O(G, \tau)$ [9]. An operation $\beta : \alpha O(G) \rightarrow P(G)$ is said to be α -monotone on $\alpha O(G)$ [2] if for all $A, B \in \alpha O(G)$, $A \subseteq B$ implies $A^\beta \subseteq B^\beta$. An operation $\beta : \alpha O(G) \rightarrow P(G)$ is said to be α -idempotent on $\alpha O(G)$ [2] if $A^{\beta\beta} = A^\beta$ for all $A \in \alpha O(G)$.

The following definitions are also used in the context.

Definition 2.1. [2] Let (G, τ) be a topological space and β an operation on $\alpha O(G)$. A point $a \in A \subseteq G$ is said to be α_β -interior point of A if there exists an α -open set N of G containing a such that $N^\beta \subseteq A$. We denote the set of all such points by $\alpha Int_\beta(A)$. Thus $\alpha Int_\beta(A) = \{x \in A : x \in N \in \alpha O(G) \text{ and } N^\beta \subseteq A\} \subseteq A$.

In [2], we have proved that, $\alpha Int_\beta(A)$ is an α -open set for any subset A of G .

Definition 2.2. [10] A space G is said to be α -compact if every α -open cover of G has a finite subcover

Definition 2.3. [6] A subset A of a topological space (G, τ) is said to be α - β -compact of G if for every α -open cover $\{V_i : i \in I\}$ of A , there exists a finite subset I_0 of I such that $A \subseteq \bigcup \{V_i^\beta : i \in I_0\}$.

Definition 2.4. [8] Two subsets A and B of a topological space (G, τ) are called α_β -separated if $(\alpha_\beta Cl(A) \cap B) \cup (A \cap \alpha_\beta Cl(B)) = \emptyset$.

Definition 2.5. [8] A subset C of a space G is said to be α_β -disconnected if there are nonempty α_β -separated subsets A and B of G such that $C = A \cup B$, otherwise C is called α_β -connected.

Definition 2.6. [4] A topological space (G, τ) with an operation β on $\alpha O(G)$ is said to be:

1. An α - β - T_0 space if for any two distinct points $x, y \in G$, there exists an α -open set U such that either $x \in U$ and $y \notin U^\beta$ or $y \in U$ and $x \notin U^\beta$.
2. An α - β - T_1 space if for any two distinct points $x, y \in G$, there exist two α -open sets U and V containing x and y , respectively, such that $y \notin U^\beta$ and $x \notin V^\beta$.
3. An α - β - T_2 space if for any two distinct points $x, y \in G$, there exist two α -open sets U and V containing x and y , respectively, such that $U^\beta \cap V^\beta = \emptyset$.

Definition 2.7. A topological space (G, τ) is said to be:

1. α_β -regular [3], if for each $x \in G$ and for each α -open set V in G containing x , there exists an α -open set U in G containing x such that $U^\beta \subseteq V$.
2. weakly α_β -regular [5], if for any α_β -closed set A and $x \notin A$, there exist α_β -open sets U, V such that $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$.

Definition 2.8. [7] A function $f : (G, \tau) \rightarrow (G, \tau)$ is said to be:

1. α - β -continuous if for each point x in G and for each α -open set V of G containing $f(x)$, there exists an α -open set U of G containing x such that $f(U) \subseteq V^\beta$.

2. $\alpha-(\beta, \beta)$ -continuous if for each $x \in G$ and each α -open set V containing $f(x)$, there exists an α -open set U such that $x \in U$ and $f(U^\beta) \subseteq V^\beta$.
3. $\alpha-(\beta, \beta)$ -homeomorphic, if f is bijective, $\alpha-(\beta, \beta)$ -continuous and f^{-1} is $\alpha-(\beta, \beta)$ -continuous.
4. $\alpha_{(\beta, \beta)}$ -open if for any α_β -open set A of (G, τ) , $f(A)$ is α_β -open in (G, τ) .

Definition 2.9. [9] A mapping $f : (G, \tau) \rightarrow (G, \tau)$ is said to be $\alpha_{(\beta, \beta)}$ -continuous if for each x of G and each α_β -open set V containing $f(x)$, there exists an α_β -open set U such that $x \in U$ and $f(U) \subseteq V$.

Throughout this paper, $(G, *, \tau)$, or simply G , will denote a group $(G, *)$ endowed with a topology τ and the identity element of G is denoted by e . The operation $*$: $G \times G \rightarrow G$, $(x, y) \rightarrow x * y$ is called the multiplication mapping and sometimes denoted by m , and the inverse operation $G \rightarrow G$, $x \rightarrow x^{-1}$ is denoted by i .

3 β - α -topological Group

Definition 3.1. [1] Let $(G, *)$ be a group and τ be a topology on G . Then, the multiplication map is continuous in the first variable if given $a, b \in G$ and $O \in \tau$ such that $a * b \in O$, there is $U \in \tau$ with $a \in U$ and $U * b \subseteq O$. Similarly, the multiplication map is continuous in the second variable if given $a, b \in G$ and $O \in \tau$ such that $a * b \in O$, there is $V \in \tau$ with $b \in V$ and $a * V \subseteq O$.

Definition 3.2. Let $(G, *)$ be a group and τ be a topology on G . Then, the multiplication map is β - α -continuous in the first variable if given $a, b \in G$ and $O \in \alpha O(G, \tau)$ such that $a * b \in O$, there is $U \in \alpha O(G, \tau)$ with $a \in U$ and $U^\beta * b \subseteq O^\beta$.

Similarly, the multiplication map is β - α -continuous in the second variable if given $a, b \in G$ and $O \in \alpha O(G, \tau)$ such that $a * b \in O$, then there is $V \in \alpha O(G, \tau)$ with $b \in V$ and $a * V^\beta \subseteq O^\beta$.

The proof of the following result is clear, so it is omitted.

Proposition 3.3. For a commutative group G . The multiplication map is β - α -continuous in the first variable if and only if it is β - α -continuous in the second variable.

Example 3.4. Consider the permutation group S_3 defined on the set $A = \{1, 2, 3\}$, the elements of this group are:

$$\begin{aligned} f_1 &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = 1, f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12) \\ f_3 &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (23), f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13) \\ f_5 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123), f_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132). \end{aligned}$$

Then, $S_3 = \{1, (12), (13), (23), (123), (132)\}$ with the usual composition of maps, $(S_3, \circ)$, forms a non-commutative group.

Let $\tau = \{\phi, S_3, \{f_1\}, \{f_2, f_3, f_4\}, \{f_1, f_2, f_3, f_4\}, \{f_1, f_5, f_6\}\}$ be a topology on S_3 . Then, the multiplication map is not continuous neither in the first nor in the second variable because for $i = 2, 3, 4$, we have

$f_i \circ f_i = f_1$ and $\{f_2, f_3, f_4\} \circ f_i \not\subseteq \{f_1\}$ also $f_i \circ \{f_2, f_3, f_4\} \not\subseteq \{f_1\}$.

We have

$$\alpha O(S_3, \tau) = \{\phi, S_3, \{f_1\}, \{f_1, f_5\}, \{f_1, f_6\}, \{f_2, f_3, f_4\}, \{f_1, f_2, f_3, f_4\}, \{f_1, f_5, f_6\}\}.$$

For each $A \in \alpha O(S_3, \tau)$, we define β on $\alpha O(S_3, \tau)$ by

$$A^\beta = Cl A \text{ for each } A \in \alpha O(S_3, \tau).$$

To check that the multiplication map is β - α -continuous in the first variable for every element in S_3 . Consider $f_i \circ f_j = f_k$ for $i, j, k = \{1, 2, 3, 4, 5, 6\}$. Now fixing $j = 2$, if $i \in \{2, 3, 4\}$, then $k \in \{1, 5, 6\}$ and so, we have $\{f_2, f_3, f_4\} \circ f_2 \subseteq \{f_1, f_5, f_6\}$. If $i \in \{1, 5, 6\}$, then $k \in \{2, 3, 4\}$ and so, we have $\{f_1, f_5, f_6\} \circ f_2 \subseteq \{f_2, f_3, f_4\}$. In a similar way we can check for $j \in \{1, 3, 4, 5, 6\}$, so the multiplication map is β - α -continuous in the first variable for every element in S_3 . By the same way we can show that the multiplication map is β - α -continuous in the second variable also for every element in S_3 .

Theorem 3.5. *Let $(G, *)$ be a group and τ be a topology on G . If the multiplication map is β - α -continuous in the second variable, then for any $a \in G$, $A \subseteq G$ and $B \subseteq G$ the following statements are true:*

1. $a * \alpha Cl_\beta(B) = \alpha Cl_\beta(a * B)$ and $A * \alpha Cl_\beta(B) \subseteq \alpha Cl_\beta(A * B)$.
2. $a * \alpha Int_\beta(B) = \alpha Int_\beta(a * B)$ and $A * \alpha Int_\beta(B) \subseteq \alpha Int_\beta(A * B)$.
3. $A * B$ is α -open for any A and any α_β -open B .

Proof.

1. Let $y \in a * \alpha Cl_\beta(B)$, then $y = a * x$ where $x \in \alpha Cl_\beta(B)$ and let $O \in \alpha O(G, \tau)$ with $a * x \in O$. Since the multiplication map is β - α -continuous in the second variable, then there is $V \in \alpha O(G, \tau)$ with $x \in V$ and $a * V^\beta \subseteq O^\beta$. Since $x \in \alpha Cl_\beta(B)$, there is $b \in B \cap V^\beta$, thus $a * b \in a * V^\beta \subseteq O^\beta$. Therefore, $a * b \in a * B \cap O^\beta$ which implies $y = a * x \in \alpha Cl_\beta(a * B)$. Hence, $a * \alpha Cl_\beta(B) \subseteq \alpha Cl_\beta(a * B)$.
Conversely, let $a * x \in \alpha Cl_\beta(a * B)$ and $O \in \alpha O(G, \tau)$ such that $x \in O$. Since $x = a^{-1} * (a * x) \in O$, there is $V \in \alpha O(G, \tau)$ with $a * x \in V$ and $a^{-1} * V^\beta \subseteq O^\beta$. Since $a * x \in \alpha Cl_\beta(a * B) \cap V$, there is $a * b \in a * B \cap V^\beta$, thus $a^{-1} * (a * b) = b \in a^{-1} * V^\beta \subseteq O^\beta$. Hence $b \in B \cap O^\beta$, and $x \in \alpha Cl_\beta(B)$. Therefore, $a * x \in a * \alpha Cl_\beta(B)$, and $\alpha Cl_\beta(a * B) \subseteq a * \alpha Cl_\beta(B)$. Consequently $a * \alpha Cl_\beta(B) = \alpha Cl_\beta(a * B)$.
Now, let $a \in A$, then $a * \alpha Cl_\beta(B) = \alpha Cl_\beta(a * B) \subseteq \alpha Cl_\beta(A * B)$ as $a * B \subseteq A * B$. Thus, $A * \alpha Cl_\beta(B) = \bigcup_{a \in A} a * \alpha Cl_\beta(B) = \bigcup_{a \in A} \alpha Cl_\beta(a * B) \subseteq \alpha Cl_\beta(A * B)$.
2. Let $a * x \in a * \alpha Int_\beta(B)$, then there is an α -open set O that contains x such that $O^\beta \subseteq B$. Since $x = a^{-1} * (a * x) \in O$, there is an α -open set V with $a * x \in V$ and $a^{-1} * V^\beta \subseteq O^\beta$, thus $a * x \in V \subseteq V^\beta \subseteq a * O^\beta \subseteq a * B$. Therefore, $a * x \in \alpha Int_\beta(a * B)$ and $a * \alpha Int_\beta(B) \subseteq \alpha Int_\beta(a * B)$.
Conversely, let $a * x \in \alpha Int_\beta(a * B)$, then there is an α -open set O such that $a * x \in O$ and $O^\beta \subseteq a * B$. There is an α -open set V containing x with $a * V^\beta \subseteq O^\beta$, hence $x \in V \subseteq V^\beta \subseteq a^{-1} * O^\beta \subseteq B$ and $x \in \alpha Int_\beta(B)$. Therefore, $a * x \in a * \alpha Int_\beta(B)$ and $\alpha Int_\beta(a * B) \subseteq a * \alpha Int_\beta(B)$. Consequently $a * \alpha Int_\beta(B) = \alpha Int_\beta(a * B)$.
Now, let $a \in A$, then $a * \alpha Int_\beta(B) = \alpha Int_\beta(a * B) \subseteq \alpha Int_\beta(A * B)$. Hence, $A * \alpha Int_\beta(B) = \bigcup_{a \in A} a * \alpha Int_\beta(B) = \bigcup_{a \in A} \alpha Int_\beta(a * B) \subseteq \alpha Int_\beta(A * B)$.
3. If B is α_β -open, then $\alpha Int_\beta(B) = B$. Let $a \in A$, then $a * B = a * \alpha Int_\beta(B) = \alpha Int_\beta(a * B)$. Hence, $A * B = A * \alpha Int_\beta(B) = \bigcup_{a \in A} a * \alpha Int_\beta(B) = \bigcup_{a \in A} \alpha Int_\beta(a * B)$. Thus, $A * B$ is the union of α -open sets and therefore, $A * B$ is α -open.

Theorem 3.6. *Let $(G, *)$ be a group and (G, τ) be an α_β -regular space. If the multiplication map is β - α -continuous in the second variable, then the following statements are true:*

1. If S is a semigroup, then $\alpha_\beta Int(S)$ is also semigroup.
2. For any $a \in G$, the left translation $l_a : G \rightarrow G$ defined by $l_a(x) = a * x$, is an α - (β, β) -homeomorphism of G onto G .
3. For two elements x and y in G , there exists an α - (β, β) -homeomorphism f of G onto itself such that $f(x) = y$.

Proof.

1. Since $\alpha_\beta Int(S) \subseteq S$, hence $\alpha_\beta Int(S)$ is associative. It remains to prove that $\alpha_\beta Int(S)$ under the operation $(*)$. Let $a, b \in \alpha_\beta Int(S)$, then there is an α_β -open set V that contains b such that $V \subseteq S$. Since S is closed under the operation $(*)$, so $a * V \subseteq S$, but by Theorem 3.5(3), $a * V$ is α -open and $a * b \in a * V$. Since, G is α_β -regular, hence $a * b \in \alpha_\beta Int(S)$. Therefore, $\alpha_\beta Int(S)$ is closed under the operation $(*)$.
2. Let $a, x, y \in G$ and $l_a(x) = l_a(y)$, then $a * x = a * y$ and $x = y$. Therefore, l_a is one-to-one. Since G is a group, for every $x \in G$, $a^{-1} * x \in G$, thus $l_a(a^{-1} * x) = a * (a^{-1} * x) = x$. Hence l_a is onto. Let O be an α_β -open set, then $l_a(O) = a * O$. By Theorem 3.5 (3) and since G is α_β -regular, so $a * O \in \alpha O(G, \tau)_\beta$. Hence l_a is $\alpha_{(\beta, \beta)}$ -open. Let O be an α_β -open set, then $l_a^{-1}(O) = a^{-1} * O$. By Theorem 3.5 (3) and since G is α_β -regular, so $a^{-1} * O \in \alpha O(G, \tau)_\beta$. Hence, l_a is α - (β, β) -continuous. Thus, l_a is an α - (β, β) -homeomorphism.
3. Let $x, y \in G$, then $y * x^{-1} \in G$. Define $f = l_{y * x^{-1}} : G \rightarrow G$ as above, then $l_{y * x^{-1}}$ is α - (β, β) -homeomorphism, and $l_{y * x^{-1}}(x) = y * x^{-1} * x = y$.

Theorem 3.7. Let $(G, *)$ be a group and τ be a topology on G . If the multiplication map is β - α -continuous in the first variable, then:

1. $\alpha Cl_\beta(A) * b = \alpha Cl_\beta(A * b)$ and $\alpha Cl_\beta(A) * B \subseteq \alpha Cl_\beta(A * B)$ for any $b \in G$, $A \subseteq G$ and $B \subseteq G$.
2. $\alpha Int_\beta(A) * b = \alpha Int_\beta(A * b)$ and $\alpha Int_\beta(A) * B \subseteq \alpha Int_\beta(A * B)$.

Proof. The proofs are similar to Theorem 3.5 (1) and (2).

Theorem 3.8. Let $(G, *)$ be a group and τ be a topology on G . If the multiplication map is β - α -continuous in each variable, then:

1. If β is α -open, then $\alpha Cl_\beta(A) * \alpha Cl_\beta(B) \subseteq \alpha Cl_\beta(A * B)$ for all $A, B \subseteq G$.
2. $\alpha Int_\beta(A) * \alpha Int_\beta(B) \subseteq A * \alpha Int_\beta(B) \cap \alpha Int_\beta(A) * B \subseteq A * \alpha Int_\beta(B) \cup \alpha Int_\beta(A) * B \subseteq \alpha Int_\beta(A * B)$.
3. If A is arbitrary and B is α_β -open, then $A * B$ and $B * A$ are α -open.

Proof.

1. By Theorem 3.5 (1), $A * \alpha Cl_\beta(B) \subseteq \alpha Cl_\beta(A * B)$ which implies $\alpha Cl_\beta(A * \alpha Cl_\beta(B)) \subseteq \alpha Cl_\beta(A * B)$. Since the multiplication map is β - α -continuous in the first variable, then, by Theorem 3.7(1), we have $\alpha Cl_\beta(A) * \alpha Cl_\beta(B) \subseteq \alpha Cl_\beta(A * \alpha Cl_\beta(B)) \subseteq \alpha Cl_\beta(A * B)$.
2. Since $\alpha Int_\beta(A) \subseteq A$, thus $\alpha Int_\beta(A) * \alpha Int_\beta(B) \subseteq A * \alpha Int_\beta(B)$. Similarly, $\alpha Int_\beta(A) * \alpha Int_\beta(B) \subseteq \alpha Int_\beta(A) * B$. Thus, $\alpha Int_\beta(A) * \alpha Int_\beta(B) \subseteq A * \alpha Int_\beta(B) \cap \alpha Int_\beta(A) * B \subseteq A * \alpha Int_\beta(B) \cup \alpha Int_\beta(A) * B$. By Theorem 3.5 (2), $A * \alpha Int_\beta(B) \subseteq \alpha Int_\beta(A * B)$. Since the multiplication map is β - α -continuous in the first variable, so $\alpha Int_\beta(A) * B \subseteq \alpha Int_\beta(A * B)$. Therefore, $A * \alpha Int_\beta(B) \cup \alpha Int_\beta(A) * B \subseteq \alpha Int_\beta(A * B)$.
3. By Theorem 3.5 (3), $A * B$ is α -open. Let $a \in A$, and since the multiplication map is β - α -continuous in the first variable, then by Theorem 3.7 (2), we have $B * a = \alpha Int_\beta(B) * a = \alpha Int_\beta(B * a)$. Hence, $B * A = \alpha Int_\beta(B) * A = \alpha Int_\beta(B) * \bigcup_{a \in A} a = \bigcup_{a \in A} \alpha Int_\beta(B * a)$. Thus, $B * A$ is the union of α -open sets and therefore, $B * A$ is α -open.

Recall that a subset S of a group $(G, *)$ is normal if $x * s * x^{-1} \in S$ for each $s \in S$ and each $x \in G$.

Theorem 3.9. Let $(G, *)$ be a group, τ be a topology on G and (G, τ) be α_β -regular. If the multiplication map is β - α -continuous in each variable, then:

1. If S is a semigroup of G , then $\alpha_\beta Int(S)$ and $\alpha_\beta Cl(S)$ are also semigroups, where β is both α -monotone and α -idempotent operation.
2. If S is a normal set in G , then both $\alpha_\beta Int(S)$ and $\alpha_\beta Cl(S)$ are also normal sets.

Proof.

1. By Theorem 3.6 (1), $\alpha_\beta Int(S)$ is semigroup. Since $\alpha_\beta Cl(S) \subseteq G$, $\alpha_\beta Cl(S)$ is associative. Let $x, y \in \alpha_\beta Cl(S)$ and $O \in \alpha O(G, \tau)_\beta = \alpha O(G, \tau)$ such that $x * y \in O$, then there is an α -open set U containing x such that $U^\beta * y \subseteq O^\beta$. Since $x \in U \cap \alpha_\beta Cl(S)$, there is $a \in G$ such that $a \in U \cap S$, thus $a * y \in U * y \subseteq U^\beta * y \subseteq O^\beta$. By Theorem 3.8 (3), $U * y$ is α -open, then there is an α -open set V containing y such that $a * V^\beta \subseteq (U * y)^\beta \subseteq (O^\beta)^\beta = O^\beta$. Since $y \in V \cap \alpha_\beta Cl(S)$, there is $b \in G$ such that $b \in V \cap S$. Hence, $a * b \in a * V^\beta \subseteq O^\beta$ and since S is a semigroup. Therefore, $a * b \in O^\beta \cap S$ and so $x * y \in \alpha Cl_\beta(S) \subseteq \alpha_\beta Cl(S)$.
2. Let $x \in G$, then $x * \alpha_\beta Int(S) * x^{-1}$ is α -open as the multiplication map is β - α -continuous in both variables. Since S is normal, so $x * \alpha_\beta Int(S) * x^{-1} \subseteq x * S * x^{-1} \subseteq S$ and thus $x * \alpha_\beta Int(S) * x^{-1} \subseteq \alpha_\beta Int(S)$ and so $\alpha_\beta Int(S)$ is normal.

To prove that $\alpha_\beta Cl(S)$ is normal, let $x \in G$, $y \in \alpha_\beta Cl(S)$ and $O \in \alpha O(G, \tau)$ with $x * y * x^{-1} \in O$, then there is an α -open set U containing $x * y$ such that $U^\beta * x^{-1} \subseteq O^\beta$. Since U is α -open, there is $V \in \alpha O(G, \tau)$ with $y \in V$ and $x * V^\beta \subseteq U^\beta$. Now, we have $y \in V \cap \alpha_\beta Cl(S)$, then there is $b \in S \cap V$ such that $x * b \in x * V \subseteq x * V^\beta \subseteq U^\beta$ and $x * b * x^{-1} \in U^\beta * x^{-1} \subseteq O^\beta$, thus $x * b * x^{-1} \in O^\beta \cap S$ and $x * y * x^{-1} \in \alpha Cl_\beta(S) \subseteq \alpha_\beta Cl(S)$. Therefore, $\alpha_\beta Cl(S)$ is normal.

Corollary 3.10. *Suppose that the multiplication map is β - α -continuous in each variable on G . Let β be α -open, α -monotone and α -idempotent. If S is a semigroup of G , then $\alpha Cl_\beta(S)$ is also a semigroup.*

Proof. Since $\alpha Cl_\beta(S) \subseteq G$, so $\alpha Cl_\beta(S)$ is associative.

Let $x, y \in \alpha Cl_\beta(S)$ and $O \in \alpha O(G, \tau)$ such that $x * y \in O$, then there is an α -open set U containing x such that $U^\beta * y \subseteq O^\beta$. By β is α -open, there is an α_β -open set L containing x such that $L \subseteq U^\beta$ and $L * y \subseteq U^\beta * y \subseteq O^\beta$ and $\alpha Cl_\beta(S) = \alpha_\beta Cl(S)$. Since $x \in L \cap \alpha_\beta Cl(S)$, there is $a \in G$ such that $a \in L \cap S$, thus $a * y \in L * y \subseteq U^\beta * y \subseteq O^\beta$. By Theorem 3.8 (3), $L * y$ is α -open, then there is an α -open set V containing y such that $a * V^\beta \subseteq (L * y)^\beta \subseteq (O^\beta)^\beta = O^\beta$ because β is both α -monotone and α -idempotent. Again, by β is α -open, there is an α_β -open set K containing y such that $K \subseteq V^\beta$ and $a * K \subseteq a * V^\beta \subseteq O^\beta$. Since $y \in K \cap \alpha_\beta Cl(S)$, there is $b \in G$ such that $b \in K \cap S$. Hence, $a * b \in a * K \subseteq O^\beta$ and since S is a semigroup. Therefore, $a * b \in O^\beta \cap S$ and so $x * y \in \alpha Cl_\beta(S)$.

Theorem 3.11. *Let $(G, *)$ be a group, τ be a topology on G and (G, τ) be α_β -regular. If the multiplication map is β - α -continuous in each variable and $x * S = S * x$ for all $x \in G$, then $x * \alpha_\beta Int(S) = \alpha_\beta Int(S) * x$ and $x * \alpha_\beta Cl(S) = \alpha_\beta Cl(S) * x$ for all $x \in G$.*

Proof. Follows from Theorem 3.9 (2).

Definition 3.12. *Let $(G, *)$ be a group and τ be a topology on G . The inversion map is β - α -continuous if given $a \in G$ and $O \in \alpha O(G, \tau)$ such that $a^{-1} \in O$, then there is $U \in \alpha O(G, \tau)$ with $a \in U$ and $(U^\beta)^{-1} \subseteq O^\beta$, where $(U^\beta)^{-1} = \{x^{-1} : x \in U^\beta\}$.*

Example 3.13. *Let $(S_3, \circ)$ be a group and $\tau = \{\phi, S_3, \{1, (12), (13), (23)\}\}$ be a topology on S_3 , then $\alpha O(S_3, \tau) = \{\phi, S_3, \{1, (12), (13), (23)\}, \{1, (12), (13), (23), (123)\}, \{1, (12), (13), (23), (132)\}\}$. For each $A \in \alpha O(S_3, \tau)$, we define β on $\alpha O(S_3, \tau)$ by*

$$A^\beta = \begin{cases} A & \text{if } A = \{1, (12), (13), (23)\}, \\ Cl(A) & \text{if } A \neq \{1, (12), (13), (23)\}. \end{cases}$$

Then, the inversion is β - α -continuous.

Remark 3.14. *For the group $(G, \odot_8)$ where $G = \{1, 3, 5, 7\}$, if τ is any topology on G , then the inversion map is β - α -continuous, because $a = a^{-1}$ for any $a \in G$.*

Definition 3.15. *Let $(G, *)$ be a group and τ be a topology on G . A mapping $\beta : \alpha O(G, \tau) \rightarrow P(G)$ is called α -left operation if for any $a \in G$ and whenever $V, (a * V)^{-1} \in \alpha O(G, \tau)$, then $((a * V)^{-1})^\beta = (a * V^\beta)^{-1}$.*

Corollary 3.16. *If $\beta : \alpha O(G, \tau) \rightarrow P(G)$ is an α -left operation, then $(V^{-1})^\beta = (V^\beta)^{-1}$ for all $V, V^{-1} \in \alpha O(G, \tau)$.*

In the next results we denote the collection of all α -open set containing the identity e by Ω .

Theorem 3.17. *Let $(G, *)$ be a group, τ be a topology on G , (G, τ) be α_β -regular and β be an α -left operation. If the inversion map is β - α -continuous and multiplication is β - α -continuous in one variable, then:*

1. *The multiplication is β - α -continuous in each variable.*
2. *For any subset A of G , $\alpha Cl_\beta(A) = \bigcap_{V \in \Omega} A * V^\beta = \bigcap_{V \in \Omega} V^\beta * A$.*

Proof.

Assume multiplication is β - α -continuous in the second variable. Let $f : G \rightarrow G$ defined by $f(x) = x^{-1}$ for all $x \in G$. Let O be an α -open set in G containing $f(x) = x^{-1}$. Then, by inversion is β - α -continuous, there is $U \in \alpha O(G, \tau)$ with $x \in U$ and $f(U^\beta) = U^{\beta-1} \subseteq O^\beta$, then f is α -(β, β)-continuous, so if O is α_β -open, then $f^{-1}(O) = O^{-1}$ is α_β -open.

1. Let $x, y \in G$ and O be an α -open set containing $x * y$, then there is an α -open set V containing y with $x * V^\beta \subseteq O^\beta$. Since $x * V$ is α -open, so $f^{-1}(x * V) = V^{-1} * x^{-1}$ is α -open. Because $y^{-1} * x^{-1} \in V^{-1} * x^{-1}$, there is an α -open set U containing x^{-1} with $y^{-1} * U^\beta \subseteq (V^{-1} * x^{-1})^\beta$, thus by β is α -left, we have $x^{-1} \in U \subseteq U^\beta \subseteq y * (V^{-1} * x^{-1})^\beta = y * ((x * V)^{-1})^\beta = y * (x * V^\beta)^{-1} \subseteq y * O^{\beta-1}$, implies that $x \in U^{-1} \subseteq U^{\beta-1} \subseteq (x * V^\beta) * y^{-1} \subseteq O^\beta * y^{-1}$ and $x * y \in U^{-1} * y \subseteq U^{\beta-1} * y \subseteq x * V^\beta \subseteq O^\beta$. Since U is α -open, so U^{-1} is α -open. Thus, U^{-1} is an α -open set containing x such that $U^{-1\beta} * y \subseteq O^\beta$. Therefore multiplication is β - α -continuous in the first variable.

2. Let $x \in \alpha Cl_\beta(A)$ and $V \in \Omega$, then $e \in V$ implies that $e \in V^{-1}$ which is α -open. Since $e = x^{-1} * x$, there is an α -open set U containing x with $x^{-1} * U^\beta \subseteq V^{-1\beta}$. There is $a \in A \cap U^\beta$, thus $x^{-1} * a \in V^{-1\beta}$ and $a^{-1} * x \in V^\beta$. Hence, $a * (a^{-1} * x) = x \in A * V^\beta$ and $\alpha Cl_\beta(A) \subseteq A * V^\beta$. Since V was chosen arbitrarily, so $\alpha Cl_\beta(A) \subseteq \bigcap_{V \in \Omega} A * V^\beta$.

Conversely, let $x \in \bigcap_{V \in \Omega} A * V^\beta$ and $O \in \alpha O(G, \tau)$ such that $x = x * e \in O$, then there is $V \in \Omega$ with $x * V^\beta \subseteq O^\beta$. Since $e \in V$ and V is α -open, so $e \in V^{-1}$ and V^{-1} is also α -open, hence $V^{-1} \in \Omega$ and $x \in A * V^{-1\beta}$. For some $a \in A$, $a^{-1} * x \in V^{-1\beta}$, thus $x^{-1} * a \in V^\beta$. Since $a = x * (x^{-1} * a) \subseteq x * V^\beta \subseteq O^\beta$, $a \in A \cap O^\beta$, hence $x \in \alpha Cl_\beta(A)$. Therefore, $\bigcap_{V \in \Omega} A * V^\beta \subseteq \alpha Cl_\beta(A)$ and $\alpha Cl_\beta(A) = \bigcap_{V \in \Omega} A * V^\beta$. Similarly, $\alpha Cl_\beta(A) = \bigcap_{V \in \Omega} V^\beta * A$, thus $\alpha Cl_\beta(A) = \bigcap_{V \in \Omega} A * V^\beta = \bigcap_{V \in \Omega} V^\beta * A$.

Remark 3.18. Every β - α -continuous inversion map is α - β -continuous. But, the converse is not true in general as the next example shows.

Example 3.19. From Example 3.13, if for each $A \in \alpha O(S_3, \tau)$, we define β on $\alpha O(S_3, \tau)$ by

$$A^\beta = \begin{cases} A & \text{if } A = \{1, (12), (13), (23), (132)\}, \\ Cl(A) & \text{if } A \neq \{1, (12), (13), (23), (132)\}. \end{cases}$$

Then, the inversion map i is α - β -continuous, but not β - α -continuous because $\{1, (12), (13), (23), (132)\}$ is an α -open set in G containing $(123)^{-1}$, but there exist no α -open set U in G containing (123) such that $U^{\beta^{-1}} \subseteq \{1, (12), (13), (23), (132)\}^\beta = \{1, (12), (13), (23), (132)\}$.

Corollary 3.20. Let $(G, *)$ be a group and τ be a topology on G . If the inversion map is β - α -continuous, then $(\alpha_\beta Cl(A))^{-1} \subseteq \alpha_\beta Cl(A^{-1})$ for any subset A of G .

Proof. Let $x \in (\alpha_\beta Cl(A))^{-1}$ and let U be an α_β -open set containing x . Since the inverse mapping is β - α -continuous, then the set U^{-1} is α_β -open containing x^{-1} . Since $x^{-1} \in \alpha_\beta Cl(A)$, so $U^{-1} \cap A \neq \emptyset$. Therefore, $U \cap A^{-1} \neq \emptyset$, that is $x \in \alpha_\beta Cl(A^{-1})$ and so $(\alpha_\beta Cl(A))^{-1} \subseteq \alpha_\beta Cl(A^{-1})$.

Definition 3.21. Let $(G, *)$ be a group and τ be a topology on G . The multiplication is jointly β - α -continuous in both variables if given $a, b \in G$ and $O \in \alpha O(G, \tau)$ such that $a * b \in O$, then there exist $U, V \in \alpha O(G, \tau)$ with $a \in U$, $b \in V$ and $U^\beta * V^\beta \subseteq O^\beta$.

Example 3.22. Let $(S_3, \circ)$ be a group and τ be the discrete topology on S_3 . For each $A \in \alpha O(S_3, \tau)$, we define β on $\alpha O(S_3, \tau)$ by

$$A^\beta = \begin{cases} A & \text{if } A \text{ is singleton set,} \\ S_3 & \text{otherwise.} \end{cases}$$

Then, the multiplication is jointly β - α -continuous in both variables.

The proof of the following corollary follows directly from the definition.

Corollary 3.23. If multiplication is jointly β - α -continuous in both variables, then multiplication is β - α -continuous in each variable.

Corollary 3.24. If multiplication map is jointly β - α -continuous, then multiplication is α - β -continuous.

Proof. Let f denote multiplication, then $f : (G, \tau) \times (G, \tau) \rightarrow (G, \tau)$ defined by $f(a, b) = a * b$ for all $a, b \in G$. Let O be an α -open set in G containing $f(x, y) = x * y$. Since multiplication is jointly β - α -continuous in both variables, then there are α -open sets U containing x and V containing y such that $U^\beta * V^\beta \subseteq O^\beta$ implies that $f(U \times V) \subseteq f(U^\beta \times V^\beta) = U^\beta * V^\beta \subseteq O^\beta$ and $(x, y) \in U \times V \in \alpha O(G \times G, \tau \times \tau)$, hence f is α - β -continuous.

The converse of the above corollary need not be true in general as seen from the following example.

Example 3.25. Let $(Z_3, +_3)$ be a group and τ be the discrete topology on Z_3 . For each $A \in \alpha O(Z_3, \tau)$, we define β on $\alpha O(Z_3, \tau)$ by

$$A^\beta = \begin{cases} A & \text{if } A = \{0, 1\}, \\ Z_3 & \text{if } A \neq \{0, 1\}. \end{cases}$$

Then, the multiplication map f is α - β -continuous, but not jointly β - α -continuous because $\{0, 1\}$ is an α -open set in G containing $0 + 1$, but there exist no $U, V \in \alpha O(Z_3, \tau)$ with $0 \in U$, $1 \in V$ and $U^\beta +_3 V^\beta \subseteq \{0, 1\}^\beta = \{0, 1\}$.

Theorem 3.26. *Let $(G, *)$ be a group and τ be a topology on G . Then inversion is β - α -continuous and multiplication is jointly β - α -continuous in both variables if and only if for any elements a, b of G and α -open set O with $a * b^{-1} \in O$, there exist α -open sets U and V containing a and b respectively such that $U^\beta * (V^\beta)^{-1} \subseteq O^\beta$.*

Proof. Let $a, b \in G$, hence $a * b^{-1} \in G$. Let O be an α -open set containing $a * b^{-1}$. Since multiplication is jointly β - α -continuous in both variables, then there are α -open sets U containing a and R containing b^{-1} such that $U^\beta * R^\beta \subseteq O^\beta$. Since inversion is β - α -continuous, so there is an α -open set V containing b such that $(V^\beta)^{-1} \subseteq R^\beta$. Hence, $U^\beta * (V^\beta)^{-1} \subseteq O^\beta$.

Conversely, let $a \in G$ and O be an α -open set containing a^{-1} . Then, O contains the element $e * a^{-1}$ and hence there exist α -open sets U containing e and V containing a such that $U^\beta * (V^\beta)^{-1} \subseteq O^\beta$ and $(V^\beta)^{-1} = e * (V^\beta)^{-1} \subseteq U * (V^\beta)^{-1} \subseteq U^\beta * (V^\beta)^{-1} \subseteq O^\beta$, thus the inversion is β - α -continuous. Now let $a, b \in G$ and $O \in \alpha O(G, \tau)$ such that $a * b \in O$. By hypothesis there exist α -open sets U and R containing a and b^{-1} respectively such that $U^\beta * (R^\beta)^{-1} \subseteq O^\beta$ and there is an α -open set V containing b such that $(V^\beta)^{-1} \subseteq R^\beta$. Now, $U^\beta * V^\beta = U^\beta * ((V^\beta)^{-1})^{-1} \subseteq U^\beta * (R^\beta)^{-1} \subseteq O^\beta$. Therefore, the multiplication is jointly β - α -continuous in both variables.

Theorem 3.27. *Let $(G, *)$ be a group, τ be a topology on G , (G, τ) be α_β -regular and β be both α -monotone and α -left operation. If the inversion is β - α -continuous and multiplication is jointly β - α -continuous in both variables, then $A \subseteq G$ implies*

$$\alpha Cl_\beta(A) = \bigcap_{V \in \Omega} A * V^\beta * (V^{-1})^\beta = \bigcap_{V \in \Omega} V^\beta * (V^{-1})^\beta * A = \bigcap_{V \in \Omega} \alpha Cl_\beta(A * V^\beta * (V^{-1})^\beta) = \bigcap_{V \in \Omega} \alpha Cl_\beta(V^\beta * (V^{-1})^\beta * A).$$

Proof. Since multiplication is jointly β - α -continuous in both variables, so it is β - α -continuous in one variable, thus Theorem 3.17 (2) applies. Let $V \in \Omega$, then $A * V^\beta \subseteq A * V^\beta * (V^{-1})^\beta$, hence $\alpha Cl_\beta(A) = \bigcap_{V \in \Omega} A * V^\beta \subseteq \bigcap_{V \in \Omega} A * V^\beta * (V^{-1})^\beta$.

Let $x \in \bigcap_{V \in \Omega} A * V^\beta * (V^{-1})^\beta$ and $O \in \alpha O(G, \tau)$ such that $x = x * e \in O$, then there are α -open sets U and V with $x \in U$, $e \in V$ and $U^\beta * V^\beta \subseteq O^\beta$. Since $e * e^{-1} \in V$, by Theorem 3.26, there exist $V_1, V_2 \in \Omega$ such that $V_1^\beta * (V_2^\beta)^{-1} \subseteq V^\beta$. Let $V_3 = V_1 \cap V_2$, then V_3 is α -open, $e \in V_3$ and $V_3^\beta * (V_3^{-1})^\beta \subseteq V_1^\beta * (V_2^\beta)^{-1} \subseteq V^\beta$, thus $U^\beta * V_3^\beta * (V_3^{-1})^\beta \subseteq O^\beta$. Since $x \in A * V_3^\beta * (V_3^{-1})^\beta$, there is $a \in A$ and $v_1, v_2 \in V_3^\beta$ such that $x = a * v_1 * v_2^{-1}$, hence $a = (a * v_1 * v_2^{-1}) * (v_2 * v_1^{-1}) = x * v_2 * v_1^{-1} \in U^\beta * V_3^\beta * (V_3^{-1})^\beta \subseteq O^\beta$. Therefore, $a \in O^\beta \cap A$, $x \in \alpha Cl_\beta(A)$ and $\bigcap_{V \in \Omega} A * V^\beta * (V^{-1})^\beta \subseteq \alpha Cl_\beta(A)$. Hence, $\alpha Cl_\beta(A) = \bigcap_{V \in \Omega} A * V^\beta * (V^{-1})^\beta$. Similarly, $\alpha Cl_\beta(A) = \bigcap_{V \in \Omega} V^\beta * (V^{-1})^\beta * A$.

Let $V \in \Omega$, then $A * V^\beta * (V^{-1})^\beta \subseteq \alpha Cl_\beta(A * V^\beta * (V^{-1})^\beta)$, thus $\alpha Cl_\beta(A) = \bigcap_{V \in \Omega} A * V^\beta * (V^{-1})^\beta \subseteq \bigcap_{V \in \Omega} \alpha Cl_\beta(A * V^\beta * (V^{-1})^\beta)$.

Let $x \in \bigcap_{V \in \Omega} \alpha Cl_\beta(A * V^\beta * (V^{-1})^\beta)$ and $O \in \alpha O(G, \tau)$ such that $x \in O$, then, as above, there is $U \in \alpha O(G, \tau)$ and $V \in \Omega$ with $x \in U$ and $U^\beta * V^\beta * (V^{-1})^\beta \subseteq O^\beta$. Since there is $a \in A$ and $v_1, v_2 \in V^\beta$ such that $a * v_1 * v_2^{-1} \in A * V^\beta * (V^{-1})^\beta \cap U^\beta$ and $a = (a * v_1 * v_2^{-1}) * (v_2 * v_1^{-1}) \in U^\beta * V^\beta * (V^{-1})^\beta \subseteq O^\beta$. Thus, $a \in O^\beta \cap A$, $x \in \alpha Cl_\beta(A)$ and $\bigcap_{V \in \Omega} \alpha Cl_\beta(A * V^\beta * (V^{-1})^\beta) \subseteq \alpha Cl_\beta(A)$. Hence, $\alpha Cl_\beta(A) = \bigcap_{V \in \Omega} \alpha Cl_\beta(A * V^\beta * (V^{-1})^\beta)$. Similarly, $\alpha Cl_\beta(A) = \bigcap_{V \in \Omega} \alpha Cl_\beta(V^\beta * (V^{-1})^\beta * A)$.

Definition 3.28. *Let $(G, *)$ be a group and (G, τ) be a topological space. A triple $(G, *, \tau)$ is called a β - α -topological group if the inversion map is β - α -continuous and the multiplication map is jointly β - α -continuous in both variables.*

The following example is a group space in which the inversion map is β - α -continuous, but not jointly β - α -continuous.

Example 3.29. *Let $(Z, +)$ be a group of integers numbers under addition and τ be the discrete topology on Z . For each $A \in \alpha O(Z, \tau)$, we define β on $\alpha O(Z, \tau)$ by*

$$A^\beta = \begin{cases} A & \text{if } A \text{ is infinite and } Z \setminus A \text{ is finite,} \\ Z & \text{otherwise.} \end{cases}$$

Then, if $O \in \alpha O(Z, \tau)$ and $a \in Z$ such that $-a \in O$, then we have two cases:

1. If O is finite set or O is infinite set and $Z \setminus O$ is infinite, then $Z \in \alpha O(Z, \tau)$, $a \in Z$ and $-Z^\beta = Z \subseteq O^\beta = Z$.

2. If O is infinite set and $Z \setminus O$ is finite, then $-O$ is an infinite α -open set containing a , $Z \setminus -O$ is finite, and $-(-O)^\beta = -(-O) = O \subseteq O^\beta = O$.

Thus in both the cases, we obtain that inversion is β - α -continuous.

Now, let $O = \{\dots, -5, -4, -3, -1, 1, 3, 4, 5, \dots\}$, then $Z \setminus O = \{-2, 0, 2\}$ which is finite, thus $O^\beta = O \in \alpha O(G, \tau)$. By definition of O , $1 + 0 = 1 \in O$. Suppose there are α -open sets U and V with $1 \in U$, $0 \in V$ and $U^\beta + V^\beta \subseteq O^\beta = O$. Hence, U and V are must be infinite and also $Z \setminus U$ and $Z \setminus V$ are both be finite. If $a \in U$, then $-a \notin V$ as if so, $a + (-a) = 0 \in O$, but this contradicts the definition of O . Therefore, $-U \cap V = \emptyset$. Since U is α -open, U is infinite, thus $-U$ is infinite, but $-U \subseteq Z \setminus V$ implies $Z \setminus V$ is infinite. This contradicts the definition of V . Hence, there do not exist α -open sets U and V with $1 \in U$, $0 \in V$ and $U^\beta + V^\beta \subseteq O^\beta$ which proves the operation is not jointly β - α -continuous.

Example 3.30. For every group G and any operation β on $\alpha O(G, \tau)$ the indiscrete topology on G is β - α -topological group.

Theorem 3.31. Let $(G, *)$ be a group and (G, τ) be a topological space. A triple $(G, *, \tau)$ is a β - α -topological group if and only if the function $(x, y) \rightarrow x^{-1} * y$ is jointly β - α -continuous in both variables.

Proof. The proof is similar to Theorem 3.26.

Theorem 3.32. Let $(G, *, \tau)$ be a β - α -topological group. If V is an α_β -open set of G and $e \in V$, then $\alpha Cl(V) \subseteq V^2$.

Proof. Let $x \in \alpha Cl(V)$, since the inverse mapping is β - α -continuous, then V^{-1} is α_β -open containing e . By Theorem 3.5 (3), $x * V^{-1}$ is an α -open set containing x and so $x * V^{-1} \cap V \neq \emptyset$. Thus, there is $t \in V$ such that $t = x * v^{-1}$, where $v \in V$, but $x = t * v \in V * V = V^2$ and hence $\alpha Cl(V) \subseteq V^2$.

Recalling the well known definition, a subset A of a group G is symmetric if $A = A^{-1}$.

Theorem 3.33. Let $(G, *, \tau)$ be a β - α -topological group, (G, τ) be α_β -regular, β be α -monotone and $K = \{V \in \Omega : V \text{ is symmetric}\}$, then the following statements are true:

1. For any $O \in \alpha O(G, \tau)$ with $e \in O$, there exists $B \in K$ such that $B \subseteq O^\beta$.
2. $\alpha Cl(A) \subseteq A * U^\beta$, for every subset A of G and $U \in \Omega$.

Proof.

1. Let $K = \{V \in \Omega : V = V^{-1}\}$, then $K \subseteq \Omega \subseteq \alpha O(G, \tau)$. Let $O \in \alpha O(G, \tau)$ such that $e \in O$, then $e * e^{-1} \in O$, thus by Theorem 3.26, there are $V_1, V_2 \in \Omega$ such that $V_1^\beta * (V_2^\beta)^{-1} \subseteq O^\beta$. Let $V = V_1 \cap V_2$, then $e \in V$, $V \in \alpha O(G, \tau)$ and $V^\beta * V^{\beta-1} \subseteq V_1^\beta * V_2^{\beta-1} \subseteq O^\beta$. Let $B = V \cup V^{-1}$, then $e \in B$ and $B \in \alpha O(G, \tau)$ as $V \in \alpha O(G, \tau)$ implies $V^{-1} \in \alpha O(G, \tau)$. Since $e \in V$ and $e \in V^{-1} \subseteq (V^\beta)^{-1}$, so $B = V \cup V^{-1} = V * e \cup e * V^{-1} \subseteq V^\beta * (V^\beta)^{-1} \subseteq O^\beta$. Therefore, B is a symmetric α -open set and $e \in B$, thus $B \in K$ and $B \subseteq O^\beta$.
2. Since U is an α -open set with $e \in U$, then by (1), there exists $B \in K$ such that $B \subseteq U^\beta$. Now, let $r \in \alpha Cl(A)$, then $r * B$ is an α -open set containing r . Therefore, there is $a \in A \cap r * B$, that is $a \in r * B$, this implies that $a = r * b$ for some $b \in B$. Then, $r = a * b^{-1} \in a * B^{-1} = a * B \subseteq A * U^\beta$. Hence, $\alpha Cl(A) \subseteq A * U^\beta$.

Theorem 3.34. Let $(G, *, \tau)$ be a β - α -topological group. If for any $V \in \Omega$, there exists $B \in \Omega$ such that $B^\beta \subseteq V$, then for any $x \in G$ and $O \in \alpha O(G, \tau)_\beta$ containing x we have $x * B^\beta \subseteq O$.

Proof. Let $x \in G$ and $O \in \alpha O(G, \tau)_\beta$ containing x , then $e = x^{-1} * x \in x^{-1} * O$ and by Remark 3.23 and Theorem 3.5 (3), $O \in \alpha O(G, \tau)_\beta$ implies $x^{-1} * O \in \alpha O(G, \tau)$. By hypothesis there is $B \in \Omega$ such that $B^\beta \subseteq x^{-1} * O$, therefore $x \in x * B^\beta \subseteq x * x^{-1} * O = O$.

Definition 3.35. Let $(G, *)$ be a group and τ be a topology on G . A mapping $\beta : \alpha O(G, \tau) \rightarrow P(G)$ is called α_β -left operation if $(a * V)^\beta = a * V^\beta$ for all $V, a * V \in \alpha O(G, \tau)$ and $a \in G$.

Theorem 3.36. Let $(G, *, \tau)$ be a β - α -topological group, (G, τ) be α_β -regular and β be α -monotone, α -left and α_β -left operation. The group G is α - β - T_2 if and only if $\{e\}$ is α_β -closed.

Proof. Assume G is α - β - T_2 , then G is α - β - T_1 , hence singleton sets are α_β -closed. Therefore, $\{e\}$ is α_β -closed.

Conversely, let $\{e\}$ is α_β -closed. Suppose G is not α - β - T_2 , then there are $x, y \in G$ such that $x \neq y$ and if $U, V \in \alpha O(G, \tau)$ such that $x \in U$ and $y \in V$, then $U^\beta \cap V^\beta \neq \emptyset$.

Hence, $x * V$ and $y * V$ are α -open sets containing x and y , respectively, thus there exist v_1 and v_2 elements of V^β , such that $x * v_1 = y * v_2 \in x * V^\beta \cap y * V^\beta = (x * V)^\beta \cap (y * V)^\beta$ and then $y = x * v_1 * v_2^{-1} \in \{x\} * V^\beta * (V^{-1})^\beta$. Since V was selected arbitrarily, $y \in \{x\} * V^\beta * (V^{-1})^\beta$ for every $V \in \Omega$, hence $y \in \bigcap_{V \in \Omega} \{x\} * V^\beta * (V^{-1})^\beta$. By Theorem 3.27, $\alpha Cl_\beta(\{x\}) = \bigcap_{V \in \Omega} \{x\} * V^\beta * (V^{-1})^\beta$, thus $y \in \alpha Cl_\beta(\{x\})$. Since $\{e\}$ is α_β -closed, so by Theorem 3.5, $\{x\} = x * \{e\} = x * \alpha Cl_\beta(\{e\}) = \alpha Cl_\beta(x * \{e\}) = \alpha Cl_\beta(\{x\})$ implies that $\{x\}$ is α_β -closed. Then, $y \in \alpha Cl_\beta(\{x\}) = \{x\}$ implies $y = x$, but this contradicts that $x \neq y$. Therefore, G is α - β - T_2 .

Theorem 3.37. Let $(G, *, \tau)$ be a β - α -topological group, (G, τ) be α_β -regular and β be α -left operation. Then, $\{e\}$ is α_β -closed if and only if $\bigcap_{V \in \Omega} V^\beta = \{e\}$.

Proof. Assume that $\{e\}$ is α_β -closed, then by Theorem 3.17 (2), $\{e\} = \alpha Cl_\beta(\{e\}) = \bigcap_{V \in \Omega} \{e\} * V^\beta = \bigcap_{V \in \Omega} V^\beta$.

Conversely, let $\bigcap_{V \in \Omega} V^\beta = \{e\}$, then $V^\beta = \{e\} * V^\beta$ for every $V \in \Omega$ implies $\alpha Cl_\beta(\{e\}) = \bigcap_{V \in \Omega} \{e\} * V^\beta = \bigcap_{V \in \Omega} V^\beta = \{e\}$, therefore $\{e\}$ is α_β -closed.

Theorem 3.38. If $(G, *, \tau)$ is a β - α -topological group and β is identity, then the following statements are true:

1. G is weakly α_β -regular.
2. If G is α - β - T_0 , then G is α - β - T_2 .

Proof.

1. Let U be an α_β -open set and $x \in U$, then $x * e * e \in U$. Thus there is $W \in \Omega$ such that $(x * W) * W = (x * W)^\beta * W^\beta \subseteq U^\beta = U$. Since $\alpha Cl_\beta(x * W) = \bigcap_{V \in \Omega} (x * W) * V^\beta$ and $W \in \Omega$, $x \in x * W \subseteq \alpha Cl_\beta(x * W) \subseteq (x * W) * W \subseteq U$. Therefore, G is weakly α_β -regular.
2. Assume that G is α - β - T_0 and let $x, y \in G$ such that $x \neq y$, then there is an α -open set U that contains one of the points, but not the other. Assume $x \in U$, then G being weakly α_β -regular implies there is an α -open set V containing x such that $V \subseteq \alpha Cl_\beta(V) \subseteq U$. Since $G \setminus \alpha Cl_\beta(V)$ is α -open and $y \notin U$ implies $y \notin \alpha Cl_\beta(V)$, then $y \in G \setminus \alpha Cl_\beta(V)$. Thus V and $G \setminus \alpha Cl_\beta(V)$ are disjoint α -open sets containing x and y , respectively. Therefore, G is α - β - T_2 .

Theorem 3.39. If $(G, *, \tau)$ is a β - α -topological group, and if A and B are α -compact nonempty subsets of G , then $A * B$ is α - β -compact.

Proof. Let f denote multiplication map, then by Corollary 3.24, f is α - β -continuous.

Now, since A and B are α -compact in G , so $A \times B$ is α -compact in $G \times G$. Since the α - β -continuous image of an α -compact set is α - β -compact. Hence, $f(A \times B) = A * B$ and $A * B$ is α - β -compact.

Theorem 3.40. Let $(G, *, \tau)$ be a β - α -topological group. If A is α_β -closed and B is α -compact subsets of G , then $A * B$ and $B * A$ are α -closed.

Proof. If $A * B = G$, then we are done. Since $A * B = \bigcup_{b \in B} A * b$. Let $p \in \alpha D(A * B)$ and suppose $p \notin A * B$, then $p \notin A * b$. Thus, $p * b^{-1} \notin A$ and $p * b^{-1} \in G \setminus A$ for all $b \in B$. Since $G \setminus A$ is α_β -open, there exists an α -open set K such that $p * b^{-1} \in K$ and $K^\beta \subseteq G \setminus A$ and then there exist α -open sets U_b and V_b such that $p \in U_b$, $b \in V_b$ and $U_b * V_b^{-1} \subseteq U_b^\beta * (V_b^\beta)^{-1} \subseteq K^\beta \subseteq G \setminus A$ for all $b \in B$. Thus $\{V_b : b \in B\}$ is an α -open cover of B . Since B is α -compact, so $B \subseteq \bigcup_{i=1}^n V_{b_i}$ for some finite subset of $\{V_b : b \in B\}$. Let $V = \bigcup_{i=1}^n V_{b_i}$ and $U = \bigcap_{i=1}^n U_{b_i}$, then V and U are α -open, $B \subseteq V$, $p \in U$ and $U * V^{-1} \subseteq G \setminus A$. Since $p \in U$, there is $a \in A$ and $b \in B$ such that $a * b \in U$, thus $a = a * b * b^{-1} \in U * V^{-1} \subseteq G \setminus A$. Therefore $a \in A \cap G \setminus A$. Obviously this is a contradiction. Hence $p \in A * B$ and $A * B$ is α -closed. Similarly $B * A$ is α -closed.

Theorem 3.41. Let K be an α -compact subset of a β - α -topological group and β be α -open. Then, $\alpha Cl(K) \subseteq K * \alpha Cl_\beta(\{e\}) = \alpha Cl_\beta(\{e\}) * K \subseteq \alpha Cl_\beta(K)$.

Proof. By Theorem 3.40, the sets $K * \alpha Cl_\beta(\{e\})$ and $\alpha Cl_\beta(\{e\}) * K$ are α -closed. Since $K \subseteq K * \alpha Cl_\beta(\{e\}) \cup \alpha Cl_\beta(\{e\}) * K$, we have $\alpha Cl(K) \subseteq K * \alpha Cl_\beta(\{e\}) \cup \alpha Cl_\beta(\{e\}) * K$. By Theorems 3.5 (1) and 3.7 (1), we have $K * \alpha Cl_\beta(\{e\}) = \bigcup_{k \in K} k * \alpha Cl_\beta(\{e\}) = \bigcup_{k \in K} \alpha Cl_\beta(k * \{e\}) = \bigcup_{k \in K} \alpha Cl_\beta(\{k\})$
 $= \bigcup_{k \in K} \alpha Cl_\beta(\{e\} * k) = \bigcup_{k \in K} \alpha Cl_\beta(\{e\}) * k = \alpha Cl_\beta(\{e\}) * K$.
 And also, we have $K * \alpha Cl_\beta(\{e\}) \cup \alpha Cl_\beta(\{e\}) * K \subseteq \alpha Cl_\beta(K * \{e\}) \cup \alpha Cl_\beta(\{e\} * K) = \alpha Cl_\beta(K)$. Hence, $\alpha Cl(K) \subseteq K * \alpha Cl_\beta(\{e\}) = \alpha Cl_\beta(\{e\}) * K \subseteq \alpha Cl_\beta(K)$.

Theorem 3.42. Let A and B be nonempty subsets of a β - α -topological group $(G, *, \tau)$ and β be identity, then:

1. If A and B are α_β -connected, then $A * B$ is α_β -connected.
2. If B is α_β -connected and $A * b$ is α_β -connected for some $b \in B$, then $A * B$ is α_β -connected.
3. If B is α_β -connected and $A \subseteq B^{-1}$, then $e \in A * B$ and $A * B$ is α_β -connected.
4. If A is α_β -connected, then $A^{-1} * A$ and $A * A^{-1}$ are α_β -connected.

Proof.

1. Let f denote multiplication, then $f : (G, \tau) \times (G, \tau) \rightarrow (G, \tau)$ defined by $f(a, b) = a * b$ for all $a, b \in G$ and so f is α - β -continuous implies f is $\alpha_{(\beta, \beta)}$ -continuous. Since A and B are α_β -connected in G , so $A \times B$ is α_β -connected in $G \times G$. Since the $\alpha_{(\beta, \beta)}$ -continuous image of an α_β -connected set is α_β -connected. Hence, $f(A \times B) = A * B$ and $A * B$ is α_β -connected.
2. Suppose A is not α_β -connected, then there exist α_β -separated sets U and V such that $A = U \cup V$, thus $A * b = U * b \cup V * b$. If $x \in U * b \cap V * b$, then $x = u * b$ and $x = v * b$ for some $u \in U$ and $v \in V$, $x = u * b = v * b$ implies $u = v$. This contradicts that U and V are disjoint, therefore $U * b \cap V * b = \phi$. Since $\alpha Cl_\beta(U * b) = \alpha Cl_\beta(U) * b$ and $\alpha Cl_\beta(V * b) = \alpha Cl_\beta(V) * b$, a proof similar to the above achieves $\alpha Cl_\beta(U * b) \cap V * b = \phi$ and $U * b \cap \alpha Cl_\beta(V * b) = \phi$. Thus, $U * b$ and $V * b$ are α_β -separated sets whose union is $A * b$, but this contradicts the fact that $A * b$ is α_β -connected. Therefore, A is α_β -connected. Since A and B are α_β -connected, by (1) $A * B$ is α_β -connected.
3. Let $x \in A$, then $x^{-1} \in B$ implies $e = x * x^{-1} \in A * B$.
 Let $x, y \in A$, then $\{x\}, \{y\}$ and B are α_β -connected, thus $\{x\} * B = x * B$ and $\{y\} * B = y * B$ are α_β -connected by (1). Since $x, y \in A$, $x^{-1}, y^{-1} \in B$ and $e = x * x^{-1} = y * y^{-1} \in x * B \cap y * B$. Thus $x * B$ and $y * B$ are not α_β -separated for every $x, y \in A$. Hence $\{x * B : x \in A\}$ is a collection of α_β -connected subsets of G such that no two members are α_β -separated. Therefore, $\bigcup_{x \in A} x * B = A * B$ is α_β -connected.
4. Since inversion is $\alpha_{(\beta, \beta)}$ -continuous and A is α_β -connected, so A^{-1} is α_β -connected. By (1), A and A^{-1} α_β -connected implies $A^{-1} * A$ and $A * A^{-1}$ are α_β -connected.

References

- [1] A. V. Arhangel'skii, M. Tkachenko, Topological Groups and Related Structures, Atlantis Studies in Mathematics, Atlantis Press/World Scientific, AmsterdamParis, 1 (2008). <https://doi.org/10.2991/978-94-91216-35-0>
- [2] A. B. Khalaf and H. Z. Ibrahim, Some properties of operations on $\alpha O(X)$, *International Journal of Mathematics and Soft Computing*, 6 (1) (2016), 107-120.
- [3] A. B. Khalaf, S. Jafari and H. Z. Ibrahim, Bioperations on α -open sets in topological spaces, *International Journal of Pure and Applied Mathematics*, 103 (4) (2015), 653-666. <https://doi.org/10.12732/ijpam.v103i4.5>
- [4] A. B. Khalaf, A. K. Kaymakci and H. Z. Ibrahim, Operation-separation axioms via α -open sets, *Acta Universitatis Apulensis*, (47) (2016), 99-115.
- [5] A. B. Khalaf and H. Z. Ibrahim, Weakly α_γ -regular and weakly α_γ -normal spaces, *FACTA UNIVERSITATIS, Ser. Math. Inform.*, (2017).
- [6] A. B. Khalaf and H. Z. Ibrahim, α - γ -convergence, α - γ -accumulation and α - γ -compactness, *Commun. Fac. Sci. Univ. Ank. Sr. A1 Math. Stat.*, 66 (1) (2017), 43-50. <https://doi.org/10.1501/commua10000000773>

- [7] A. B. Khalaf and H. Z. Ibrahim, Some new functions via operations defined on α -open sets, *Journal of Garmian University*, no. 12 (2017). <https://doi.org/10.24271/garmian.10>
- [8] A. B. Khalaf and H. Z. Ibrahim, α_γ -connectedness and some properties of $\alpha_{(\gamma,\beta)}$ -continuous functions, *Accepted in The First International Conference of Natural Science (ICNS) from 11th–12th July 2016*, Charmo University.
- [9] H. Z. Ibrahim, On a class of α_γ -open sets in a topological space, *Acta Scientiarum. Technology*, 35 (3) (2013), 539-545. <https://doi.org/10.4025/actascitechnol.v35i3.15788>
- [10] D. Jankovic , I. J. Reilly and M. K. Vamanamurthy , On strongly compact topological spaces, *Question and answer in General Topology*, 6 (1) (1988).
- [11] O. Njastad, On some classes of nearly open sets, *Pacific J. Math.* 15 (1965), 961-970. <https://doi.org/10.2140/pjm.1965.15.961>
- [12] M. Tkachenko, Paratopological and semitopological groups vs topological groups, *In: K.P. Hart, J. van Mill, P. Simon (eds.), Recent Progress in General Topology III, Atlantis Press*, (2014), 803-859. https://doi.org/10.2991/978-94-6239-024-9_20