A six-step Block Unification Integrator for Numerical Solution of Fourth Order Boundary Value Problems

Mark I. Modebei ^{*1} and Raphael B. Adeniyi²

^{1,2} Department of Mathematics, University of Ilorin, P.M.B 1515, Ilorin, Nigeria.

¹gmarc.ify@gmail.com

Abstract. In this paper, a new 7th order continuous finite difference methods is proposed. These methods are derived using the Chebyshev polynomials as basis functions. The collocation approach is employed to obtain the main methods and additional methods used for solving general nonlinear fourth order two and four-points boundary value problems. Several numerical examples are shown to illustrate the strength of the method. To show the robustness of this method for high accuracy, we applied the method of line to discretize PDEs into system of fourth order ODEs and thus use the derived method to obtain approximate solution for the PDEs. The approximate solution obtained using the proposed methods is compared to the exact solutions of the problem, and other methods from existing literature. The Convergence of these methods is also guaranteed.

Keywords: *Block Methods; Convergence; Finite Difference; Linear multistep.*

2010 MSC No: 65L05, 65L06, 65L10, 65L12.

1 Introduction

In this work, the numerical approximation for the μ th-order problems of the type:

$$y^{(iv)} = f(x, y, y', \dots, y'''), \quad a \leq x \leq b \quad (1)$$

$$\left. \begin{aligned} y(a) &= \alpha_0, \quad y(c_0) = \gamma_0 \\ y'(c_1) &= \gamma_1, \quad y'(b) = \beta_1 \end{aligned} \right\} \quad (2)$$

where a, b, α_i, β_i and $\gamma_i, i = 0, 1, 2$ are real numbers and $c_0, c_1 \in (a, b)$ Other forms of boundary conditions considered are $y(a) = \alpha_0, y(c_0) = \gamma_0, y(c_1) = \gamma_1, y(b) = \beta_1$ and $y(a) = \alpha_0, y'(c_0) = \gamma_0, y''(c_1) = \gamma_1, y'''(b) = \beta_1$.

Problems arising from engineering and other sciences are modeled into linear and nonlinear of at least two points boundary value problems. Modeled problems with more than two points boundary conditions are referred to as multi-point boundary value problems (MBVPs). This kind of problem arises in different applied mathematics and physics. For example, MBVP models the flow of fluid such as water, oil and gas through ground layers, where each layer constitutes a subdomain, see [29]. The vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities, see [12]. Problems in the theory of elastic stability can be modeled by multi-point problems, large size bridges are sometimes constructed with multi-point supports which correspond to a multi-point boundary value condition, see [8, 9, 10].

^{*}Corresponding author. Mark I. Modebei ¹ gmarc.ify@gmail.com

So many numerical methods have been developed for the solutions of fourth order BVPs include Optimal Homotopy Asymptotic Method (OHAM) see [6], Quasi-Newton and reproducing kernel method, see [29]. Siddiqi and Akram [26, 27] used the Quintic Spline Solutions for Fourth Order Boundary-Value Problems. The Adomian decomposition method for solving multipoint boundary value problems, was also used in [28], Runge-Kutta Type Methods for Directly Solving Special Fourth-Order Ordinary Differential Equations was also used in [14, 16]. Readers may also see [3],[4],[7] and [22], just to mention few.

we derive a continuous linear multistep method (LMM) via a block unification technique, which is used to formulate a six-step block unification integrator (SBUI) using Chebyshev polynomials as basis functions with the collocation approach. Continuous LMM is derived and additional methods which are assembled and solved sequentially to obtain approximations y_n , to the solution of (1) directly at the points $\{x\}_i$.

The paper is organized as follows. In Section 2, we derive an approximation $Y(x)$ for the exact solution $y(x)$ which is continuous. Further more we state the specification of the methods and how the SBUI is obtained. In section 3 we discussed the order, local truncation error and convergence of the method and in section 4, we give some numerical examples to show the accuracy and reliability of the method. In the last section, we discuss the conclusion.

2 Derivation of the Methods

Here, the exact solution $y(x)$ is approximated by seeking the continuous method $Y(x)$ of the form

$$Y(x) = \sum_{i=0}^{r+s-1} \rho_i T_i(x) \quad (3)$$

with the fourth derivative given by

$$Y^{(iv)}(x) = \sum_{i=0}^{r+s-1} \rho_i T_i^{(iv)}(x) \quad (4)$$

where $x \in [a, b]$, ρ_i 's are coefficient to be determined, and $T_i(x)$ are Chebyshev polynomial basis function of degree $r + s - 1$, r is the number of interpolation points that satisfies $4 < r \leq k$ and s is the number of collocation points satisfying $0 < s \leq k + 1$. and k indicates the number of steps and in this case, $k \geq 4$. We thus impose the following conditions as follows;

$$Y(x_{n+i}) = y_{n+i}, \quad i = 0, 1, 2, \dots, r-1 \quad (5)$$

$$Y^{(iv)}(x) = DY(x_{n+i}) = f_{n+i}, \quad i = 0, 1, 2, \dots, s-1 \quad (6)$$

where $D = \frac{d^4}{dx^4}$

Interpolating (3) at x_{n+i} ; $i = 0, 1, 2, \dots, r-1$ and collocating (4) at x_{n+s} ; $s = 0, 1, 2, \dots, k$. which leads to the following systems of equations

$$\sum_{i=0}^{r+s-1} \rho_i T_i(x) = y_{n+i} \quad (7)$$

$$\sum_{i=0}^{r+s-1} \rho_i T_i^{(iv)}(x) = f_{n+i} \quad (8)$$

where ρ_i 's are coefficients of the Chebyshev series. Thus we define the following interpolation and collocation in a

single matrix as follows

$$A = \begin{pmatrix} T_0(x_n) & T_1(x_n) & \cdot & \cdot & \cdot & T_{r+s-1}(x_n) \\ T_0(x_{n+1}) & T_1(x_{n+1}) & \cdot & \cdot & \cdot & T_{r+s-1}(x_{n+1}) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ T_0(x_{n+r-1}) & T_1(x_{n+r-1}) & \cdot & \cdot & \cdot & T_{r+s-1}(x_{n+r-1}) \\ T_0^{(iv)}(x_n) & T_1^{(iv)}(x_n) & \cdot & \cdot & \cdot & T_{r+s-1}^{(iv)}(x_n) \\ T_0^{(iv)}(x_{n+1}) & T_1^{(iv)}(x_{n+1}) & \cdot & \cdot & \cdot & T_{r+s-1}^{(iv)}(x_{n+1}) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ T_0^{(iv)}(x_{n+s-1}) & T_1^{(iv)}(x_{n+s-1}) & \cdot & \cdot & \cdot & T_{r+s-1}^{(iv)}(x_{n+s-1}) \end{pmatrix}$$

$$B = (\rho_0 \quad \rho_1 \quad \cdot \quad \cdot \quad \cdot \quad \rho_{r+s-1})^T$$

$$C = (y_n \quad y_{n+1} \quad \cdot \quad \cdot \quad \cdot \quad y_{n+r-1} \quad f_n \quad f_n \quad \cdot \quad \cdot \quad \cdot \quad f_{n+s-1})^T$$

$$E = (T_0(x) \quad T_1(x) \quad \cdot \quad \cdot \quad \cdot \quad T_{r+s-1}(x))^T$$

Then the system (7) and (8) is thus

$$A \cdot B = C \quad (9)$$

Hence we state the following theorem

Theorem 2.1. *Let*

$$Y(x_{n+i}) = y_{n+i}, \quad i = 0, 1, 2, \dots, r-1 \quad (10)$$

$$Y^{(iv)}(x) = DY(x_{n+i}) = f_{n+i}, \quad i = 0, 1, 2, \dots, s-1 \quad (11)$$

be satisfied, the continuous k -step LMM (17) is derived from the equation

$$Y(x) = C^T(A^{-1})^T E \quad (12)$$

Proof. Given the continuous scheme

$$Y(x) = \sum_{i=0}^{r+s-1} \rho_i T_i(x) \quad (13)$$

where $x \in [a, b]$, ρ_i 's are unknown coefficients, $T_i(x)$'s are the Chebyshev polynomial basis functions of degree $r+s-1$. We can clearly write (13) as

$$Y(x) = \rho_0 T_0(x) + \rho_1 T_1(x) + \rho_2 T_2(x) + \dots + \rho_{r+s-2} T_{r+s-2}(x) + \rho_{r+s-1} T_{r+s-1}(x) \quad (14)$$

Then (14) can be written compactly in vector form as

$$Y(x) = B^T E \quad (15)$$

From (9), by left inverse cancelation law we have

$$B = A^{-1} C \quad (16)$$

Hence, by (15) we have

$$Y(x) = C^T(A^{-1})^T E$$

as required.

The equation (14) takes the form

$$Y(x) = \sum_{i=0}^{r-1} \alpha_i(t) h^i y_n^{(i)} + h^4 \sum_{i=0}^{s-1} \beta_i(t) f_{n+i} \quad (17)$$

where $\alpha_i(t)$ and $\beta_i(t)$ are continuous coefficients. The solution of (1) is sought on the partition

$$\pi N : a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots < x_N = b$$

of the integration interval $[a, b]$ with a constant step size $h = \frac{b-a}{N}$, N is the number of subinterval of integration.

It is note worthy that the continuous methods (17) and its derivatives $Y'(x)$, $Y''(x)$ and $Y'''(x)$ will be used to produce the main and additional methods which gives a total of 24 equations and are combined to provide all approximations on the entire interval for boundary value problems as stated in (1).

Applying the theorem (2.1), with $k = 6$, $r = 4$, $s = 7$, and obtain the continuous coefficients $\alpha_i(x)$, $\beta_i(x)$, expressed as functions of t (whose expressions are not included), where $th = x - x_{k-1}$. By evaluating (17) at $x = x_{n+i}$ $i = 1(1)6$, the following 6-step LMM are obtained;

$$\begin{aligned} y_{n+1} &= y_n + \frac{3hy'_n}{2} + \frac{9}{8}h^2y''_n + \frac{9}{16}h^3y'''_n - h^4 \left(\frac{95929f_n}{3628800} - \frac{4001f_{n+1}}{129600} + \frac{101729f_{n+2}}{725760} \right. \\ &\quad \left. - \frac{811f_{n+3}}{30240} + \frac{10693f_{n+4}}{725760} - \frac{4219f_{n+5}}{907200} - \frac{2323f_{n+6}}{3628800} \right) \\ y_{n+2} &= y_n + 3hy'_n + \frac{9}{2}h^2y''_n + \frac{9}{2}h^3y'''_n + h^4 \left(\frac{4127f_n}{14175} + \frac{8782f_{n+1}}{14175} - \frac{6425f_{n+2}}{9072} \right. \\ &\quad \left. + \frac{388f_{n+3}}{945} - \frac{127f_{n+4}}{567} + \frac{998f_{n+5}}{14175} - \frac{137f_{n+6}}{14175} \right) \\ y_{n+3} &= y_n + \frac{9hy'_n}{2} + \frac{81}{8}h^2y''_n + \frac{243}{16}h^3y'''_n + h^4 \left(\frac{49239f_n}{44800} + \frac{34263f_{n+1}}{11200} - \frac{56129f_{n+2}}{26880} \right. \\ &\quad \left. + \frac{261f_{n+3}}{160} - \frac{8019f_{n+4}}{8960} + \frac{3159f_{n+5}}{11200} - \frac{1737f_{n+6}}{44800} \right) \\ y_{n+4} &= y_n + 4hy'_n + 8h^2y''_n + \frac{32}{3}h^3y'''_n + h^4 \left(\frac{628358656f_n}{837019575} + \frac{543924224f_{n+1}}{279006525} - \frac{114992431f_{n+2}}{74401740} \right. \\ &\quad \left. + \frac{184795136f_{n+3}}{167403915} - \frac{33762304f_{n+4}}{55801305} + \frac{843776f_{n+5}}{4428675} - \frac{21911552f_{n+6}}{837019575} \right) \\ y_{n+5} &= y_n + 5hy'_n + \frac{25}{2}h^2y''_n + \frac{125}{6}h^3y'''_n + h^4 \left(\frac{51613375f_n}{33480783} + \frac{803750f_{n+1}}{177147} - \frac{480390557f_{n+2}}{178564176} \right. \\ &\quad \left. + \frac{77537500f_{n+3}}{33480783} - \frac{4709375f_{n+4}}{3720087} + \frac{4453250f_{n+5}}{11160261} - \frac{1836875f_{n+6}}{33480783} \right) \\ y_{n+6} &= y_n + 6hy'_n + 18h^2y''_n + 36h^3y'''_n + h^4 \left(\frac{7808f_n}{2835} + \frac{124928f_{n+1}}{14175} - \frac{90733f_{n+2}}{22680} \right. \\ &\quad \left. + \frac{4096f_{n+3}}{945} - \frac{928f_{n+4}}{405} + \frac{2048f_{n+5}}{2835} - \frac{1408f_{n+6}}{14175} \right) \end{aligned} \quad (18)$$

Evaluating $Y'(x)$ of (17) at the points $x = x_{n+i}$ $i = 1(1)6$, we obtain the first derivative additional method as;

$$\begin{aligned} y'_{n+1} &= y'_n + hy''_n + \frac{1}{2}h^2y'''_n + h^3 \left(\frac{18685714f_n}{837019575} + \frac{6227992f_{n+1}}{279006525} - \frac{17182393f_{n+2}}{178564176} \right. \\ &\quad \left. + \frac{97504f_{n+3}}{4782969} - \frac{627026f_{n+4}}{55801305} + \frac{991432f_{n+5}}{279006525} - \frac{409901f_{n+6}}{837019575} \right) \\ y'_{n+2} &= y'_n + 2hy''_n + 2h^2y'''_n + h^3 \left(\frac{106516688f_n}{837019575} + \frac{67716224f_{n+1}}{279006525} - \frac{50951657f_{n+2}}{178564176} \right. \\ &\quad \left. + \frac{29563648f_{n+3}}{167403915} - \frac{5369008f_{n+4}}{55801305} + \frac{8434304f_{n+5}}{279006525} - \frac{496096f_{n+6}}{119574225} \right) \\ y'_{n+3} &= y'_n + 3hy''_n + \frac{9}{2}h^2y'''_n + h^3 \left(\frac{13774f_n}{42525} + \frac{11992f_{n+1}}{14175} - \frac{5875f_{n+2}}{9072} + \frac{832f_{n+3}}{1701} \right. \\ &\quad \left. - \frac{758f_{n+4}}{2835} + \frac{1192f_{n+5}}{14175} - \frac{491f_{n+6}}{42525} \right) \\ y'_{n+4} &= y'_n + 4hy''_n + 8h^2y'''_n + h^3 \left(\frac{510727936f_n}{837019575} + \frac{526440448f_{n+1}}{279006525} - \frac{904408973f_{n+2}}{892820880} \right. \\ &\quad \left. + \frac{153915392f_{n+3}}{167403915} - \frac{28274432f_{n+4}}{55801305} + \frac{6375424f_{n+5}}{39858075} - \frac{18420224f_{n+6}}{837019575} \right) \\ y'_{n+5} &= y'_n + 5hy''_n + \frac{25}{2}h^2y'''_n + h^3 \left(\frac{33034250f_n}{33480783} + \frac{5357000f_{n+1}}{1594323} - \frac{159598651f_{n+2}}{127545840} \right. \\ &\quad \left. + \frac{51980000f_{n+3}}{33480783} - \frac{9271250f_{n+4}}{11160261} + \frac{2921000f_{n+5}}{11160261} - \frac{1205125f_{n+6}}{33480783} \right) \\ y'_{n+6} &= y'_n + 6hy''_n + 18h^2y'''_n + h^3 \left(\frac{61808f_n}{42525} + \frac{74624f_{n+1}}{14175} - \frac{12227f_{n+2}}{9072} + \frac{4352f_{n+3}}{1701} \right. \\ &\quad \left. - \frac{496f_{n+4}}{405} + \frac{5504f_{n+5}}{14175} - \frac{2272f_{n+6}}{42525} \right) \end{aligned} \quad (19)$$

Similarly, evaluating $Y''(x)$ of (17) at the points $x = x_{n+i}$ $i = 1(1)6$, we obtain the second derivative additional method as;

$$\begin{aligned}
y''_{n+1} &= y''_n + hy'''_n + h^2 \left(\frac{1093814f_n}{18600435} + \frac{55528f_{n+1}}{688905} - \frac{509932f_{n+2}}{6200145} + \frac{184016f_{n+3}}{2657205} \right. \\
&\quad \left. - \frac{15698f_{n+4}}{413343} + \frac{74248f_{n+5}}{6200145} - \frac{6128f_{n+6}}{3720087} \right) \\
y''_{n+2} &= y''_n + 2hy'''_n + h^2 \left(\frac{563008f_n}{3720087} + \frac{487040f_{n+1}}{1240029} - \frac{612704f_{n+2}}{2066715} + \frac{908032f_{n+3}}{3720087} \right. \\
&\quad \left. - \frac{821728f_{n+4}}{6200145} + \frac{85888f_{n+5}}{2066715} - \frac{15136f_{n+6}}{2657205} \right) \\
y''_{n+3} &= y''_n + 3hy'''_n + h^2 \left(\frac{2054f_n}{8505} + \frac{776f_{n+1}}{945} - \frac{32f_{n+2}}{81} + \frac{3152f_{n+3}}{8505} - \frac{194f_{n+4}}{945} + \frac{184f_{n+5}}{2835} - \frac{76f_{n+6}}{8505} \right) \\
y''_{n+4} &= y''_n + 4hy'''_n + h^2 \left(\frac{6161024f_n}{18600435} + \frac{520192f_{n+1}}{413343} - \frac{1939456f_{n+2}}{6200145} + \frac{9396224f_{n+3}}{18600435} - \frac{21248f_{n+4}}{76545} \right. \\
&\quad \left. + \frac{77824f_{n+5}}{885735} - \frac{45056f_{n+6}}{3720087} \right) \\
y''_{n+5} &= y''_n + 5hy'''_n + h^2 \left(\frac{1568950f_n}{3720087} + \frac{299000f_{n+1}}{177147} - \frac{2500f_{n+2}}{15309} + \frac{2950000f_{n+3}}{3720087} - \frac{458750f_{n+4}}{1240029} \right. \\
&\quad \left. + \frac{47800f_{n+5}}{413343} - \frac{59000f_{n+6}}{3720087} \right) \\
y''_{n+6} &= y''_n + 6hy'''_n + h^2 \left(\frac{4352f_n}{8505} + \frac{6016f_{n+1}}{2835} - \frac{32f_{n+2}}{945} + \frac{10496f_{n+3}}{8505} - \frac{32f_{n+4}}{81} + \frac{128f_{n+5}}{945} - \frac{32f_{n+6}}{1701} \right)
\end{aligned} \tag{20}$$

Lastly, evaluating $Y'''(x)$ of (17) at the points $x = x_{n+i}$ $i = 1(1)6$, we obtain the third derivative additional method

$$\begin{aligned}
y'''_{n+1} &= y'''_n + h \left(\frac{1656502f_n}{18600435} + \frac{1251248f_{n+1}}{6200145} - \frac{383918f_{n+2}}{2066715} + \frac{2847328f_{n+3}}{18600435} - \frac{515594f_{n+4}}{6200145} + \frac{17968f_{n+5}}{688905} \right. \\
&\quad \left. - \frac{66506f_{n+6}}{18600435} \right) \\
y'''_{n+2} &= y'''_n + h \left(\frac{341872f_n}{3720087} + \frac{163712f_{n+1}}{413343} - \frac{1180288f_{n+2}}{6200145} + \frac{3013376f_{n+3}}{18600435} - \frac{183056f_{n+4}}{2066715} + \frac{172672f_{n+5}}{6200145} \right. \\
&\quad \left. - \frac{71104f_{n+6}}{18600435} \right) \\
y'''_{n+3} &= y'''_n + h \left(\frac{2278f_n}{25515} + \frac{752f_{n+1}}{1701} + \frac{22f_{n+2}}{8505} + \frac{2656f_{n+3}}{25515} - \frac{538f_{n+4}}{8505} + \frac{176f_{n+5}}{8505} - \frac{74f_{n+6}}{25515} \right) \\
y'''_{n+4} &= y'''_n + h \left(\frac{1680256f_n}{18600435} + \frac{2673152f_{n+1}}{6200145} + \frac{10624f_{n+2}}{76545} + \frac{3607552f_{n+3}}{18600435} - \frac{539456f_{n+4}}{6200145} + \frac{54784f_{n+5}}{2066715} \right. \\
&\quad \left. - \frac{66944f_{n+6}}{18600435} \right) \\
y'''_{n+5} &= y'''_n + h \left(\frac{336190f_n}{3720087} + \frac{178000f_{n+1}}{413343} + \frac{178750f_{n+2}}{1240029} + \frac{1420000f_{n+3}}{3720087} - \frac{1250f_{n+4}}{15309} + \frac{32240f_{n+5}}{1240029} \right. \\
&\quad \left. - \frac{13250f_{n+6}}{3720087} \right) \\
y'''_{n+6} &= y'''_n + h \left(\frac{2288f_n}{25515} + \frac{3712f_{n+1}}{8505} + \frac{1024f_{n+2}}{8505} + \frac{12032f_{n+3}}{25515} + \frac{464f_{n+4}}{8505} + \frac{128f_{n+5}}{8505} - \frac{64f_{n+6}}{25515} \right)
\end{aligned} \tag{21}$$

The main method (18) and the additional methods (19)-(21) together for the block unification integrator that will be assembled to form a single block and would be used in sequence over the interval $[x_n, x_{n+6}]$, $n = 0, 6, N - 6$, N is the number of subinterval which must be a multiple of 6.

3 Order and Truncation Error And Convergence

3.1 Local truncation error

The linear differential operator $\mathcal{L}[z(x); h]$ defined by

$$\mathcal{L}[z(x); h] = z(x + ih) - \sum_{i=0}^3 \alpha_i h^i z(x + ih) - h^4 \sum_{i=0}^6 \beta_i z^{(iv)}(x + ih) \tag{22}$$

Expanding (22) in Taylor series, we obtain

$$\mathcal{L}[y(x); h] = C_0 z(x) + C_1 h z'(x) + C_2 h^2 z''(x) + \cdots + C_p h^p z^{(p)}(x) + O(h^{(p+1)}) \tag{23}$$

where C_j are constants such that

$$C_0 = C_1 = C_2 = \cdots = C_{p+3} = 0, C_{p+4} \neq 0$$

so that

$$\mathcal{L}[z(x); h] = C_{p+4}h^{p+4}z^{(p+4)}(x) + O(h^{(p+5)}) \quad (24)$$

In this case, C_{p+4} is the error constant (see Lambert [15]). A linear difference operator $\mathcal{L}[z(x); h]$ is said to be consistent of order p if

$$\mathcal{L}[z(x); h] = O(h^{(p+4)})$$

The error constants $C_{p+4}^{\tau_j^{(\mu)}}$ for $j = 1(1)6$, $\mu = 0, 1, 2, 3$ are given below.

$$\begin{aligned} C_{p+4}^{\tau_{1,2,3,4,5,6}} &= \left(-\frac{1}{3024}, -\frac{1}{756}, -\frac{1}{336}, -\frac{229}{665280}, \frac{107}{997920}, \frac{1273}{9979200} \right)^T \\ C_{p+4}^{\tau_{1,2,3,4,5,6}} &= \left(-\frac{617}{3326400}, -\frac{761}{1425600}, -\frac{1427}{997920}, -\frac{1549}{665280}, \frac{41}{14400}, -\frac{41}{302400} \right)^T \\ C_{p+4}^{\tau_{1,2,3,4,5,6}} &= \left(\frac{1}{43200}, -\frac{1}{3024}, -\frac{23}{33600}, -\frac{53}{100800}, -\frac{1061}{302400}, -\frac{8737}{907200} \right)^T \\ C_{p+4}^{\tau_{1,2,3,4,5,6}} &= \left(\frac{3151}{1814400}, -\frac{151}{129600}, \frac{751}{1814400}, \frac{151}{129600}, \frac{3151}{1814400}, -\frac{8737}{907200} \right)^T \end{aligned} \quad (25)$$

In what follows, the convergence of the method is established in theorem:

Theorem 3.1 ([18]). *Let $\bar{Y}$ be the approximate solution vector of the exact solution Y for the system formed by combining the methods (18) through (21). Let $e_i = |y(x_i) - y_i|$, $e'_i = |y'(x_i) - y'_i|$, $e''_i = |y''(x_i) - y''_i|$, $e'''_i = |y'''(x_i) - y'''_i|$ and $e_i^{(iv)} = |y^{(iv)}(x_i) - y_i^{(iv)}|$ be defined as the error for $i = 1, \dots, N$ where the exact solution $y(x) \in C^n[a, b]$. Define $\|E\|_\infty = \|y - \bar{y}\|_\infty$, then the method is seventh-order convergent. That is $\|E\|_\infty = O(h^7)$.*

Proof We write (18) through (21) in the exact form

$$AY - h^5 BF(Y) + \bar{C} + L(h) = 0 \quad (26)$$

where A is an $4N \times 4N$ coefficient matrix defined by

$$A = \begin{bmatrix} A_{11} & \cdots & A_{14} \\ \vdots & \ddots & \vdots \\ A_{41} & \cdots & A_{44} \end{bmatrix}$$

A_{ij} are $N \times N$ matrices for $i = 1(1)6$, $j = 1$, and $A_{22} = A_{33} = A_{44} = A_{55} = I$, where I is an $N \times N$ identity matrices and $A_{ij} = 0$ for $i = 1(1)6$; $j = 2(1)6$; $i \neq j$.

Similarly, B is a $4N \times 4N$ coefficient matrix defined by

$$B = \begin{bmatrix} B_{11} & \cdots & B_{14} \\ \vdots & \ddots & \vdots \\ B_{41} & \cdots & B_{44} \end{bmatrix}$$

with B_{ij} an $N \times N$ matrices for $i = 1(1)6$, $j = 1$, and B_{ij} are an $N \times$ zero matrix for $i = 1(1)6$; $j = 2(1)6$; and

$$\bar{C} = \left(-hy'_0 - \frac{11y_0}{6} - \frac{193hf_0}{20160}, -h^2y''_0 + 2y_0 + \frac{4463h^2f_0}{60480}, -h^3y'''_0 - y_{0,m} - \frac{53h^3f_0}{160}, +y_0 - \frac{h^4f_0}{720}, -y_0 + \frac{-31h^4f_0}{60480}, \frac{1}{6}y_0 + \frac{11h^4f_0}{12096}, -\frac{1}{3}y_0 + \frac{-23h^4f_0}{20160}, \frac{11}{6}y_0 - \frac{151h^4f_0}{60480}, y_0 + \frac{-157h^4f_0}{60480}, \frac{11h^4f_0}{60480}, -y_0 + \frac{-73h^4f_0}{60480}, y_0 - \frac{409h^4f_0}{60480}, -y_0 + \frac{5h^4f_0}{288}, -y_0 + \frac{-13h^4f_0}{1440}, -y_0 + \frac{h^4f_0}{160}, y_0 - \frac{h^4f_0}{1440}, 0, \dots, 0 \right)$$

$L(h)$ is the truncation error of the formulas (18) through (21), defined as

$$L(h) = (C^{\tau_1}, \dots, C^{\tau_N}, C^{\tau'_1}, \dots, C^{\tau'_N}, C^{\tau''_1}, \dots, C^{\tau''_N}, C^{\tau'''_1}, \dots, C^{\tau'''_N}, C^{\tau^{(iv)}_1}, \dots, C^{\tau^{(iv)}_N})^T$$

and

$$Y = (y(x_1), \dots, y(x_N), y'(x_1), \dots, y'(x_N), y''(x_1), \dots, y''(x_N), y'''(x_1), \dots, y'''(x_N), y^{(iv)}(x_1), \dots, y^{(iv)}(x_N))^T$$

$$F(Y) = (f_1, \dots, f_N, f'_1, \dots, f'_N, f''_1, \dots, f''_N, f'''_1, \dots, f'''_N, f_1^{(iv)}, \dots, f_N^{(iv)})^T$$

The approximate form of the system is given as

$$A\bar{Y} - h^5 BF(\bar{Y}) + \bar{C} = 0 \quad (27)$$

where $\bar{Y}$ is the approximate solution of the vector Y Define the vector E as

$$E = \bar{Y} - Y = (e_1, \dots, e_N, e'_1, \dots, e'_N, e''_1, \dots, e''_N, e'''_1, \dots, e'''_N, e_1^{(iv)}, \dots, e_N^{(iv)})^T$$

where

$$\bar{Y} = (y_1, \dots, y_N, y'_1, \dots, y'_N, y''_1, \dots, y''_N, y'''_1, \dots, y'''_N, y_1^{(iv)}, \dots, y_N^{(iv)})^T$$

Subtracting (26) from (27) and applying the mean value theorem, we obtain the error system as

$$(A - BJ_f)E = L(h) \quad (28)$$

where J_f is the Jacobian matrix

$$J_f = \begin{pmatrix} J_{11} & \cdots & J_{14} \\ \vdots & & \vdots \\ J_{41} & \cdots & J_{44} \end{pmatrix}$$

whose entries J_{ij} are $N \times N$ matrices given as $J_{11} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_N} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial y_1} & \cdots & \frac{\partial f_N}{\partial y_N} \end{pmatrix};$

$$J_{1,j+1} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_1}{\partial y_N^{(j)}} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_N}{\partial y_N^{(j)}} \end{pmatrix} \text{ for } j = 1, 2, 3;$$

$$J_{i,j+1} = h^i \begin{pmatrix} \frac{\partial f_1^{(i-1)}}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_1^{(i-1)}}{\partial y_N^{(j)}} \\ \vdots & & \vdots \\ \frac{\partial f_N^{(i-1)}}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_N^{(i-1)}}{\partial y_N^{(j)}} \end{pmatrix}; \text{ for } j = 0, 1, 2, 3. \text{ and for } i = 2, \dots, 4.$$

Let $M = -BJ_f$ be a matrix of dimension $4N$ so that (28) becomes

$$(A + M)E = L(h) \quad (29)$$

and for sufficiently small h , $A + M$ is a monotone matrix and thus nonsingular (see [?]). Hence

$$\begin{aligned} (A + M)^{-1} &= D = (d_{ij}) \geq 0 \\ \sum_{j=1}^{4M} d_{ij} &= O(h^{-4}), \\ E &= DL(h), \\ \|E\| &= \|DL(h)\| = O(h^{-4})O(h^{11}) \\ &= O(h^7) \end{aligned} \quad (30)$$

which shows that the method is 7th order convergent, that is, with a global error of order $O(h^7)$.

Implementation: Our main method (18) through (21) is implemented efficiently by combining them in a single matrix equation simultaneously of the form

$$A_0 V_\mu = A_1 V_{\mu-1} + h^4 B_1 F_{\mu-1} + h^4 B_0 F_\mu, \quad \mu = 1, \dots, \Gamma, \quad n = 0, 4, \dots, N-6 \quad (31)$$

where

$$\begin{aligned} V_\mu &= (y_1, \dots, y_{n+6}, y'_1, \dots, y'_{n+6}, y''_1, \dots, y''_{n+6}, y'''_1, \dots, y'''_{n+6}, y_1^{(iv)}, \dots, y_{n+6}^{(iv)})^T \\ V_{\mu-1} &= (y_0, \dots, y_{n-5}, y'_0, \dots, y'_{n-5}, y''_0, \dots, y''_{n-5}, y'''_0, \dots, y'''_{n-5}, y_0^{(iv)}, \dots, y_{n-5}^{(iv)})^T \\ F_{\mu-1} &= (f_0, \dots, f_{n-5}, f'_0, \dots, f'_{n-5}, f''_0, \dots, f''_{n-5}, f'''_0, \dots, f'''_{n-5}, f_0^{(iv)}, \dots, f_{n-5}^{(iv)})^T \\ F_\mu &= (f_1, \dots, f_{n+6}, f'_1, \dots, f'_{n+6}, f''_1, \dots, f''_{n+6}, f'''_1, \dots, f'''_{n+6}, f_1^{(iv)}, \dots, f_{n+6}^{(iv)})^T \end{aligned}$$

The positive integer $\Gamma = \frac{N}{6}$ is the number of blocks.

The Method has been implemented using the system *Mathematica*, enhanced by the feature ***NSolve*** for linear problems while nonlinear problems were solved by Newton's method enhanced by the feature ***FindRoot***, as summarized in the algorithm below. We begin by noting that the solution of the problem (1) is sought in the subinterval $\pi_N = \{a = x_0 < x_1 < \dots < x_N = b\}$, where $h = \frac{b-a}{N}$ is a constant step-size of the partition of π_N , N is the number of subintervals and n the grid index.

Step 1: Use the block of (31) for $\mu = 1$, $n = 0$ to obtain V_1 on the rectangle $[y_0, y_6] \times [a, b]$, similarly, for $\mu = 2$, $n = 6$ so that V_2 is obtained on the rectangle $[y_6, y_{12}] \times [a, b]$, and on the rectangle $[y_{12}, y_{18}] \times [a, b], \dots, [y_{N-6}, y_N] \times [a, b]$ for $\mu = 3, \dots, \Gamma$, $n = 12, 18, \dots, N-6$, we thus obtain $V_3, \dots, V_\Gamma$.

Step 2: Solve the unified block given by the system $V_1 \cup V_2 \cup \dots \cup V_\Gamma$, obtained in step 1.

Step 3: The solution of (1) is approximated by the solutions in step 2 as $y = [y(x_1), y(x_2), \dots, y(x_n)]^T$, $n = 1, 2, \dots, N$.

4 Numerical Examples

In this part, we implement our derived method using numerical examples to show the high level of accuracy and efficiency of this method.

Problem 1.[6] Consider the following singular fourth order four-point boundary value problem

$$x^4(1-x)y^{(iv)}(x) + 300e^{x/2}y'''(x) + 200\sin(\sqrt{x})y''(x) + 2y'(x) + xy^3(x) = f(x),$$

$$y(0) = 0, \quad y(1/2) = \sin(1/2), \quad y(1/4) = \sin(1/4), \quad y(1) = \sin(1), \quad 0 \leq x \leq 1$$

where

$f(x) = (e^x - 1)^2 \sin^2(x) - 2\sin(\sqrt{x})\sin(x) - e^{x/2}\cos(x) + x\sin(\sin(x)) + \cos(x)\sinh(x)$ The exact solution is given by $y(x) = \sinh x$.

Table 1: error in Problem 1:

	methods in [6]	SBUI
method 1	1.8E-12	6.44E-15
method 2	5.6E-13	
method 3	2.2E-14	

Problem 2. We consider the following nonlinear boundary value problem also found in [1].

$$y^{(iv)}(x) = (1+c)y''(x) - cy(x) + \frac{1}{2}cx^2 - 1, \quad x \in [0, 1]$$

$$y(0) = 0, \quad y'(0) = 0, \quad y(1) = 1.5 + \sinh(1), \quad y'(1) = 1 + \cos(1)$$

with solution (independent of c) given as $y(x) = 1 + \frac{1}{2}x^2 + \sinh(x)$

Table 2: error in Problem 2:

x	SBUI	HBVP [1]
0.1	4.58E-16	5.44E-10
0.2	6.94E-16	6.71E-10
0.3	1.67E-16	9.72E-11
0.4	2.94E-15	2.49E-10
0.5	3.66E-15	1.85E-11
0.6	2.55E-15	2.03E-10
0.7	3.89E-15	1.65E-10
0.8	7.77E-16	7.51E-10
0.9	4.44E-15	5.90E-10

Table 2 gives a clear comparison between SBUI and HBVP in [1] in problem 2. The errors obtained show that the SBUI is more accurate.

Problem 3. We consider the following nonlinear boundary value problem also found in [1].

$$y^{(iv)}(x) = y(x) + y''(x) + e^x(x-3), \quad x \in [0, 1]$$

$$y(0) = 0, \quad y'(0) = 0, \quad y(1) = 0, \quad y'(1) = -e$$

with exact solution $e^x(1-x)$.

Table 3: Maximum error in Problem 3:

x	SBUI	HBVP[1]
0.1	9.99E-16	1.97E-13
0.2	1.44E-16	1.56E-13
0.3	1.09E-16	1.83E-13
0.4	1.77E-14	2.06E-13
0.5	2.10E-14	2.02E-13
0.6	1.67E-14	2.25E-13
0.7	1.05E-14	2.04E-13
0.8	1.67E-15	1.98E-13
0.9	3.33E-16	2.18E-13

Table 3 shows the comparison between SBUI and HBVP in [1] in problem 3. The errors obtained show that the SBUI is superior.

Problem 4. Consider the following nonlinear four points BVP, [29].

$$y^{(iv)}(x) - y(x)y'(x) = 4x^7 + 24, x \in [0, 1]$$

$$y(0) = 0, \quad y'''(0.25) = 6, \quad y''(0.5) = 3, \quad y(1) = 1$$

with exact solution $y(x) = x^4$.

Table 4: Error of methods for Problem 4, h=0.25:

x	y _{app}	y _{Ex}	Er
0.1	0.0001	0.0001	5.21521×10^{-17}
0.2	0.0016	0.0016	2.15154×10^{-17}
0.3	0.0081	0.0081	3.54854×10^{-16}
0.4	0.0256	0.0256	1.44854×10^{-16}
0.5	0.0625	0.0625	6.52587×10^{-16}
0.6	0.1296	0.1296	9.54454×10^{-16}
0.7	0.2401	0.2401	1.74447×10^{-16}
0.8	0.4096	0.4096	6.45451×10^{-16}
0.9	0.6561	0.6561	1.57321×10^{-16}

Table 5: error in Problem 4:

Method in [28]	Method in [13]	Method in [29]	SBUI
method 1: 2.2×10^{-9}	$k_1 = 1: 3.4 \times 10^{-4}$	method 1 1.8×10^{-12}	N=4: 8.1×10^{-16}
method 2: 1.3×10^{-9}	$k_1 = 2: 7.1 \times 10^{-7}$	method 2 5.6×10^{-13}	N=8: 1.4×10^{-17}
method 3: 1.0×10^{-10}	$k_1 = 3: 5.4 \times 10^{-9}$	method 3 2.2×10^{-14}	N=12: 2.1×10^{-16}

In what follows, we consider forth order parabolic PDEs which we carried a semi-discretisation using the method of line as described in [11, 17, 25].

Problem 5. Consider the linear fourth-order parabolic equation, ([24].

$$y_{tt} + y_{xxxx} = (\pi^4 - 1) \sin \pi x \cos t, \quad 0 \leq x \leq 1, \quad t \geq 0$$

subject to the initial conditions

$$y(x, 0) = \sin \pi x, \quad y_t(x, 0) = 0, \quad 0 \leq x \leq 1$$

and with appropriate boundary conditions

$$y(0, t) = y(1, t) = y_{xx}(0, t) = y_{xx}(1, t) = 0, \quad t \geq 0$$

with the exact solution $y(x, t) = \sin \pi x \cos t$.

Here, following [23], the problem on discretizing the time variable, becomes

$$\frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2} + \frac{d^4 y_m}{dx^4} = g_m, \quad 0 \leq x \leq 1, \quad m = 1, \dots, M-1 \quad (32)$$

where $\Delta t = (L_4 - L_3)/M$, $t_m = L_3 + m\Delta t$, $m = 0, 1, \dots, M$, $y = [y_1(x), \dots, y_M(x)]^T$, $y_m(x) \approx y(x, t_m)$, $g = [g_1(x), \dots, g_m(x)]^T$ and $g_m(x) \approx g(x, t_m) = (\pi^4 - 1) \sin \pi x \cos t_m$, which is expressed in the form

$$y^{(4)} = f(x, y, y', y'', y''') = Ay + g \quad (33)$$

A is an $(M-1) \times (M-1)$ matrix arising from the discretized system, and g is a vector of constants.

Problem 6. Consider the following homogenous fourth-order parabolic equation, [20].

$$y_{tt} - y_{xxxx} = 0, \quad 0 \leq x \leq 1, \quad t \geq 0$$

subject to the initial conditions

$$y(x, 0) = \sin \pi x, \quad y_t(x, 0) = -\pi^2 \sin \pi x, \quad 0 \leq x \leq 1$$

and with appropriate boundary conditions

$$\left. \begin{aligned} y(0, t) &= 0, & y(1, t) &= 0, & t &\geq 0, \\ y_{xx}(0, t) &= 0, & y_{xx}(1, t) &= 0, & t &\geq 0, \end{aligned} \right\}$$

The exact solution for this problem is $y(x, t) = e^{-\pi^2 t} \sin \pi x$.

Upon discretization of the time variable, we obtain,

$$\frac{y_{m+1} - 2y_m + y_{m-1}}{(\Delta t)^2} - \frac{d^4 y_m}{dx^4} = g_m, \quad 0 \leq x \leq 1, \quad m = 1, \dots, M-1 \quad (34)$$

where Δt , t_m , $m = 0, 1, \dots, M$, y , $y_m(x) \approx y(x, t_m)$, g and g_m are as expressed in example 1, which is expressed in the form

$$y^{(4)} = f(x, y, y', y'', y''') = Ay + g \quad (35)$$

A is as expressed in problem 5 and $g_m = 0$.

Table 6: Error of methods for Problem 5: $h=0.05$

x	y_{app}	y_{ex}	Er
0.10.3090		0.3090	1.5415×10^{-13}
0.20.5878		0.5878	5.5454×10^{-13}
0.30.8090		0.8090	8.1254×10^{-13}
0.40.9511		0.9511	1.3251×10^{-13}
0.50.9999		0.9999	8.4451×10^{-13}
0.60.9511		0.9511	8.1115×10^{-13}
0.70.8090		0.8090	9.8157×10^{-13}
0.80.5878		0.5878	8.3254×10^{-13}
0.90.3090		0.3090	2.2124×10^{-13}
$1.01.2246 \times 10^{-16}$		1.2246×10^{-16}	0

Table 7: Mximum error of methods for Problem 5, with $h = 0.05$

Method	k	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
SBUI	10	6.56E-12	6.54E-12	6.65E-10	6.12E-10	5.54E-10
	16	6.54E-13	1.57E-11	5.15E-11	1.32E-11	5.51E-11
Method in [24]	10	2.91E-6	1.73E-6	1.60E-6	2.23E-6	2.60E-7
	16	4.47E-7	2.66E-7	1.37E-7	1.55E-7	1.57E-7

Fig. 1: Graphical representation for (32) showing surface plots for the numerical solution using SBUI, the analytic solution and the error (e), for example 5, where $e = |y_{n,m} - y(x_n, t_m)|$

Table 8: Error of methods for Problem 6, $h=0.05$.

x	y_{app}	y_{ex}	Er
0.10.2913		0.2912	8.6565×10^{-6}
0.20.5540		0.5540	4.5455×10^{-6}
0.30.7625		0.7625	6.5451×10^{-6}
0.40.8964		0.8964	9.2555×10^{-6}
0.50.9425		0.9425	1.7545×10^{-6}
0.60.8964		0.8964	1.8458×10^{-6}
0.70.7625		0.7625	7.1111×10^{-6}
0.80.5540		0.5540	3.4545×10^{-6}
0.90.2912		0.2912	1.4545×10^{-6}
$1.01.1542 \times 10^{-16}$		5.8454×10^{-16}	0

Table 9: Maximum error of methods for Problem 6

Method	Time	Parameter	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
SBUI	$t = 0.02$	$h = 0.05, k = 0.005$	2.10E-7	4.00E-9	5.51E-9	6.48E-9	6.81 E-9
	$t = 0.05$	$h = 0.05, k = 0.005$	1.12E-8	2.14E-8	2.94E-8	3.46E-8	3.64 E-8
Method in [20]	$t = 0.02$	$n = 91, k = 0.005$	3.34E-5	6.35E-5	8.74E-5	1.03E-4	1.08 E-4
	$t = 0.05$	$n = 91, k = 0.005$	3.33E-5	6.34E-5	8.73E-5	1.03E-4	1.08 E-4

Fig. 2: Graphical representation for (34) showing surface plots for the numerical solution using SBUI, the analytic solution and the error (e), for example 6, where $e = |y_{n,m} - y(x_n, t_m)|$

Table 4 show the approximate and exact numerical solutions to problem 4. On the other hand, Table 5 shows the maximum absolute error obtained in methods in [28], [13] and [29], as compared with the SBUI derived in the this work. For small values of $N = 4, 8, 12$, it shows the efficiency in terms of the accuracy (as seen in the errors)

Table 6 shows the absolute errors obtained for problem 5. In Table 7, at different values of k and points of x , shows the maximum absolute errors obtained, in comparison with the SBUI and the method in [24]. This shows the superiority of the

SBUI. Figure 1 shows the graphical representation with surface plot for the numerical solution, analytical solution and residue (errors), for problem 5.

Table 8 shows the absolute errors obtained in problem 6. In Table 9, for different values of the parameters k , t , h and n and at different points of x , for $t = 0.02$ and $t = 0.05$ respectively, shows the maximum absolute errors obtained when the SBUI was compared with the method in [20]. This shows that the SBUI performed better. Figure 2 shows the graphical representation with surface plot for the numerical solution, analytical solution and residue (errors), for problem 6.

5 Conclusion

Six-step Block Unification Integrator, SBUI, have been derived from unifying linear multistep Methods (LMMs), and applied to solve BVPs and parabolic PDEs with appropriate boundary conditions, that have applications in mathematical physics. This method was applied directly without first reducing ODEs to an equivalent first order system. Standard numerical examples in literature were used to show the efficiency in terms of the techniques, accuracy in terms of the errors obtained, of the derived method when compared to other methods. It was also observed that the proposed method compare favourably and superior to existing methods in the literature cited.

References

- [1] F. A. Costabile, A. Napoli. Collocation for high order differential equations with two-points Hermite boundary conditions. *J. Appl. Numerical Mathematics* 87 (2015) 157–167 <https://doi.org/10.1016/j.apnum.2014.09.008>
- [2] M. K. Jain and T. Aziz, Cubic spline solution of two-point boundary Value with significant first derivatives, *Comp.Meth. Appl.Mech. Eng.*, 39 (1983), pp. 83–91 [https://doi.org/10.1016/0045-7825\(83\)90075-0](https://doi.org/10.1016/0045-7825(83)90075-0)
- [3] S. N. Jator, Numerical Integrators For Fourth Order Initial And Boundary Value Problems. *Inter. J. Pure and Appl. Math.* 47, (4), (2008), 563–576
- [4] H. B. Keller, Numerical methods for two-point boundary value problems, Blaisdell, Waltham, Mass.; 184 pp. QA372.K42 (1968).
- [5] D. Krajcinovic, Sandwich Beam Analysis, *Journal of Applied Mechanics* 39 (1972) 773–778. <https://doi.org/10.1115/1.3422787>
- [6] X.Y. Li., B.Y. Wu, A Novel Method For Nonlinear Singular Fourth Order Four-Point Boundary Value Problems. *Comp. Math Appl* 62 (2011) 27–31. <https://doi.org/10.1016/j.camwa.2011.04.029>
- [7] A.A. Salama, Higher-Order Method For Solving Free Boundary Value Problems, *Numerical Heat Transfer, Part B: Fundamentals:An International Journal of Computation and Methodology* 45 (2004) 385–394. <https://doi.org/10.1080/10407790490278002>
- [8] D.X. Ma, X.Z. Yang, Upper and lower solution method for fourth-order four-point boundary value problems, *Journal of Computational and Applied Mathematics* 223 (2009) 543–551. <https://doi.org/10.1016/j.cam.2007.10.051>
- [9] Q. Zhang, S.H. Chen, J.H. Lv, Upper and lower solution method for fourth-order four-point boundary value problems, *Journal of Computational and Applied Mathematics* 196 (2006) 387–393. <https://doi.org/10.1016/j.cam.2005.09.007>
- [10] Y.L. Zhong, S.H. Chen, C.P. Wang, Existence results for a fourth-order ordinary differential equation with a four-point boundary condition, *Applied Mathematics Letters* 21 (2008) 465–470 <https://doi.org/10.1016/j.aml.2007.03.029>
- [11] T. A. Biala, S.N. Jator, R.B. Adeniyi, Numerical approximations of second order PDEs by boundary value methods and the method of lines, Springer Plus (2015) 4:588 <https://doi.org/10.1007/s13370-016-0458-4>
- [12] M. Devakar, D. Sreenivasu, B. Shankar, Analytical solutions of couple stress fluid flows with slip boundary conditions, *Alexandria Engineering Journal* (2014) 53, 723–730 <https://doi.org/10.1016/j.aej.2014.06.005>
- [13] J. Duan, R. Rach, A new modification of the Adomian decomposition method for solving boundary value problems for higher order nonlinear differential equations, *Appl. Math. Comput.* (2011) 218, 4090–4118. <https://doi.org/10.1016/j.amc.2011.09.037>

- [14] K. Hussain, F. Ismail and N. Senua, Runge-Kutta Type Methods for Directly Solving Special Fourth-Order Ordinary Differential Equations: Mathematical Problems in Engineering (2015), Article ID 893763.
<https://doi.org/10.1155/2015/893763>
- [15] J.D. Lambert, Computational Methods in Ordinary Differential Equations, John Wiley, New York (1973).
- [16] K. Hussain, F. Ismail and N. Senua, Solving directly special fourth-order ordinary differential equations using Runge-Kutta type method. Journal of Computational and Applied Mathematics (2016) 306, 179–199.
<https://doi.org/10.1016/j.cam.2016.04.002>
- [17] S.N. Jator, Block Unification Scheme for Elliptic, Telegraph, and Sine-Gordon Partial Differential Equations. American Journal of Computational Mathematics (2015) 5, 175–185. <http://dx.doi.org/10.4236/ajcm.2015.52014>.
- [18] S.N. Jator, V. Manathunga, Block Nystrm type integrator for Bratus equation: Journal of Computational and Applied Mathematics (2018) 327, 341–349 <https://doi.org/10.1016/j.cam.2017.06.025> .
- [19] R. C. Mittal, R. K. Jain, B-splines methods with redefined basis functions for solving fourth order parabolic partial differential equations, Appl. Math. and Comp.(2011) 217, 9741–9755.
<https://doi.org/10.1016/j.amc.2011.04.061>
- [20] R. K. Mohanty, D. Kaur, High accuracy implicit variable mesh methods for numerical study of special types of fourth order nonlinear parabolic equations, Appl. Math. and Comp. (2016) 273, 678–696.
<https://doi.org/10.1016/j.amc.2015.10.036>
- [21] R. K. Mohanty, D. Kaur, Numerov type variable mesh approximations for 1D unsteady quasi-linear biharmonic problem: application to Kuramoto-Sivashinsky equation, Numerical Algorithms , (2017) 74,427–459.
<https://doi.org/10.1007/s11075-016-0154-3>
- [22] M.A Noor, S. T. Mohyud-Din, An efficient method for fourth order boundary value problems, Comput. Math. Appl. (2007) 54, 1101–1111.
- [23] F.F. Ngwane and S.N. Jator, A Trigonometrically Fitted Block Method for Solving Oscillatory Second-Order Initial Value Problems and Hamiltonian Systems, Inter. J. of Diff. Eqn. (2017), Article ID 9293530, 14 pages.
<https://doi.org/10.1155/2017/9293530>
- [24] J. Rashidinia and R. Mohammadi, Sextic spline solution of variable coefficient fourth-order parabolic equations, Int. J. of Comp. Math. (2010) 87 (15) 3443–3454
<https://doi.org/10.1080/00207160903085820>
- [25] P. Saucez, A. Vande Wouwer, W.E. Schiesser, P. Zegeling, Method of lines study of nonlinear dispersive waves, J of Computational and App Math . (2014) 168, 413–423.
<https://doi.org/10.1016/j.cam.2003.12.012>
- [26] Siddiqi S. S. and Akram G. (2008), Quintic Spline Solutions Of Fourth Order Boundary-Value Problems, Inter. Journal of Numerical Analysis And Modeling, 5, (1), 101–111.
- [27] Siddiqi S. S., Arshed S. (2014), Quintic B-spline for the numerical solution of the good Boussinesq equation, J. Egypt. Math. Soc. 22, 209–213. <https://doi.org/10.1016/j.joems.2013.06.015>
- [28] Tatari M., Dehghan M. (2006), The use of the Adomian decomposition method for solving multipoint boundary value problems, Phys. Scr. 73, 672–676. <https://doi.org/10.1088/0031-8949/73/6/023>
- [29] Xu. M. Q., Lin Y. Z., Wang Y. H., (2016), A new algorithm for nonlinear fourth order multi-point boundary value problems, Applied Mathematics and Computation 274, 163–168 <https://doi.org/10.1016/j.amc.2015.10.041>