An Oscillation Criterion in Delay Differential Equations

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Abstract. Consider the first order linear delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0,$$

where p is a continuous function of nonnegative real numbers and the argument $\tau(t)$ is not necessarily monotone. Based on an iterative technique, a new oscillation criterion is established when the well-known oscillation conditions

$$\limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p(s)ds > 1 \quad \text{and} \quad \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s)ds > \frac{1}{e}$$

are not satisfied. An example, numerically solved in MATLAB, is also given to illustrate the applicability and strength of the obtained condition over known ones.

Keywords: *differential equation, non-monotone argument, oscillatory solution, nonoscillatory solution, Grönwall inequality.*

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1 Introduction

Consider the first-order delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{E}$$

where p and τ are continuous on $[t_0, \infty)$, $p(t) \geq 0$, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t \rightarrow \infty} \tau(t) = \infty$.

Definition 1.1. By a solution of (E) we mean a function which is continuous on $[\bar{t}_*, \infty)$ for some $t_* \geq t_0$, where $\bar{t}_* = \inf \{\tau(t) : t \geq t_*\}$.

Definition 1.2. A solution of (E) is said to be oscillatory, if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory. An equation is oscillatory if all its solutions oscillate.

In 1972, Ladas, Lakshmikantham and Papadakis [9], and in 1982, Koplatadze and Chanturiya [8] proved that, if

$$\limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p(s)ds > 1 \quad \text{or} \quad \alpha := \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s)ds > \frac{1}{e}, \tag{1.1}$$

then all solutions of (E) are oscillatory.

Assume that the argument $\tau(t)$ is not necessarily monotone. Set

$$h(t) := \sup_{s \leq t} \tau(s), \quad t \geq t_0. \quad (1.2)$$

Clearly, the function $h(t)$ is nondecreasing and $\tau(t) \leq h(t) < t$ for all $t \geq t_0$.

In 2011 Braverman and Karpuz [1] proved that, if

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left(\int_{\tau(s)}^{h(t)} p(u) du \right) ds > 1, \quad (1.3)$$

then all solutions of (E) oscillate.

Several improvements were made to the above condition, see [2–5] to arrive at the recent form [5]

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left(\int_{\tau(s)}^{h(t)} p(u) \exp \left(\int_{\tau(u)}^u d_\ell(\xi) d\xi \right) du \right) ds > 1, \quad (1.4)$$

where

$$d_\ell(t) = p(t) \left[1 + \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(u) \exp \left(\int_{\tau(u)}^u d_{\ell-1}(\xi) d\xi \right) du \right) ds \right],$$

with

$$d_0(t) = p(t) \left[1 + \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(\omega) \exp \left(\lambda_0 \int_{\tau(\omega)}^\omega p(u) du \right) d\omega \right) ds \right]$$

and λ_0 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$.

In the present paper we establish a new iterative sufficient condition, involving $\limsup$, for the oscillation of all solutions of (E) when the argument is not necessarily monotone. The result obtained essentially improves known results in the literature.

2 Main Results

The proof of our main result is essentially based on the following lemmas.

Lemma 2.1. [6, Lemma 2.1.1] Assume that $h(t)$ is defined by (1.2). Then

$$\alpha := \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds = \liminf_{t \rightarrow \infty} \int_{h(t)}^t p(s) ds.$$

Lemma 2.2. [10] Assume that $h(t)$ is defined by (1.2), $\alpha \in (0, 1/e]$ and $x(t)$ is an eventually positive solution of (E). Then

$$\liminf_{t \rightarrow \infty} \frac{x(h(t))}{x(t)} \geq \lambda_0, \quad (2.1)$$

where λ_0 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$.

Theorem 2.3. Assume that $h(t)$ is defined by (1.2) and for some $\ell \in \mathbb{N}$

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left(\int_{\tau(s)}^{h(t)} p(u) \exp \left(\int_{\tau(u)}^u b_\ell(\xi) d\xi \right) du \right) ds > 1, \quad (2.2)$$

where

$$b_\ell(t) = p(t) \left[1 + \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(u) \exp \left(\int_{\tau(u)}^u b_{\ell-1}(\xi) d\xi \right) du \right) ds \right],$$

with

$$b_0(t) = p(t) \left[1 + \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(y) \exp \left(\int_{\tau(y)}^y p(\omega) \exp \left(\lambda_0 \int_{\tau(\omega)}^\omega p(u) du \right) d\omega \right) dy \right) ds \right]$$

and λ_0 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$. Then all solutions of (E) are oscillatory.

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (E). Since $-x(t)$ is also a solution of (E), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then there exists a $t_1 > t_0$ such that $x(t)$ and $x(\tau(t)) > 0$ for all $t \geq t_1$. Thus, from (E) we have $x'(t) = -p(t)x(\tau(t)) \leq 0$ for all $t \geq t_1$, which means that $x(t)$ is an eventually nonincreasing function of positive numbers. Now we divide (E) by $x(t) > 0$ and integrate on $[\tau(t), t]$, so

$$\int_{\tau(t)}^t \frac{x'(u)}{x(u)} du = - \int_{\tau(t)}^t p(u) \frac{x(\tau(u))}{x(u)} du,$$

or

$$x(\tau(t)) = x(t) \exp \left(\int_{\tau(t)}^t p(u) \frac{x(\tau(u))}{x(u)} du \right). \quad (2.3)$$

Combining (E) and (2.3), we have

$$x'(t) + p(t)x(t) \exp \left(\int_{\tau(t)}^t p(u) \frac{x(\tau(u))}{x(u)} du \right) = 0. \quad (2.4)$$

We divide (2.4) by $x(t) > 0$ and integrate on $[\tau(t), t]$, so

$$\int_{\tau(t)}^t \frac{x'(\omega)}{x(\omega)} d\omega = - \int_{\tau(t)}^t p(\omega) \exp \left(\int_{\tau(\omega)}^{\omega} p(u) \frac{x(\tau(u))}{x(u)} du \right) d\omega,$$

or

$$x(\tau(t)) = x(t) \exp \left(\int_{\tau(t)}^t p(\omega) \exp \left(\int_{\tau(\omega)}^{\omega} p(u) \frac{x(\tau(u))}{x(u)} du \right) d\omega \right). \quad (2.5)$$

Combining (E) and (2.5), we have

$$x'(t) + p(t)x(t) \exp \left(\int_{\tau(t)}^t p(\omega) \exp \left(\int_{\tau(\omega)}^{\omega} p(u) \frac{x(\tau(u))}{x(u)} du \right) d\omega \right) = 0. \quad (2.6)$$

We divide (2.6) by $x(t) > 0$ and integrate on $[\tau(s), t]$, so

$$\int_{\tau(s)}^t \frac{x'(y)}{x(y)} dy = - \int_{\tau(s)}^t p(y) \exp \left(\int_{\tau(y)}^y p(\omega) \exp \left(\int_{\tau(\omega)}^{\omega} p(u) \frac{x(\tau(u))}{x(u)} du \right) d\omega \right) dy,$$

or

$$x(\tau(s)) = x(t) \exp \left(\int_{\tau(s)}^t p(y) \exp \left(\int_{\tau(y)}^y p(\omega) \exp \left(\int_{\tau(\omega)}^{\omega} p(u) \frac{x(\tau(u))}{x(u)} du \right) d\omega \right) dy \right). \quad (2.7)$$

Integrating (E) from $\tau(t)$ to t , using (2.7), multiplying by $p(t)$ and taking into account the fact that $x'(t) = -p(t)x(\tau(t))$, we obtain

$$\begin{aligned} & x'(t) + p(t)x(t) \\ & + p(t)x(t) \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(y) \exp \left(\int_{\tau(y)}^y p(\omega) \exp \left(\int_{\tau(\omega)}^{\omega} p(u) \frac{x(\tau(u))}{x(u)} du \right) d\omega \right) dy \right) ds = 0. \end{aligned}$$

Since $\tau(u) \leq h(u)$, clearly

$$\begin{aligned} & x'(t) + p(t)x(t) \\ & + p(t)x(t) \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(y) \exp \left(\int_{\tau(y)}^y p(\omega) \exp \left(\int_{\tau(\omega)}^{\omega} p(u) \frac{x(h(u))}{x(u)} du \right) d\omega \right) dy \right) ds \leq 0. \end{aligned}$$

Taking into account the fact that Lemmas 1 and 2 are satisfied, the last inequality becomes

$$x'(t) + p(t) \left[1 + \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(y) \exp \left(\int_{\tau(y)}^y p(\omega) \exp \left((\lambda_0 - \epsilon) \int_{\tau(\omega)}^\omega p(u) du \right) d\omega \right) dy \right) ds \right] x(t) \leq 0,$$

or

$$x'(t) + b_0(t, \epsilon)x(t) \leq 0, \quad (2.8)$$

with

$$b_0(t, \epsilon) = p(t) \left[1 + \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(y) \exp \left(\int_{\tau(y)}^y p(\omega) \exp \left((\lambda_0 - \epsilon) \int_{\tau(\omega)}^\omega p(u) du \right) d\omega \right) dy \right) ds \right].$$

Applying the Grönwall inequality in (2.8), we obtain

$$x(\tau(u)) \geq x(u) \exp \left(\int_{\tau(u)}^u b_0(\xi, \epsilon) d\xi \right). \quad (2.9)$$

Now we divide (E) by $x(t) > 0$, integrate on $[\tau(s), t]$ and use (2.9), so

$$- \int_{\tau(s)}^t \frac{x'(u)}{x(u)} du = \int_{\tau(s)}^t p(u) \frac{x(\tau(u))}{x(u)} du \geq \int_s^t p(u) \exp \left(\int_{\tau(u)}^u b_0(\xi, \epsilon) d\xi \right) du,$$

or

$$x(\tau(s)) \geq x(t) \exp \left(\int_{\tau(s)}^t p(u) \exp \left(\int_{\tau(u)}^u b_0(\xi, \epsilon) d\xi \right) du \right). \quad (2.10)$$

Integrating (E) from $\tau(t)$ to t and using (2.10), we obtain

$$x(t) - x(\tau(t)) + x(t) \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(u) \exp \left(\int_{\tau(u)}^u b_0(\xi, \epsilon) d\xi \right) du \right) ds \leq 0.$$

Multiplying the last inequality by $p(t)$, we find

$$p(t)x(t) - p(t)x(\tau(t)) + p(t)x(t) \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(u) \exp \left(\int_{\tau(u)}^u b_0(\xi, \epsilon) d\xi \right) du \right) ds \leq 0,$$

which, in view of (E), becomes

$$x'(t) + p(t)x(t) + p(t)x(t) \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(u) \exp \left(\int_{\tau(u)}^u b_0(\xi, \epsilon) d\xi \right) du \right) ds \leq 0.$$

Hence, for sufficiently large t

$$x'(t) + p(t) \left[1 + \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(u) \exp \left(\int_{\tau(u)}^u b_0(\xi, \epsilon) d\xi \right) du \right) ds \right] x(t) \leq 0,$$

or

$$x'(t) + b_1(t, \epsilon)x(t) \leq 0,$$

where

$$b_1(t, \epsilon) = p(t) \left[1 + \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(u) \exp \left(\int_{\tau(u)}^u b_0(\xi, \epsilon) d\xi \right) du \right) ds \right].$$

Following the above procedure, we can inductively construct the inequalities

$$x'(t) + b_\ell(t)x(t, \epsilon) \leq 0, \quad (\ell \in \mathbb{N}),$$

where

$$b_\ell(t, \epsilon) = p(t) \left[1 + \int_{\tau(t)}^t p(s) \exp \left(\int_{\tau(s)}^t p(u) \exp \left(\int_{\tau(u)}^u b_{\ell-1}(\xi, \epsilon) d\xi \right) du \right) ds \right]$$

and

$$x(\tau(s)) \geq x(h(t)) \exp \left(\int_{\tau(s)}^{h(t)} p(u) \exp \left(\int_{\tau(u)}^u b_\ell(\xi, \epsilon) d\xi \right) du \right). \quad (2.11)$$

Integrating (E) from $h(t)$ to t , using (2.11) and if we omit $x(t) > 0$, we have

$$x(h(t)) \left[\int_{h(t)}^t p(s) \exp \left(\int_{\tau(s)}^{h(t)} p(u) \exp \left(\int_{\tau(u)}^u b_\ell(\xi, \epsilon) d\xi \right) du \right) ds - 1 \right] < 0,$$

which means that

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left(\int_{\tau(s)}^{h(t)} p(u) \exp \left(\int_{\tau(u)}^u b_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq 1.$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (2.2).

The proof of the theorem is complete. $\square$

Example 2.4. Consider the delay differential equation

$$x'(t) + \frac{1}{8}x(\tau(t)) = 0, \quad t \geq 0, \quad (2.12)$$

with

$$\tau(t) = \begin{cases} -t + 12k - 1, & \text{if } t \in [6k, 6k + 1] \\ 6k - 2, & \text{if } t \in [6k + 1, 6k + 2] \\ 3t - 12k - 8, & \text{if } t \in [6k + 2, 6k + 3] \\ -0.5t + 9k + 2.5, & \text{if } t \in [6k + 3, 6k + 5] \\ 5t - 24k - 25, & \text{if } t \in [6k + 5, 6k + 6] \end{cases}$$

where $k \in \mathbb{N}_0$ and $\mathbb{N}_0$ is the set of nonnegative integers. By (1.2), we see that

$$h(t) = \begin{cases} 6k - 1, & \text{if } t \in [6k, 6k + 7/3] \\ 3t - 12k - 8, & \text{if } t \in [6k + 7/3, 6k + 3] \\ 6k + 1, & \text{if } t \in [6k + 3, 6k + 26/5] \\ 5t - 24k - 25, & \text{if } t \in [6k + 26/5, 6k + 6] \end{cases}.$$

It is easy to see that $\alpha = \liminf_{k \rightarrow \infty} \int_{6k+5}^{6k+6} \frac{1}{8} ds = 0.125$ and $\lambda_0 \simeq 1.15537$.

Observe that the function $F_\ell : \mathbb{R}_0 \rightarrow \mathbb{R}_+$ defined as

$$F_\ell(t) = \int_{h(t)}^t p(s) \exp \left(\int_{\tau(s)}^{h(t)} p(u) \exp \left(\int_{\tau(u)}^u b_\ell(\xi) d\xi \right) du \right) ds,$$

attains its maximum at $t = 6k + 26/5$, for every $\ell \in \mathbb{N}$. Specifically, by using an algorithm on MATLAB software, we obtain

$$F_1(t = 6k + 26/5) = \int_{6k+1}^{6k+26/5} p(s) \exp \left(\int_{\tau(s)}^{6k+1} p(u) \exp \left(\int_{\tau(u)}^u b_1(\xi) d\xi \right) du \right) ds \simeq 1.037$$

and therefore, condition (2.2) of Theorem 1 is satisfied for $\ell = 1$. Thus, all solutions of (2.12) are oscillatory. Observe, however, that

$$\limsup_{k \rightarrow \infty} \int_{6k+1}^{6k+26/5} \frac{1}{8} ds \simeq 0.525 < 1 \quad \text{and} \quad \alpha = 0.125 < \frac{1}{e},$$

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left(\int_{\tau(s)}^{h(t)} p(u) du \right) ds \simeq 0.6226 < 1$$

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t p(s) \exp \left(\int_{\tau(s)}^{h(t)} p(u) \exp \left(\int_{\tau(u)}^u d_1(\xi) d\xi \right) du \right) ds \simeq 0.9269 < 1$$

That is, none of the conditions (1.1), (1.3) and (1.4) (for $\ell = 1$) is satisfied.

Comments. It is worth noting that the improvement of condition (2.2) to the corresponding condition (1.1) is significant, approximately 97.52%, if we compare the values on the left-side of these conditions. Also, the improvement compared to conditions (1.3) and (1.4) is very satisfactory, around 66.56%, and 11.88%, respectively. In addition, observe that condition (1.4) does not lead to oscillation for first iteration. On the contrary, condition (2.2) is satisfied from the first iteration. This means that our condition is better and much faster than (1.4).

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