A Note on Some Unified Types of Open and Locally Closed Sets

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Abstract. Recently, new types of (open) sets have been studied by some topologist. This note shows that these sets are identical to some other sets already exist in the literature.

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1 Introduction

It is known that open sets are the fundamental building blocks of topology. They have experienced rapid development during the past sixty years, and nowadays it is a primary topic for many general topologists. The introduction of first type of open sets goes back to Levine [11] under the name of semiopen in 1963. Since then, β -open, preopen, α -open, and b-open sets have been defined by Abd El-Monsef et al. [1], Mashhour et al. [13], Njastad [14], and Andrijevic [2], respectively. These types of open sets are the most well-known weak type of open sets. Then too many types of open sets (strong and weak) have been defined and studied. Some authors defined types of open sets without studying them well or comparing them to the closest types of open sets exist in the literature. In 1982, Maheshwari [12] introduced the class of feebly open sets, and in 2012, Indira and Rekha [8] introduced $*b$ -open and $**b$ -open sets. We prove that feebly open, $*b$ -open and $**b$ -open sets are similar to the class of α -open sets, see Theorem 2.6. In 2011, Bharathi et al. [3] defined the class of b^{**} -open sets, and then, in 2016, Sayed and Mansour [16] introduced the class of b^* -open sets. But it turns out that these two classes are the same as the class of β -open sets, see Theorem 2.3.

The concept of locally closed sets is due to Kuratowski and Sierpinski [10], in 1921. Also, this type of sets has received a rapid growth. Many researchers used different types of open and closed sets to introduce new class of locally closed sets, including ([3], [6], [7], [8], [9], [15]). We show that several classes of these sets are equal. For more details, see the results in Section III.

2 Equivalence Types Of Open Sets

Let A be any subset of a topological space (X, τ) , $Int(A)$ and $Cl(A)$ denote the interior and the closure of A , respectively.

Definition 2.1. A subset A of a topological space (X, τ) is said to be:

1. β -open [1] if $A \subseteq Cl(Int(Cl(A)))$.
2. preopen [13] if $A \subseteq Int(Cl(A))$.
3. semiopen [11] if $A \subseteq Cl(Int(A))$.
4. semiclosed [4] if $Int(Cl(A)) \subseteq A$.
5. α -open [14] if $A \subseteq Int(Cl(Int(A)))$.
6. b -open [2] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$.
7. b^* -open [16] if $A \subseteq Cl(Int(Cl(A))) \cup Int(Cl(A))$.
8. b^{**} -open [3] if $A \subseteq Int(Cl(Int(A))) \cup Cl(Int(Cl(A)))$.
9. $*b$ -open [8] if $A \subseteq Cl(Int(A)) \cap Int(Cl(A))$.
10. $**b$ -open [8] if $A \subseteq Int(Cl(Int(A))) \cap Cl(Int(Cl(A)))$.
11. feebly open [12] if there exists an open set U such that $U \subseteq A \subseteq sCl(U)$, where $sCl(A)$ is the intersection of all semiclosed sets containing A .

The complement of β -open (respectively, preopen, α -open, b -open, b^* -open, b^{**} -open, $*b$ -open, $**b$ -open and feebly open) is said to be β -closed (respectively, preclosed, α -closed, b -closed, b^* -closed, b^{**} -closed, $*b$ -closed, $**b$ -closed and feebly closed).

The results contained in this paper are based on the following lemma. We know that it is an elementary result but also important.

Lemma 2.2. [17, Ch.2] For any subsets A, B of a topological space (X, τ) , the following statements hold:

1. $Int(A) \subseteq A$.
2. $A \subseteq Cl(A)$.
3. $Int(Int(A)) = Int(A)$.
4. $Cl(Cl(A)) = Cl(A)$.
5. If $A \subseteq B$, then $Int(A) \subseteq Int(B)$.
6. If $A \subseteq B$, then $Cl(A) \subseteq Cl(B)$.
7. $Int(Cl(Int(Cl(A)))) = Int(Cl(A))$.
8. $Cl(Int(Cl(Int(A)))) \subseteq Cl(Int(A))$.
9. $Int(A) \subseteq Int(Cl(Int(A))) \subseteq Int(Cl(A)) \subseteq Int(Cl(A)) \cup Cl(Int(A)) \subseteq Cl(Int(Cl(A))) \subseteq Cl(A)$.
10. $Int(A) \subseteq Int(Cl(Int(A))) \subseteq Cl(Int(A)) \subseteq Int(Cl(A)) \cup Cl(Int(A)) \subseteq Cl(Int(Cl(A))) \subseteq Cl(A)$.

From the above lemma, one can have the following relation between the most well-known weak types of open sets.

Diagram (A)

The above results tell us no matter how many times you take closure and interior of a set in the same manner as in Definition 2.1, or their unions or intersections, you will end up with one of the open sets in the diagram.

Now, it is time to give our main results.

Theorem 2.3. *For a subset A of a topological space (X, τ) , the following are equivalent:*

1. A is b^{**} -open;
2. A is b^* -open; and
3. A is β -open.

Proof. (1) $\implies$ (2) Let A be b^{**} -open. Then $A \subseteq Cl(Int(Cl(A))) \cup Int(Cl(Int(A)))$. Since $Int(Cl(Int(A))) \subseteq Int(Cl(A))$ by Lemma 2.2 (9), so $A \subseteq Cl(Int(Cl(A))) \cup Int(Cl(A))$. Hence A is b^* -open.

(2) $\implies$ (3) If A is b^* -open, then $A \subseteq Cl(Int(Cl(A))) \cup Int(Cl(A))$. But $Int(Cl(A)) \subseteq Cl(Int(Cl(A)))$, so $A \subseteq Cl(Int(Cl(A)))$, which shows that A is β -open.

(3) $\implies$ (1) Let A be a β -open set. So $A \subseteq Cl(Int(Cl(A)))$. By Lemma 2.2 (10), $Int(Cl(Int(A))) \subseteq Cl(Int(Cl(A)))$, therefore $A \subseteq Cl(Int(Cl(A))) \cup Int(Cl(Int(A)))$. This proves that A is b^{**} -open. We are done. $\square$

Lemma 2.4. *A subset A of a space X is α -open if and only if there is an open set U such that $U \subseteq A \subseteq Int(Cl(U))$.*

Proof. Suppose that A is α -open. So $A \subseteq Int(Cl(Int(A)))$. Since $Int(A) \subseteq A$ by Lemma 2.2 (1), then $Int(A) \subseteq A \subseteq Int(Cl(Int(A)))$. Set $U = Int(A) \implies U \subseteq A \subseteq Int(Cl(U))$ and U is open.

Conversely; assume that there exists an open set U such that $U \subseteq A \subseteq Int(Cl(U))$. We need to show that $A \subseteq Int(Cl(Int(A)))$. Since U is open and by assumption, we have $U \subseteq A$, so $U \subseteq Int(A) \subseteq Int(Cl(U)) \subseteq Int(Cl(Int(A)))$. But $A \subseteq Int(Cl(U))$. Therefore $A \subseteq Int(Cl(U)) \subseteq Int(Cl(Int(A)))$. Thus $A \subseteq Int(Cl(Int(A)))$. This shows that A is α -open. $\square$

Lemma 2.5. [5, Lemma 2.1] *If A is an open subset of a space X , then $sCl(A) = Int(Cl(A))$.*

Theorem 2.6. *For a subset A of a topological space (X, τ) , the following are equivalent:*

1. A is $^{**}b$ -open;
2. A is *b -open;
3. A is α -open; and
4. A is feebly open.

Proof. (1) $\implies$ (2) Let A be $^{**}b$ -open. Then $A \subseteq Int(Cl(Int(A))) \cap Cl(Int(Cl(A)))$. Since $Int(Cl(Int(A))) \subseteq Cl(Int(Cl(A)))$ by Lemma 2.2 (9), so $A \subseteq Int(Cl(Int(A)))$. By Lemma 2.2 (9) & (10), $Int(Cl(Int(A))) \subseteq Int(Cl(A))$ and $Int(Cl(Int(A))) \subseteq Cl(Int(A))$, respectively. Therefore $A \subseteq Int(Cl(A)) \cap Cl(Int(A))$. Hence A is *b -open.

(2) $\implies$ (3) If A is *b -open, then $A \subseteq Cl(Int(A)) \cap Int(Cl(A))$. Therefore $A \subseteq Cl(Int(A))$ and $A \subseteq Int(Cl(A))$. From the former, we have $Cl(A) \subseteq Cl(Int(A))$ and then $Int(Cl(A)) \subseteq Int(Cl(Int(A)))$. But $A \subseteq Int(Cl(A))$. So $A \subseteq Int(Cl(Int(A)))$, which shows that A is α -open.

(3) $\implies$ (4) Follows from Lemmas 2.4 and 2.5.

(4) $\implies$ (1) By (4) there exists an open set U such that $U \subseteq A \subseteq sCl(A)$. Since $U \subseteq A$ and U is open, so $U \subseteq Int(A) \implies Cl(U) \subseteq Cl(Int(A)) \implies Int(Cl(U)) \subseteq Int(Cl(Int(A)))$. Therefore, by assumption and Lemma 2.5, $A \subseteq sCl(U) = Int(Cl(U)) \subseteq Int(Cl(Int(A)))$. But, by Lemma 2.2 (10), $Int(Cl(Int(A))) \subseteq Cl(Int(Cl(A)))$, so $Int(Cl(Int(A))) = Int(Cl(Int(A))) \cap Cl(Int(Cl(A)))$. Hence $A \subseteq Int(Cl(Int(A))) \cap Cl(Int(Cl(A)))$. This proves that A is $^{**}b$ -open. $\square$

3 Equivalence Types Of Localized Closed Sets

In this section, we define some concepts without studying as we were not able to find them in the literature during preparing the present paper. Probably some authors defined them under odd names. We follow topologists for naming them to be standard.

Definition 3.1. *A subset A of a topological space (X, τ) is called:*

1. locally b^{**} -closed [3] if $A = U \cap F$, where U is open and F is b^{**} -closed.
2. locally b^* -closed if $A = U \cap F$, where U is open and F is b^* -closed.

3. locally $*b$ -closed [8, 9], if $A = U \cap F$, where U is open and F is $*b$ -closed.
4. locally $**b$ -closed [8] if $A = U \cap F$, where U is open and F is $**b$ -closed.
5. locally β -closed if $A = U \cap F$, where U is open and F is β -closed.
6. locally α -closed if $A = U \cap F$, where U is open and F is α -closed.

Theorem 3.2. For a subset A of a topological space (X, τ) , the following are equivalent:

1. A is locally b^{**} -closed;
2. A is locally b^* -closed; and
3. A is locally β -closed.

Proof. Follows from Theorem 2.3. □

Theorem 3.3. For a subset A of a topological space (X, τ) , the following are equivalent:

1. A is locally $**b$ -closed;
2. A is locally $*b$ -closed; and
3. A is locally α -closed.

Proof. Follows from Theorem 2.6. □

If we are allowed to introduce the following types of locally closed sets, then we would have the results below:

Definition 3.4. A subset A of a topological space (X, τ) is called:

1. b^{**} -locally closed if $A = U \cap F$, where U is b^{**} -open and F is b^{**} -closed.
2. b^* -locally closed if $A = U \cap F$, where U is b^* -open and F is b^* -closed.
3. $*b$ -locally closed if $A = U \cap F$, where U is $*b$ -open and F is $*b$ -closed.
4. $**b$ -locally closed if $A = U \cap F$, where U is $**b$ -open and F is $**b$ -closed.
5. β -locally closed [6] if $A = U \cap F$, where U is β -open and F is β -closed.
6. α -locally closed [7] if $A = U \cap F$, where U is α -open and F is α -closed.
7. feebly locally closed [15] if $A = U \cap F$, where U is feebly open and F is feebly closed.

Theorem 3.5. For a subset A of a topological space (X, τ) , the following are equivalent:

1. A is b^{**} -locally closed;
2. A is b^* -locally closed; and
3. A is β -locally closed.

Proof. Follows from Theorem 2.3. □

Theorem 3.6. For a subset A of a topological space (X, τ) , the following are equivalent:

1. A is $**b$ -locally closed;
2. A is $*b$ -locally closed; and
3. A is α -locally closed.
4. A is feebly locally closed.

Proof. Follows from Theorem 2.6. □

Remark 3.7. Notice that the results presented in this note can be generalized to the notions of fuzzy, nano, ideal, soft topology etc.

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