Exact Solution of a Nonlinear Time-Caputo Fabrizio Fractional Dispersive Equation

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Abstract. In this paper, the invariant subspace method is used to solve the nonlinear fifth order dispersive equation involving time-Caputo Fabrizio fractional derivative. A non trivial exact solution of this nonlinear fractional partial differential equation is obtained and it is employed to demonstrate the memory effect when the fractional order is close to zero.

Keywords: *Invariance subspace method, Caputo-Fabrizio fractional derivative, fractional fifth order dispersive equation.*

1 Introduction

During the last two decades, many researchers were attracted by fractional calculus. As the concept of derivative is of great interest to describe phenomena with different differential equations, this concept was generalized to fractional derivative according to several approaches. The use of derivative of fractional order appeared in many fields of sciences and have arisen from certain requirements in applications, in biology [4] , economy [9], geophysics [5], bioengineering [12] and so on.

In the literature, there are three major approaches to define fractional derivative [6, 13]. The first is the Riemann-Liouville definition in which fractional derivative of a constant is not zero. The second one is the Caputo approach. This one, is characterized by fractional derivative of a constant to be equal to zero and it is usually used in cases of initial value problems of fractional differential equations. The last one is the Grünwald-Letnikov approach that is useful in numerical methods cases. However, all these approaches are based on Gamma function properties to define fractional derivative which plays an important role to conserve the semi group properties in some cases.

Recently, Caputo-Fabrizio derivative is introduced. It can be used for modeling of processus with various types of memory effects and the interest for this new approach is also due to the necessity of using a model describing the behavior of classical viscoelastic materials, thermal media, etc [8]. As the existence, uniqueness and finding accurate and efficient methods for solving fractional differential equations in the classical cases had been an active research undertaking [14, 11]. So, our aim in this paper is to construct an exact solution of a fifth order nonlinear time-Caputo-Fabrizio fractional dispersive equation (CFF) by means of the invariant subspace method [14, 11, 1, 2].

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This paper is arranged as follows: In section 2, we recall some main results of the Caputo-Fabrizio derivative. Section 3 is devoted to describe the invariance subspace method. While in section 4, we use the described method to construct an exact solution of the time-Caputo-Fabrizio fractional dispersive equation. In section 5, a particular exact solution corresponding to some restrictions is found and we draw 2-D and 3-D graphs of some particular values of fractional parameter α . Finally, some remarks and comments are in order.

2 Preliminary

This section is devoted to recall briefly some definitions and basic results on the Caputo-Fabrizio fractional derivative. Let $f \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1]$ then, the Caputo-Fabrizio derivative of fractional derivative is defined as

$$D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\tau) \exp \left[-\alpha \frac{t-\tau}{1-\alpha} \right] d\tau, \quad (1)$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$ [8]. In the particular case $f(t) = t$, and $a = 0$, we get

$$D_t^\alpha t = \frac{M(\alpha)}{1-\alpha} \left(1 - \exp \left(-\frac{\alpha}{1-\alpha} t \right) \right). \quad (2)$$

In the general case, we get the following result. For any natural number $n \geq 2$, the Caputo-Fabrizio fractional derivative of order $0 < \alpha < 1$ of $f(t) = t^n$ is given by

$$D_t^\alpha t^n = \frac{M(\alpha)}{1-\alpha} \sum_{k=1}^{n-1} (-1)^{k+1} \frac{n!}{(n-k)!} \left(\frac{1-\alpha}{\alpha} \right)^k t^{n-k} + (-1)^{n-1} n! \left(\frac{1-\alpha}{\alpha} \right)^{n-1} D_t^\alpha t. \quad (3)$$

Proof. If $n = 2$, using integration by part and by putting $\beta = \frac{\alpha}{1-\alpha}$ and $H = \frac{M(\alpha)}{1-\alpha}$, we have

$$\begin{aligned} D_t^\alpha t^2 &= H \int_0^t 2\tau \exp \left[-\alpha \frac{t-\tau}{1-\alpha} \right] d\tau \\ &= 2H \exp(-\beta t) \left\{ \left[\frac{\tau}{\beta} \exp(\beta\tau) \right]_0^t - \frac{1}{\beta} \int_0^t \exp(\beta\tau) d\tau \right\} \\ &= 2H \left(\frac{1-\alpha}{\alpha} \right) t - 2 \left(\frac{1-\alpha}{\alpha} \right) D_t^\alpha t \\ &= H \sum_{k=1}^{2-1} (-1)^{k+1} \frac{2!}{(2-k)!} \left(\frac{1-\alpha}{\alpha} \right)^k t^{2-k} + (-1)^{2-1} 2! \left(\frac{1-\alpha}{\alpha} \right)^{2-1} D_t^\alpha t. \end{aligned} \quad (4)$$

Then it holds for $n = 2$. Suppose that 3 is true, for some integer $n \geq 2$. Hence,

$$D_t^\alpha t^{n+1} = H \int_0^t (n+1)\tau^n \exp \left[-\alpha \frac{t-\tau}{1-\alpha} \right] d\tau \quad (5)$$

$$\begin{aligned} &= (n+1)H \exp(-\beta t) \left\{ \left[\frac{\tau^n}{\beta} \exp(\beta\tau) \right]_0^t - \frac{n}{\beta} \int_0^t \tau^{n-1} \exp(\beta\tau) d\tau \right\} \\ &= H \frac{(n+1)!}{(n+1-1)!} \left(\frac{1-\alpha}{\alpha} \right)^1 t^n + H \sum_{k=1}^{n-1} (-1)^{k+2} \frac{(n+1)!}{(n-k)!} \times \\ &\quad \left(\frac{1-\alpha}{\alpha} \right)^{k+1} t^{n-k} + (-1)^n (n+1)! \left(\frac{1-\alpha}{\alpha} \right)^n D_t^\alpha t \\ &= \frac{M(\alpha)}{1-\alpha} \sum_{k=1}^n (-1)^{k+1} \frac{(n+1)!}{(n+1-k)!} \left(\frac{1-\alpha}{\alpha} \right)^k t^{n+1-k} \\ &\quad + (-1)^n (n+1)! \left(\frac{1-\alpha}{\alpha} \right)^n D_t^\alpha t, \end{aligned} \quad (6)$$

which is what we needed to show. $\square$

If the function does not belong to $H^1(a, b)$ then, the derivative can be reformulated as [8]

$$D_t^\alpha f(t) = \frac{\alpha M(\alpha)}{1 - \alpha} \int_a^t (f(t) - f(\tau)) \exp \left[-\alpha \frac{t - \tau}{1 - \alpha} \right] d\tau. \quad (7)$$

The associated integral of the new derivative with fractional order was proposed by Losada and Nieto [7]. Let $0 < \alpha < 1$, the fractional integral of order α of a function f is defined by

$$I_t^\alpha f(t) = \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} f(t) + \frac{2\alpha}{(1 - \alpha)M(\alpha)} \int_0^t f(\tau) d\tau, \quad t \geq 0. \quad (8)$$

It is shown in [3] that

$$\lim_{\alpha \rightarrow 1} D_t^\alpha f(t) = f'(t), \quad \lim_{\alpha \rightarrow 0} D_t^\alpha f(t) = f(t) - f(0). \quad (9)$$

As the definition of the new derivative with fractional order, is a convolution of $\frac{df(x)}{dx}$ and $\exp \left[-\alpha \frac{t}{1 - \alpha} \right]$ so, using the Laplace transform of a convolution leads

$$\mathfrak{L}(D_t^\alpha f(t)) = \frac{M(\alpha)}{1 - \alpha} \mathfrak{L} \left(\frac{df(x)}{dx} \right) \mathfrak{L} \left(\exp \left[-\alpha \frac{t}{1 - \alpha} \right] \right) \quad (10)$$

For more details and some important results and properties of Caputo-Fabrizio fractional derivative see for example [8, 1, 7].

3 Description of the invariant subspace method

The aim of this section is to recall the basic idea of the invariant subspace method. Firstly, it was introduced to construct particular exact solutions of evolutionary partial differential equations of the form:

$$\frac{\partial u}{\partial t} = F(u, u_{1x}, u_{2x}, \dots, u_{kx}), \quad k \in \mathbb{N}, \quad (11)$$

where $u = u(t, x)$, $u_{ix} = \frac{\partial^i u}{\partial x^i}$ is the i -th order derivative of u with respect to the space variable x and F is a nonlinear differential operator.

The invariant subspace method was employed to evolutionary equations in time derivative both in integer order and recently in fractional order [14, 11]. In this case, we consider the time-Caputo-Fabrizio fractional partial differential equation given by

$$D_t^\alpha u(t, x) = F[u], \quad (12)$$

where $F[u] = F(u, u_{1x}, u_{2x}, \dots, u_{kx})$ and D_t^α is the time-Caputo-Fabrizio fractional derivative. The invariant subspace method is based on the following basic definitions and results [14, 11]. Let $f_1(x), \dots, f_n(x)$ be an n linearly independent functions and W_n is the n -dimensional linear space namely $W_n = \langle f_1(x), \dots, f_n(x) \rangle$. W_n is said to be invariant under the given operator $F[u]$ if $F[u] \in W_n$ whenever $u \in W_n$.

Proposition 3.1. *Let W_n be an invariant subspace of $F[u]$. A function $u(t, x) = \sum_{i=1}^n f_i(x) u_i(t)$ is a solution of equation (12) if and only if the expansion coefficients $u_i(t)$ satisfy the following system of fractional ordinary differential equations:*

$$\begin{cases} D_t^\alpha u_1 &= F_1(u_1, \dots, u_n), \\ D_t^\alpha u_2 &= F_2(u_1, \dots, u_n), \\ \vdots &\vdots \\ D_t^\alpha u_n &= F_n(u_1, \dots, u_n), \end{cases}$$

where $F_1, \dots, F_n$ are given by:

$$F(c_1 f_1(x) + \dots + c_n f_n(x)) = F_1(c_1, \dots, c_n) f_1(x) + \dots + F_n(c_1, \dots, c_n) f_n(x). \quad (13)$$

To characterize invariant subspaces admitted by a given differential operator, we recall the following Proposition [14]

Proposition 3.2. *Let $f_1(x), \dots, f_n(x)$ form the fundamental set of solutions of a linear n -th order ordinary differential equation*

$$L[u] = y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_{n-1}(x)y' + a_n(x)y = 0, \quad (14)$$

and $F[y] = F(x, y, y', \dots, y^{(k)})$ a given differential operator of order $k \leq n-1$, then the subspace $W_n = \langle f_1(x), \dots, f_n(x) \rangle$ is invariant with respect to F if and only if:

$$L[F[y]] = 0, \quad (15)$$

whenever y satisfies equation (14).

4 Exact solution of CFF equation

In this section we are interested in the following fifth order nonlinear time-Caputo-Fabrizio fractional dispersive equation

$$D_t^\alpha u(t, x) = \nu \frac{\partial^5 u^2}{\partial x^5} + \beta \frac{\partial^3 u^2}{\partial x^3} + \gamma \frac{\partial u^2}{\partial x}, \quad t \geq 0, \quad 0 < \alpha < 1, \quad (16)$$

where

$$D_t^\alpha u(t, x) = \frac{M(\alpha)}{1-\alpha} \int_0^t \frac{\partial u(\tau, x)}{\partial \tau} \exp \left[-\alpha \frac{t-\tau}{1-\alpha} \right] d\tau,$$

is the new fractional derivative. In the case of $\alpha = 1$, this equation was studied in [3] and for $D_t^\alpha u$ taken in the Caputo-fractional derivative approach, it was studied in [10].

The use of the invariant subspace method in order to construct an exact solution for equation (16), requires to recall an important result characterizing an invariant subspace admitted by this equation. The following lemma gives a condition to satisfy by parameters ν, β and γ , so that $W_3 = \langle 1, \cos x, \sin x \rangle$ is an invariant subspace. Equation (16) admits $W_3 = \langle 1, \cos x, \sin x \rangle$ as invariant sub-space if and only if [14]

$$16\nu - 4\beta + \gamma = 0. \quad (17)$$

Proof. For any function

$$h(t, x) = C_1 + C_2 \cos x + C_3 \sin x, \quad (18)$$

with $C_i = C_i(t)$ arbitrary functions, we get

$$\begin{aligned} F[h] &= (-16\nu + 4\beta - \gamma) [C_2^2 \sin 2x - C_3^2 \sin 2x - 2C_2 C_3 \sin x \cos 2x] \\ &\quad - 2(\nu - \beta + \gamma) C_1 C_2 \sin x + 2(\nu - \beta + \gamma) C_1 C_3 \cos x, \end{aligned}$$

then, $F[h] \in W_3$ if and only if $16\nu - 4\beta + \gamma = 0$. □

Now, we search for an exact solution for the time-Caputo-Fabrizio fractional dispersive equation (16) of the form

$$u(t, x) = C_1(t) + C_2(t) \cos x + C_3(t) \sin x. \quad (19)$$

Hence, a function $u(t, x)$ of the above form is a solution of the equation (16) if the expansion coefficients $C_i(t)$ satisfy the following system of ordinary Caputo-Fabrizio fractional differential equations

$$\begin{cases} D_t^\alpha C_1(t) &= 0, \\ D_t^\alpha C_2(t) &= K C_1 C_3(t), \\ D_t^\alpha C_3(t) &= -K C_1 C_2(t), \end{cases} \quad (20)$$

where $K = 2(\nu - \beta + \gamma)$. To get a non trivial solution needs to assume the condition $K \neq 0$. In addition $C_2(0) \neq 0$ for the same reason, as it will be clear when the Laplace transform is applied. The first equation in the system (20)

yields, $C_1(t)$ is a constant λ . By applying the new fractional derivative to the second of equations (20), we get the new system

$$\begin{cases} C_1(t) &= \lambda, \\ D_t^\alpha (D_t^\alpha C_2(t)) &= -(K\lambda)^2 C_2(t), \\ D_t^\alpha C_3(t) &= -K\lambda C_2(t), \end{cases} \quad (21)$$

By putting $K_1 = K\lambda$ and applying the Laplace transform on both sides of the second equation appearing in the fractional ordinary differential system (21), we obtain

$$\mathfrak{L}\{D_t^\alpha (D_t^\alpha C_2(t))\} = -K_1^2 \mathfrak{L}(C_2(t)), \quad (22)$$

as

$$\mathfrak{L}(D_t^\alpha f(t)) = \frac{M(\alpha)}{1-\alpha} \mathfrak{L}\left(\frac{df(x)}{dx}\right) \mathfrak{L}\left(\exp\left[-\alpha \frac{t}{1-\alpha}\right]\right), \quad (23)$$

then equation (22) becomes

$$\frac{M(\alpha)}{1-\alpha} \mathfrak{L}\left\{\frac{d}{dt}(D_t^\alpha C_2(t))\right\} \mathfrak{L}\left(\exp\left[-\alpha \frac{t}{1-\alpha}\right]\right) = -K_1^2 \mathfrak{L}(C_2(t)), \quad (24)$$

hence, using the Laplace transform formulas [1]

$$\mathfrak{L}\left\{\frac{d}{dt}(D_t^\alpha C_2(t))\right\}(p) = p \mathfrak{L}(D_t^\alpha C_2(t)) - D_t^\alpha C_2(0), \quad (25)$$

and

$$\mathfrak{L}\left(\exp\left[-\alpha \frac{t}{1-\alpha}\right]\right)(p) = \frac{1}{\frac{-\alpha}{1-\alpha} - p}, \quad (26)$$

we get,

$$\left(\frac{M(\alpha)}{1-\alpha}\right)^2 \frac{p}{\left(\frac{-\alpha}{1-\alpha} - p\right)^2} [pG_2(p) - C_2(0)] = -K_1^2 G_2(p), \quad (27)$$

where $G_2(p) = \mathfrak{L}(C_2(t))(p)$. After some manipulations, we get

$$G_2(p) = \frac{C_2(0)}{A} \frac{p}{(p+s)^2 + w^2}, \quad \text{with } s = \frac{B}{2A}, \quad w = \sqrt{\frac{4AC - B^2}{4A^2}}, \quad (28)$$

where

$$A = (K_1\theta_\alpha)^2 + 1, \quad B = \frac{2\alpha}{1-\alpha} (K_1\theta_\alpha)^2, \quad C = (K_1\theta_\alpha)^2 \left(\frac{\alpha}{1-\alpha}\right)^2, \quad (29)$$

with $\theta_\alpha = \frac{1-\alpha}{M(\alpha)}$. Using the inverse Laplace transform formulae,

$$\mathfrak{L}^{-1}(G_2(p))(t) = \exp(-st) \mathfrak{L}^{-1}(G_2(p-s)), \quad (30)$$

consequently, the expression of the coefficient $C_2(t)$ takes the following form

$$C_2(t) = \frac{C_2(0)}{A} \exp(-st) \left[\cos \omega t - \frac{s}{\omega} \sin \omega t \right], \quad (31)$$

where

$$A = (K\lambda\theta_\alpha)^2 + 1, \quad s = \frac{\alpha}{1-\alpha} \frac{(K\lambda\theta_\alpha)^2}{(K\lambda\theta_\alpha)^2 + 1}, \quad \omega = \frac{\alpha}{1-\alpha} \frac{|K\lambda\theta_\alpha|}{(K\lambda\theta_\alpha)^2 + 1}, \quad (32)$$

with $\theta_\alpha = \frac{1-\alpha}{M(\alpha)}$. Replacing Equation (31) in the third equation of the system (21), we obtain

$$D_t^\alpha C_3(t) = -K_1 \frac{C_2(0)}{A} \exp(-st) \left[\cos \omega t - \frac{s}{\omega} \sin \omega t \right]. \quad (33)$$

Applying the Laplace transform on both sides of the above equation, we get

$$\begin{aligned}\mathfrak{L}(C_3(t))(p) &= K_1 \frac{1-\alpha}{M(\alpha)} G_2(p) + K_1 \frac{\alpha C_2(0)}{M(\alpha)A} \frac{1}{(p+s)^2 + \omega^2} \\ &\quad + \frac{M(\alpha)C_3(0)}{1-\alpha} \frac{1}{p} = G_3(p).\end{aligned}\quad (34)$$

Consequently,

$$C_3(t) = K_1 \frac{1-\alpha}{M(\alpha)} C_2(t) + K_1 \frac{\alpha g_2(0)}{M(\alpha)A\omega} \exp(-st) \sin \omega t - \frac{M(\alpha)C_3(0)}{1-\alpha}.\quad (35)$$

Hence, the obtained solution of fractional ordinary differential system (21) yields the following exact solution of the nonlinear time-Caputo-Fabrizio fractional equation (16) in the form

$$\begin{aligned}u(t, x) &= \lambda + \left(\frac{C_2(0)}{A} \exp(-st) \left[\cos \omega t - \frac{s}{\omega} \sin \omega t \right] \right) \cos x \\ &\quad + \left(K_1 \frac{1-\alpha}{M(\alpha)} C_2(t) + K_1 \frac{\alpha C_2(0)}{M(\alpha)A\omega} \exp(-st) \sin \omega t \right) \sin x \\ &\quad - \frac{M(\alpha)C_3(0)}{1-\alpha} \sin x,\end{aligned}\quad (36)$$

which can be reformulated as

$$\begin{aligned}u(t, x) &= \lambda + \left(\frac{C_2(0)}{A} \exp(-st) \left[\cos \omega t - \frac{s}{\omega} \sin \omega t \right] \right) \cos x \\ &\quad + \left\{ K_1 \frac{1-\alpha}{M(\alpha)} \frac{C_2(0)}{A} \exp(-st) \left[\cos \omega t - \frac{s}{\omega} \sin \omega t \right] \right\} \sin x \\ &\quad + \left\{ K_1 \frac{\alpha C_2(0)}{M(\alpha)A\omega} \exp(-st) \sin \omega t - \frac{M(\alpha)C_3(0)}{1-\alpha} \right\} \sin x.\end{aligned}\quad (37)$$

Obviously, for $\alpha = 1$ and if we suppose that $C_3(0) = 0$ and $C_2(0) = 1$, we retrieve the solution given in [14] by

$$u(t, x) = 1 + \cos(x - 2(\nu - \beta + \gamma)t).\quad (38)$$

5 Particular cases

In this section, we look for solution of nonlinear studied equation (16) corresponding to $\lambda = C_2(0) = 1$, $K\theta_\alpha = 1$ and $C_3(0) = 0$.

According to the above conditions, we get

$$A = 2, \quad s = \omega = \frac{\alpha}{2(1-\alpha)}, \quad B = \frac{2\alpha}{1-\alpha}, \quad C = \left(\frac{\alpha}{1-\alpha} \right)^2.\quad (39)$$

Substituting those expressions in the equation (37), and after some manipulations, an exact solution of equation (16) is obtained to be of the form

$$u(t, x) = 1 + \frac{\sqrt{2}}{2} \exp\left(-\frac{\alpha}{2(1-\alpha)}t\right) \cos\left(\frac{\pi}{4} + x - \frac{\alpha t}{2(1-\alpha)}\right).\quad (40)$$

The above solution is represented in the case of $\alpha = 0.9$ and for $\alpha = 0.4$ by the following graphs

Fig. 1: Solution with $\alpha = 0.9$

Fig. 2: Solution with $\alpha = 0.4$

To show the effect of fractional order α , let us present numerical simulations for different values of α when $x = 0$ corresponding to the obtained particular solution (40).

Fig. 3: Solution with $x = 0$ and particular values of α

So that, for α close to 0, we notice that the solution is more affected by the past. However, for α close to 1, the solution showed a fast stabilisation.

6 Conclusion and comments

The aim of this work was to construct an exact solution of nonlinear fifth order dispersive equation with time-Caputo-Fabrizio fractional derivative. To achieve this goal, we used the invariant subspace method and some useful properties of Laplace transforms of the new derivative. We have also presented numerical simulations for different values of α when $x = 0$ for a particular solution and we have seen that in the case of α close to zero the solution is more affected by the past (memory effect). However, when α is close to 1, the solution shows a fast stabilisation. Finally, we note that the invariant subspace method can be extended to other nonlinear fractional differential equations in Caputo-Fabrizio approach and in other generalized form of fractional derivative.

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