Multiple Homoclinic Solutions for a Class of Superquadratic Fourth-order Differential Equations

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Abstract. Applying a Symmetric Mountain Pass Theorem, we prove the existence of infinitely many homoclinic solutions for a class of fourth-order differential equations

$$u^{(4)}(x) + \omega u''(x) + a(x)u(x) = f(x, u(x)), \quad \forall x \in \mathbb{R}$$

where $a \in C(\mathbb{R}, \mathbb{R})$ may be negative on a bounded interval and $F(x, u) = \int_0^u f(x, t)dt$ is superquadratic at infinity in the second variable but does not need to satisfy the well-known Ambrosetti-Rabinowitz superquadratic growth condition.

Keywords: *Fourth-order differential equation; homoclinic solutions; variational methods; Symmetric Mountain Pass Theorem.*

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1 Introduction

We consider the nonperiodic fourth-order differential equation

$$(1.1) \quad u^{(4)}(x) + \omega u''(x) + a(x)u(x) = f(x, u(x)), \quad \forall x \in \mathbb{R}$$

where ω is a constant, $a \in C(\mathbb{R}, \mathbb{R})$ and $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. It is well-known that the mathematical modeling of important questions in different fields of research such as mechanical engineering, control systems, economics and many others, leads naturally to the consideration of the nonlinear differential equations. In particular, the fourth order differential equations, like (1.1) have been put forward as mathematical model for the study of pattern formation in physics and mechanics. For example the well-known extended Fisher-Kolmogorov equation proposed by Coullet et al. in 1987 [5], in the study of phase transitions, the fourth-order elastic beam equation in describing a large class of elastic deflection [14], the Swift-Hohenberg equation which is a general model for pattern-forming process derived in [6] to describe random thermal fluctuations in the Boussinesque equation and in the propagation of lasers [7].

As usual, we say that a solution u of equation (1.1) is homoclinic (to 0) if $u(x) \rightarrow 0$ as $x \rightarrow \pm\infty$. In addition, if $u \neq 0$, then u is called a nontrivial homoclinic solution.

During the previous years, the existence and multiplicity of homoclinic solutions for (1.1) have been extensively investigated via critical point theory and variational methods. From the beginning, most of them treated the case where $a(x)$ and $f(x, u)$ are either independent of x or periodic in x , see [1,3,4,8,16] and the references cited therein. In this kind of problems, the function a plays an important role. Compared with the case of $a(x)$ and $f(x, u)$ being periodic in x , there is less literature available for the case where $a(x)$ and $f(x, u)$ are nonperiodic in x , see [9-13,17-20]. We notice that, for the case that equation (1.1) is not periodic, to obtain the existence of homoclinic solutions, the following coercive condition on a is often needed:

(A_0) $a : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and there exists a constant r_0 such that

$$0 < r_0 \leq a(x) \rightarrow +\infty \text{ as } |x| \rightarrow \infty$$

and

$$\omega \leq 2\sqrt{r_0}$$

which is used to establish the corresponding compact embedding lemmas on suitable functional spaces, see [9,10,17,18]. Most of these known results were obtained for the case where F is superquadratic at infinity in u and satisfies the usual assumption:

$$(1.2) \quad \lim_{|u| \rightarrow 0} \frac{F(x, u)}{|u|^2} = 0 \text{ uniformly for } t \in \mathbb{R}.$$

In this case, the well-known Ambrosetti-Rabinowitz superquadratic condition was usually assumed on F , see [8,13,16,20]. In this paper, we study the existence of infinitely many homoclinic solutions for (1.1) in the case where a is unnecessarily required to be positive, and F satisfies some weak superquadratic conditions at infinity with respect to u . More precisely, we make the following assumptions:

$$(\mathcal{A}) \quad \lim_{|x| \rightarrow \infty} a(x) = +\infty;$$

(F_1) There exist constants $a_0, R_0 > 0$ such that

$$|f(x, u)| \leq a_0 |u|, \quad \forall (x, u) \in \mathbb{R} \times \mathbb{R} \text{ with } |u| \leq R_0;$$

$$(F_2) \quad \lim_{|u| \rightarrow +\infty} \frac{|F(x, u)|}{|u|^2} = +\infty, \quad \forall x \in \mathbb{R},$$

and

$$F(x, u) \geq 0, \quad \forall (x, u) \in \mathbb{R} \times \mathbb{R} \text{ with } |u| \geq R_0;$$

$$(F_3) \quad F(x, -u) = F(x, u), \quad \forall (x, u) \in \mathbb{R} \times \mathbb{R};$$

(F_4) There exist $g \in L^1(\mathbb{R})$ and constants $b_0, c_0 > 0$ and $\nu \in]0, 2[$ such that

$$\widehat{F}(x, u) = \frac{1}{2}f(x, u)u - F(x, u) \geq \begin{cases} g(x), & \forall x \in \mathbb{R}, |x| \leq R_0 \\ b_0 |u|^\nu, & \forall x \in \mathbb{R}, |u| \geq R_0; \end{cases}$$

$$|F(x, u)| \leq c_0 |u|^{2-\nu} \widehat{F}(x, u), \quad \forall (x, u) \in \mathbb{R} \times \mathbb{R} \text{ with } |u| \geq R_0;$$

(F_5) There exist constants $\mu > 2$ and $\rho > 0$ such that

$$\mu F(x, u) \leq f(x, u)u + \rho |u|^2, \quad \forall (x, u) \in \mathbb{R} \times \mathbb{R}.$$

Our main results read as follows.

Theorem 1.1. Assume that $(\mathcal{A})$ and $(F_1) - (F_4)$ are satisfied. Then $(\mathcal{FODE})$ possesses infinitely many nontrivial homoclinic orbits.

Theorem 1.2. Assume that $(\mathcal{A})$, $(F_1) - (F_3)$ and (F_5) are satisfied. Then $(\mathcal{FODE})$ possesses infinitely many nontrivial homoclinic orbits.

Remarks 1.1. a) In our results, $a(x)$ is unnecessarily positive. For example $a(x) = |x| - 1$ satisfies $(\mathcal{A})$ but it doesn't satisfy $(\mathcal{A}_0)$.

b) Let $F(x, u) = \chi(x) |x|^2 \ln(\frac{1}{2} + |x|)$, where χ is a continuous bounded function with positive lower bound. Then an easy computation shows that F satisfies the superquadratic conditions $(F_1) - (F_4)$. However, F does not satisfy the $(\mathcal{AR})$ -condition.

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. Section 3 is devoted to the proofs of our results.

2 Variational setting and preliminaries

To prove our main result via critical point theory, we need to establish the variational setting for (1.1). In the following, we shall use $\|\cdot\|_s$ to denote the norm of $L^s(\mathbb{R})$ for any $s \in [2, \infty]$. Let $H^2(\mathbb{R})$ be the Sobolev space with inner product and norm given respectively by

$$\langle u, v \rangle_{H^2} = \int_{\mathbb{R}} [u''(x)v''(x) + u'(x)v'(x) + u(x)v(x)]dx$$

and

$$\|u\|_{H^2} = \langle u, u \rangle_{H^2}^{\frac{1}{2}}$$

for all $u, v \in H^2(\mathbb{R})$.

Lemma 2.1 ([4, Lemma 8]). Assume that a satisfies $(\mathcal{A}_0)$. Then there exists a constant $b > 0$ such that

$$\int_{\mathbb{R}} [u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2]dx \geq b \|u\|_{H^2}^2, \quad \forall u \in H^2(\mathbb{R}).$$

By Lemma 2.1, we define

$$E = \left\{ u \in H^2(\mathbb{R}) / \int_{\mathbb{R}} [u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2]dx < \infty \right\}$$

with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}} [u''(x)v''(x) - \omega u'(x)v'(x) + a(x)u(x)v(x)]dx$$

and the corresponding norm

$$\|u\| = \left(\int_{\mathbb{R}} [u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2]dx \right)^{\frac{1}{2}}.$$

It is easy to verify that E is a Hilbert space.

In order to prove our result, the following compactness result is necessary.

Lemma 2.2 ([17, Lemma 2.2]). Assume that a satisfies $(\mathcal{A}_0)$. Then E is compactly embedded in $L^s(\mathbb{R})$ for all $s \in [2, \infty]$. Moreover, for all $s \in [2, \infty]$, there exists $\eta_s > 0$ such that

$$(2.1) \quad \|u\|_{L^s(\mathbb{R})} \leq \eta_s \|u\|, \quad \forall u \in E.$$

To study the critical points of the variational functional associated with $(\mathcal{FODE})$, we need to recall the Symmetric Mountain Pass Theorem [15].

Definition 2.1. Let E be a Banach space with the norm $\|\cdot\|$, we say that $\Phi \in C^1(E, \mathbb{R})$ satisfies the

a) $(PS)_c$ -condition, $c \in \mathbb{R}$, if any sequence $(u_n) \subset E$ satisfying

$$\Phi(u_n) \longrightarrow c \text{ and } \Phi'(u_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

possesses a convergent subsequence,

b) $(C)_c$ -condition, $c \in \mathbb{R}$, if any sequence $(u_n) \subset E$ satisfying

$$\Phi(u_n) \longrightarrow c \text{ and } \|\Phi'(u_n)\| (1 + \|u_n\|) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

possesses a convergent subsequence.

Lemma 2.3. Let E be an infinite dimensional Banach space, $E = Y \oplus Z$, where Y is finite dimensional space. Suppose that $\Phi \in C^1(E, \mathbb{R})$ satisfies the $(PS)_c$ -condition and

$$(a) \quad \Phi(0) = 0, \quad \Phi(-u) = \Phi(u), \quad \forall u \in E;$$

- (b) There exist constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho \cap Z} \geq \alpha$;
(c) For any finite dimensional subspace $\tilde{E} \subset E$, there is $R = R(\tilde{E}) > 0$ such that $\Phi(u) \leq 0$ on $\tilde{E} \setminus B_R$, where $B_R = \{u \in E / \|u\| < R\}$.
Then Φ possesses an unbounded sequence of critical values.

Remark 2.1. As shown in [2], a deformation lemma can be proved with $(C)_c$ -condition replacing the $(PS)_c$ -condition, and it turns out that Lemma 2.3 still holds true with the $(C)_c$ -condition instead of the $(PS)_c$ -condition.

3 Proof of Theorems

From $(\mathcal{A})$, (F_1) and (F_2) , we know that there exists a positive constant r_0 such that $a(x) + 2r_0 \geq r_0$ for all $x \in \mathbb{R}$. Let $\bar{a}(x) = a(x) + 2r_0$ and $\bar{F}(x, u) = F(x, u) + r_0 |u|^2$. Consider the following fourth-order differential equation

$$(\overline{\mathcal{FODE}}) \quad u^{(4)}(x) + \omega u''(x) + \bar{a}(x)u(x) = \bar{f}(x, u(x)), \quad \forall x \in \mathbb{R},$$

then $(\overline{\mathcal{FODE}})$ is equivalent to $(\mathcal{FODE})$. Moreover, it is easy to check that the hypotheses $(F_1) - (F_5)$ still hold for $\bar{F}$ provided that those hold for F , and $\bar{a}$ satisfies the condition $(\mathcal{A}_0)$. Hence, in what follows, we always assume without loss of generality that a satisfies $(\mathcal{A}_0)$ instead of $(\mathcal{A})$.

Consider the variational functional Φ associated to $(\mathcal{FODE})$:

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}} [u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2] dx - \int_{\mathbb{R}} F(x, u(x)) dx$$

defined on the space E introduced in Section 2.

Lemma 3.1. Under assumptions $(\mathcal{A}_0)$ and (F_1) , the functional

$$\psi(u) = \int_{\mathbb{R}} F(x, u) dx$$

is continuously differentiable on E and

$$(3.1) \quad \psi'(u)v = \int_{\mathbb{R}} f(x, u) \cdot v dx, \quad \forall u, v \in E.$$

Proof. For any given $u \in E$, we know that $u \in H^2(\mathbb{R})$ and hence there exists a constant $R > 0$ such that

$$(3.2) \quad |u(x)| \leq \frac{R_0}{2}, \quad \forall |x| \geq R.$$

By (2.1), for any $v \in E$ with $\|v\| \leq \frac{R_0}{2\eta_\infty}$, we have

$$(3.3) \quad \|v\|_{L^\infty} \leq \frac{R_0}{2}.$$

Combining (3.2), (3.3) and (F_1) , by the Mean Value Theorem and Hölder's inequality, for any $r \geq R$ and $v \in E$ with $\|v\| \leq \frac{R_0}{2\eta_\infty}$, one has

$$\begin{aligned} & \left| \int_{|x| \geq r} [F(x, u+v) - F(x, u) - f(x, u)v] dx \right| \\ &= \left| \int_{|x| \geq r} \int_0^1 (f(x, u+sv) - f(x, u)) v ds dx \right| \\ &\leq 2a_0 \int_{|x| \geq r} (|u| + |v|) |v| dx \leq 2a_0 \left(\int_{|x| \geq r} (|u| + |v|)^2 dx \right)^{\frac{1}{2}} \|v\|_{L^2} \end{aligned}$$

$$(3.4) \quad \leq 2a_0\eta_2[(\int_{|x|\geq r} |u|^2 dx)^{\frac{1}{2}} + \eta_2 \|v\|] \|v\|.$$

Since $u \in L^2(\mathbb{R})$, for any $\epsilon > 0$, there exist $0 < \alpha_1 < \frac{R_0}{2\eta_\infty}$ and $R_\epsilon \geq R$ such that for all $v \in E$ with $\|v\| \leq \alpha_1$

$$(3.5) \quad 2a_0\eta_2[(\int_{|x|\geq r} |u|^2 dx)^{\frac{1}{2}} + \eta_2 \|v\|] \leq \frac{\epsilon}{2}.$$

It is well known that the functional

$$\psi_\epsilon(u) = \int_{[-R_\epsilon, R_\epsilon]} F(x, u) dx$$

is continuously differentiable on $H^2([-R_\epsilon, R_\epsilon])$. Thus, since E is compactly embedded in $H^2(\mathbb{R})$, there exists a constant $\alpha_2 > 0$ such that for all $\|v\| \leq \alpha_2$

$$(3.6) \quad \left| \int_{[-R_\epsilon, R_\epsilon]} [F(x, u+v) - F(x, u) - f(x, u)v] dx \right| \leq \frac{\epsilon}{2} \|v\|.$$

Taking $\alpha = \min\{\alpha_1, \alpha_2\}$, then (3.4)-(3.6) imply

$$\left| \int_{\mathbb{R}} [F(x, u+v) - F(x, u) - f(x, u)v] dx \right| \leq \epsilon \|v\|$$

for all $v \in E$ with $\|v\| \leq \alpha$. Therefore, ψ is differentiable on E and satisfies (3.1).

It remains to prove that ψ' is continuous. Let $u_n \rightarrow u$ in E . By Hölder's inequality, we have

$$\begin{aligned} \|\psi'(u_n) - \psi'(u)\|_{E'} &= \sup_{\|v\|=1} |\psi'(u_n)v - \psi'(u)v| \\ &= \sup_{\|v\|=1} \left| \int_{\mathbb{R}} [f(x, u_n) - f(x, u)]v dx \right| \\ &\leq \sup_{\|v\|=1} \left(\int_{\mathbb{R}} |f(x, u_n) - f(x, u)|^2 dx \right)^{\frac{1}{2}} \|v\|_{L^2} \\ &\leq \eta_2 \left(\int_{\mathbb{R}} |f(x, u_n) - f(x, u)|^2 dx \right)^{\frac{1}{2}}. \end{aligned} \quad (3.7)$$

Lemma 2.1 implies that $u_n \rightarrow u$ in $L^s(\mathbb{R})$ for $s = 1$ and $s = \infty$. Let M be a positive constant such that $\|u_n\|_{L^1} \leq M$ for all integer n . By (F_1) , for any $\epsilon > 0$, there exists a constant $0 < r < R_0$ such that for all $x \in \mathbb{R}$ and $|u| \leq r$

$$(3.8) \quad |f(x, u)| \leq \frac{\epsilon}{2(M^{\frac{1}{2}} + \|u\|_{L^1}^{\frac{1}{2}})} |u|^{\frac{1}{2}}.$$

Due to (3.8) and the facts that $u \in H^2(\mathbb{R})$ and $u_n \rightarrow u$ in $L^\infty(\mathbb{R})$, there exists $R_\epsilon > R_0$ and $N_1 \in \mathbb{N}$ such that for all $|x| \geq R_\epsilon$ and $n \geq N_1$

$$\begin{aligned} |f(x, u_n(x))| &\leq \frac{\epsilon}{2(M^{\frac{1}{2}} + \|u\|_{L^1}^{\frac{1}{2}})} |u_n(x)|^{\frac{1}{2}} \\ |f(x, u(x))| &\leq \frac{\epsilon}{2(M^{\frac{1}{2}} + \|u\|_{L^1}^{\frac{1}{2}})} |u(x)|^{\frac{1}{2}}. \end{aligned}$$

Thus

$$\begin{aligned} &\left(\int_{|t|\geq R_\epsilon} |f(x, u_n) - f(x, u)|^2 dx \right)^{\frac{1}{2}} \\ &\leq \left(\int_{|x|\geq R_\epsilon} |f(x, u_n)|^2 dx \right)^{\frac{1}{2}} + \left(\int_{|x|\geq R_\epsilon} |f(x, u)|^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

$$(3.9) \quad \leq \frac{\epsilon}{2(M^{\frac{1}{2}} + \|u\|_{L^1}^{\frac{1}{2}})} (\|u_n\|_{L^1}^{\frac{1}{2}} + \|u\|_{L^1}^{\frac{1}{2}}) \leq \frac{\epsilon}{2}.$$

Observing that $u_n \rightarrow u$ in $L^\infty(\mathbb{R})$, then by Lebesgue's Dominated Convergence Theorem, we have

$$\left(\int_{[-R_\epsilon, R_\epsilon]} |f(x, u_n) - f(x, u)|^2 dx \right)^{\frac{1}{2}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, there is $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$

$$(3.10) \quad \left(\int_{[-R_\epsilon, R_\epsilon]} |f(x, u_n) - f(x, u)|^2 dx \right)^{\frac{1}{2}} \leq \frac{\epsilon}{2},$$

which together with (3.9) implies that for all $n \geq \max\{N_1, N_2\}$

$$\left(\int_{\mathbb{R}} |f(x, u_n) - f(x, u)|^2 dx \right)^{\frac{1}{2}} \leq \epsilon.$$

Combining this with (3.7) implies that $\psi'(u_n) \rightarrow \psi'(u)$ as $n \rightarrow \infty$ and then $\psi \in C^1(E, \mathbb{R})$. The proof of Lemma 3.1 is completed.

From Lemma 3.1, we deduce that $\Phi \in C^1(E, \mathbb{R})$ and

$$(3.11) \quad \Phi'(u)v = \langle u, v \rangle - \int_{\mathbb{R}} f(x, u) \cdot v dx, \quad \forall u, v \in E.$$

Moreover, we can infer that $u \in E$ is a critical point of Φ if and only if it is a (classical) solution of problem $(\mathcal{FODE})$.

Lemma 3.2. Under assumptions $(\mathcal{A}_0)$, (F_1) and (F_2) , for any finite dimensional subspace $\tilde{E} \subset E$, there is $R = R(\tilde{E}) > 0$ such that

$$(3.12) \quad \Phi(u) \leq 0, \quad \forall u \in \tilde{E}, \quad \|u\| \geq R.$$

Proof. We will prove the following

$$(3.13) \quad \Phi(u) \rightarrow -\infty \text{ as } \|u\| \rightarrow \infty, \quad u \in \tilde{E}.$$

Arguing indirectly, assume that there exists a sequence $(u_n) \subset \tilde{E}$ with $\|u_n\| \rightarrow \infty$ and a constant $M > 0$ such that $\Phi(u_n) \geq -M$ for all $n \in \mathbb{N}$. Set $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Passing to a subsequence if necessary, we may assume that $v_n \rightarrow v$ in E . Since $\tilde{E}$ is finite dimensional, we have $v_n \rightarrow v$ in $\tilde{E}$ and $v_n \rightarrow v$ a.e. on $\mathbb{R}$. It follows that $\|v\| = 1$. For $0 \leq a < b$, let

$$\Omega_n(a, b) = \{x \in \mathbb{R} / a \leq |u_n(x)| < b\}$$

$$A = \{x \in \mathbb{R} / v(x) \neq 0\}.$$

Since $v \neq 0$, then $\text{meas}(A) > 0$. For a.e. $t \in \mathbb{R}$, we have $\lim_{n \rightarrow \infty} |u_n(x)| = \infty$, hence $x \in \Omega_n(R_0, \infty)$ for n large enough. Since $v_n(x) \rightarrow v(x)$ a.e. $x \in \mathbb{R}$, we have $\chi_{\Omega_n(R_0, \infty)}(x) |v_n(x)| \rightarrow |v(x)|$ a.e. $x \in A$, where χ_Ω denotes the characteristic function of Ω . Hence, it follows from (F_1) , (F_2) and Fatou's Lemma that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \frac{-M}{\|u_n\|^2} \leq \lim_{n \rightarrow \infty} \frac{\Phi(u_n)}{\|u_n\|^2} = \lim_{n \rightarrow \infty} \left[\frac{1}{2} - \int_{\mathbb{R}} \frac{F(x, u_n) |v_n|^2}{|u_n|^2} dx \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{2} - \int_{\Omega_n(0, R_0)} \frac{F(x, u_n) |v_n|^2}{|u_n|^2} dx - \int_{\Omega_n(R_0, \infty)} \frac{F(x, u_n) |v_n|^2}{|u_n|^2} dx \right] \\ &\leq \limsup_{n \rightarrow \infty} \left[\frac{1}{2} + \frac{a_0}{2} \int_{\mathbb{R}} |v_n|^2 dx - \int_{\Omega_n(R_0, \infty)} \frac{F(x, u_n) |v_n|^2}{|u_n|^2} dx \right] \end{aligned}$$

$$\begin{aligned}
& \leq \frac{1}{2} + \frac{a_0}{2} \eta_2^2 - \liminf_{n \rightarrow \infty} \int_{\Omega_n(R_0, \infty)} \frac{F(x, u_n) |v_n|^2}{|u_n|^2} dx \\
(3.14) \quad & \leq \frac{1}{2} + \frac{a_0}{2} \eta_2^2 - \int_{\mathbb{R}} \liminf_{n \rightarrow \infty} \frac{F(x, u_n) |v_n|^2}{|u_n|^2} \chi_{|\Omega_n(R_0, \infty)}(x) dx = -\infty
\end{aligned}$$

which is a contradiction. Hence (3.13) and then (3.12) is verified. The proof of Lemma 3.2 is completed.

Let $(e_j)_{j \in \mathbb{N}}$ be an orthonormal basis of E and define $X_j = \mathbb{R}e_j$

$$Y_k = \oplus_{j=1}^k X_j, \quad Z_k = \overline{\oplus_{j=k+1}^{\infty} X_j}, \quad k \in \mathbb{N}.$$

Lemma 3.3. Suppose $(\mathcal{A}_0)$ hold. Then for any $p \in [2, \infty]$

$$(3.15) \quad l_p(k) = \sup_{u \in Z_k, \|u\|=1} \|u\|_{L^p} \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

Proof. It is clear that $0 < l_p(k+1) \leq l_p(k)$, so that $l_p(k) \longrightarrow \bar{l}_p$ as $k \longrightarrow \infty$. For every $k \geq 1$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_{L^p} > \frac{1}{2} l_p(k)$. For any $v \in E$, let $v = \sum_{i=1}^{\infty} v_i e_i$. By the Cauchy-Schwartz inequality, one has

$$\begin{aligned}
| \langle u_k, v \rangle | &= \left| \langle u_k, \sum_{i=1}^{\infty} v_i e_i \rangle \right| = \left| \langle u_k, \sum_{i=k+1}^{\infty} v_i e_i \rangle \right| \\
&\leq \|u_k\| \left\| \sum_{i=k+1}^{\infty} v_i e_i \right\| \leq \sum_{i=k+1}^{\infty} |v_i| \|e_i\| \longrightarrow 0 \text{ as } k \longrightarrow \infty
\end{aligned}$$

which implies that $u_k \rightharpoonup 0$. Without loss of generality, Lemma 2.2 implies that $u_k \longrightarrow 0$ in $L^2(\mathbb{R})$. Thus we have proved that $\bar{l}_p = 0$. The proof of Lemma 3.3 is completed.

By (3.15), we can choose an integer $m \geq 1$ such that

$$(3.16) \quad \|u\|_{L^p} \leq \frac{1}{2a_0} \|u\|, \quad \forall u \in Z_m.$$

In the following, we will apply Lemma 2.2 with $Y = Y_m$ and $Z = Z_m$.

Lemma 3.4. Under assumptions $(\mathcal{A}_0)$ and (F_1) , there exist constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho \cap Z} \geq \alpha$.

Proof. If $\|u\| = \frac{R_0}{\eta_\infty}$, then by Lemma 2.2, we have $\|u\|_{L^\infty} \leq R_0$. Hence, it follows from (F_1) that

$$(3.17) \quad F(x, u(x)) \leq \frac{a_0}{2} |u(x)|^2, \quad \forall u \in E, \quad \|u\| = \frac{R_0}{\eta_\infty}.$$

Combining (3.16) with (3.17) yields for all $u \in Z$ with $\|u\| = \frac{R_0}{\eta_\infty} = \rho$,

$$\begin{aligned}
\Phi(u) &= \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}} F(x, u) dx \\
&\geq \frac{1}{2} \|u\|^2 - \frac{a_0}{2} \int_{\mathbb{R}} |u|^2 dx \geq \frac{1}{2} \|u\|^2 - \frac{a_0}{2} \|u\|_{L^2}^2 \\
&\geq \frac{1}{4} \|u\|^2 = \frac{1}{4} \left(\frac{R_0}{\eta_\infty} \right)^2 = \alpha.
\end{aligned}$$

The proof of Lemma 3.4 is completed.

Proof of Theorem 1.1. By assumptions (F_1) and (F_3) , it is clear that

$$(3.18) \quad \Phi(0) = 0 \text{ and } \Phi(-u) = \Phi(u), \quad \forall u \in E.$$

Thus the condition (a) of Lemma 2.3 is satisfied. Lemmas 3.2, 3.4 imply that conditions (b) and (c) of Lemma 2.3 are satisfied. It remains to prove that Φ satisfies the $(C)_c$ -condition.

Lemma 3.5. Assume that $(\mathcal{A}_0)$, (F_1) , (F_2) and (F_4) are satisfied. Then Φ verifies the $(C)_c$ -condition for all $c \in \mathbb{R}$.

Proof. Let (u_n) be a $(C)_c$ sequence, that is

$$(3.19) \quad \Phi(u_n) \longrightarrow c \text{ and } \|\Phi'(u_n)\| (1 + \|u_n\|) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Firstly, we prove that (u_n) is bounded in E . Arguing by contradiction, suppose that $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$. Observe that for n large enough

$$(3.20) \quad c + 1 \geq \Phi(u_n) - \frac{1}{2} \Phi'(u_n)u_n = \int_{\mathbb{R}} \widehat{F}(x, u_n) dx.$$

It follows from (3.19) that

$$(3.21) \quad \frac{1}{2} \leq \limsup_{n \longrightarrow \infty} \int_{\mathbb{R}} \frac{|F(x, u_n)|}{\|u_n\|^2} dx.$$

Passing to a subsequence if necessary, we may assume that $v_n \rightharpoonup v$ in E . Then by Lemma 2.2, without loss of generality, we have $v_n \longrightarrow v$ in $L^2(\mathbb{R})$ and $v_n \longrightarrow v$ a.e. on $\mathbb{R}$.

If $v = 0$, then $v_n \longrightarrow 0$ in $L^2(\mathbb{R})$ and $v_n \longrightarrow 0$ a.e. on $\mathbb{R}$. Hence, it follows from (F_1) that

$$(3.22) \quad \begin{aligned} \int_{\Omega_n(0, R_0)} \frac{|F(x, u_n)|}{\|u_n\|^2} dx &= \int_{\Omega_n(0, R_0)} \frac{|F(x, u_n)|}{|u_n|^2} |v_n|^2 dx \\ &\leq \frac{a_0}{2} \int_{\Omega_n(0, R_0)} |v_n|^2 dx \leq \frac{a_0}{2} \|v_n\|_{L^2}^2 \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

From (F_4) and (3.19), one has

$$(3.23) \quad \begin{aligned} \int_{\Omega_n(R_0, \infty)} \frac{|F(x, u_n)|}{\|u_n\|^2} dx &= \int_{\Omega_n(R_0, \infty)} \frac{|F(x, u_n)|}{|u_n|^2} |v_n|^2 dx \\ &\leq \frac{c_0 \|v_n\|_{\infty}^{2-\nu}}{\|u_n\|^\nu} \int_{\Omega_n(R_0, \infty)} \widehat{F}(x, u_n) dx \\ &\leq \frac{c_0 \|v_n\|_{L^\infty}^{2-\nu}}{\|u_n\|^\nu} [1 + c - \int_{\Omega_n(0, R_0)} \widehat{F}(x, u_n) dx] \\ &\leq \frac{c_0 \eta_\infty^{2-\nu}}{\|u_n\|^\nu} [1 + c - \int_{\Omega_n(0, R_0)} g(x) dx] \\ &\leq \frac{c_0 \eta_\infty^{2-\nu}}{\|u_n\|^\nu} [1 + c + \int_{\mathbb{R}} |g(x)| dx] \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Combining (3.22) with (3.23) yields

$$\int_{\mathbb{R}} \frac{|F(x, u_n)|}{\|u_n\|^2} dx = \int_{\Omega_n(0, R_0)} \frac{|F(x, u_n)|}{\|u_n\|^2} dx + \int_{\Omega_n(R_0, \infty)} \frac{|F(x, u_n)|}{\|u_n\|^2} dx \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

which contradicts (3.21).

If $v \neq 0$. By a similar fashion as for (3.14), we can get a contradiction. Therefore, (u_n) is bounded in E .

Next, we prove that (u_n) possesses a convergent subsequence. Without loss of generality, we can assume by Lemma

2.2 that $u_n \rightarrow u$ in $L^s(\mathbb{R})$ for $s = 1$ and $s = \infty$. Using Hölder's inequality, we can show as in the proof of Lemma 3.1 that

$$(3.24) \quad \int_{\mathbb{R}} [f(x, u_n) - f(x, u)](u_n - u) dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Observe that

$$(3.25) \quad \|u_n - u\|^2 = (\Phi'(u_n) - \Phi'(u))(u_n - u) + \int_{\mathbb{R}} [f(x, u_n) - f(x, u)](u_n - u) dx.$$

It is clear that

$$(3.26) \quad (\Phi'(u_n) - \Phi'(u))(u_n - u) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Combining (3.24) – (3.26), we get $\|u_n - u\| \rightarrow 0$ as $n \rightarrow \infty$. The proof of Lemma 3.5 is completed.

Consequently, Lemma 2.3 together with Remark 2.1 imply that Φ possesses an unbounded sequence of critical points. Therefore $(\mathcal{FOD}\mathcal{E})$ possesses infinitely many homoclinic solutions. The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2.

Lemma 3.6. Under assumptions $(\mathcal{A}_0)$, $(F_1) - (F_3)$ and (F_5) , Φ satisfies the $(C)_c$ -condition for all $c \in \mathbb{R}$.

Proof. Let $c \in \mathbb{R}$ and $(u_n) \subset E$ satisfying (3.19). First, we prove that (u_n) is bounded in E . Arguing by contradiction, suppose that $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$ and $\|v_n\| \leq \eta_p \|v_n\|_{L^p}$ for $2 \leq p \leq \infty$. By (F_5) , one has for n large enough

$$\begin{aligned} c + 1 &\geq \Phi(u_n) - \frac{1}{\mu} \Phi'(u_n) u_n \\ &= \frac{\mu - 2}{2\mu} \|u_n\|^2 + \int_{\mathbb{R}} \left[\frac{1}{\mu} f(x, u_n) u_n - F(x, u_n) \right] dx \\ &\geq \frac{\mu - 2}{2\mu} \|u_n\|^2 - \frac{\rho}{\mu} \|u_n\|_{L^2}^2, \end{aligned}$$

which implies

$$(3.27) \quad \frac{\mu - 2}{2\rho} \leq \limsup_{n \rightarrow \infty} \|v_n\|_{L^2}^2.$$

Taking a subsequence if necessary, we may assume that $v_n \rightarrow v$ in $L^2(\mathbb{R})$ and $v_n \rightarrow v$ a.e. on $\mathbb{R}$. Hence, it follows from (3.27) that $v \neq 0$. By a similar fashion as for (3.14), we can get a contradiction. Therefore (u_n) is bounded in E . The rest of the proof is the same as that in Lemma 3.5 and the proof of Lemma 3.6 is completed.

We conclude as in the proof of Theorem 1.1 that Φ possesses an unbounded sequence of critical points and the proof of Theorem 1.2 is completed.

4 Conclusion

Assume that the function a is coercive unnecessary positive and $F(x, u) = \int_0^u f(x, t) dt$ satisfies some kinds of superquadratic conditions at infinity in the second variable unnecessary satisfying the well-known Ambrosetti-Rabinowitz superquadratic growth condition., we prove that the fourth-order equation

$$u^{(4)}(x) + \omega u''(x) + a(x)u(x) = f(x, u(x)), \quad \forall x \in \mathbb{R}$$

possesses infinitely many nontrivial homoclinic solutions by applying critical point theory to the associated least action integral over an appropriate functional space.

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