Existence Solutions of the Complex Linear Differential Equations in $Q_{K,\omega}(p, q)$ Spaces

A. Kamal ^{*1,2} and T.I. Yassen¹

¹ Port Said University, Faculty of Science, Department of Mathematics, Port Said, Egypt

² Current address: Qassim University, College of Sciences and Arts in Muthnib, Department of Mathematics

¹ ala_mohamed1@yahoo.com ² taha_hmour@yahoo.com

Abstract. In this article, we study the existence of solutions of the complex linear differential equation of the form $f^{(n)} + A_{n-1}(z)f^{(n-1)} + \dots + A_1(z)f' + A_0(z)f = 0$, where the coefficients $A_0, A_1, \dots, A_n$ are analytic functions in the unit disc. To carry out the existence of these solutions, we obtain some conditions for the coefficients which grantee the existence of the solutions in $Q_{K,\omega}(p, q)$ spaces.

Keywords: Linear differential equation, Carleson measure characterization, $Q_{K,\omega}(p, q)$ spaces.
MSC2010 47 B 33, 46 E 15.

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $\partial\mathbb{D}$ be it's boundary. Let $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$. Let dA denote the Lebesgue measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$.

For $a \in \mathbb{D}$ the Möbius transformations $\varphi_a(z)$ is defined by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \text{ for } z \in \mathbb{D}.$$

The following identity is easily verified:

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}z|^2} = (1 - |z|^2)|\varphi'_a(z)|.$$

(see [4, 19]) Let f be an analytic function in $\mathbb{D}$ and

$0 < \alpha < \infty$. The α -Bloch space $\mathcal{B}^\alpha$ is defined by

$$\mathcal{B}^\alpha = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty\},$$

the little α -Bloch space $\mathcal{B}_0^\alpha$ is given as follows:

$$\mathcal{B}_0^\alpha = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_0^\alpha} = \lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha |f'(z)| = 0\}.$$

(see [15]) Let $0 < \alpha < \infty$ and $\omega : (0, 1] \rightarrow (0, \infty)$. For an analytic function f in $\mathbb{D}$, we define the weighted α -Bloch space $\mathcal{B}_\omega^\alpha$, as follows:

$$\mathcal{B}_\omega^\alpha = \left\{ f : f \text{ analytic in } \mathbb{D} \text{ and } \|f\|_{\mathcal{B}_\omega^\alpha} = \sup_{z \in \mathbb{D}} \frac{(1 - |z|)^\alpha |f'(z)|}{\omega(1 - |z|)} < \infty \right\}.$$

*Corresponding author. A. Kamal ¹ ala_mohamed1@yahoo.com

[12] Let f be an analytic function in $\mathbb{D}$ and let $0 < p < \infty$. If

$$\|f\|_{H^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty,$$

then f belongs to the Hardy space H^p . If $\|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} |f(z)| < \infty$, then f belongs to the Hardy space H^∞ . Moreover, $f \in H^2$ if and only if

$$\int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2) dA(z) < \infty.$$

(see [2, 17]) Let $K : [0, \infty) \rightarrow [0, \infty)$ be a right continuous and nondecreasing function. For $0 < p < \infty$ and $-2 < q < \infty$. The space $Q_K(p, q)$ is defined by

$$Q_K(p, q) = \{f \in H(\mathbb{D}) : \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \infty\}.$$

If

$$\lim_{|a| \rightarrow 1^-} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) = 0,$$

then $f \in Q_{K,0}(p, q)$. Let the Green's function of $\mathbb{D}$ with logarithmic singularity at a be

$$g(z, a) = \log \left| \frac{1 - \bar{a}z}{z - a} \right| = \log \frac{1}{|\varphi_a(z)|}.$$

(see [3, 15]) Let $K : [0, \infty) \rightarrow [0, \infty)$ be right-continuous and nondecreasing function, $0 < p < \infty$, $-2 < q < \infty$ and for given reasonable function $\omega : (0, 1] \rightarrow (0, \infty)$, an analytic function f in $\mathbb{D}$ is said to belong to the space $Q_{K,\omega}(p, q)$ if

$$\|f\|_{Q_{K,\omega}(p,q)} = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \frac{(1 - |z|)^q}{\omega^p(1 - |z|)} K(g(z, a)) dA(z) < \infty.$$

Active research in field of of complex differential equations was started by H. Wittich in 1950. An important result due to Wittich [7] concerning the growth of the solution of linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_1(z)f' + A_0(z)f = 0. \quad (1)$$

The coefficients $A_0(z), \dots, A_{k-1}(z)$ in (1) are polynomials in the complex plane if and only if all solutions of (1) are entire functions of finite order of growth.

A homogeneous complex differential equation of order k as in (1) where $z \in \mathbb{D} \subseteq \mathbb{C}$ and the coefficients $A_j(z)$ are analytic functions in $\mathbb{D}$ for $j = 0, \dots, k-1$, the solutions of (1) are analytic in $\mathbb{D}$ too, and there are exactly k linearly independent solutions of (1). This means that if we have k linearly independent solutions, all other solution can be represented as a linear combination of these solutions.

In 1982 Chr. Pommerenke [14] studied the second-order equation

$$f'' + A(z)f = 0, \quad (2)$$

where $A(z)$ is an analytic function in $\mathbb{D}$. We pick two of these results that seem to of major importance. (Theorem 2, [14]) Let $A(z)$ be the analytic coefficients of (2). There is an absolute constant $\alpha > 0$ with the following property: If

$$\int_{\mathbb{D}} |A(z)|^2 (1 - |z|^2)^3 \frac{1 - |a|^2}{|1 - \bar{a}z|^2} dA(z) \leq \alpha,$$

then all solutions of (2) belong to H^2 and

$$A(z) \leq \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{(1 - |a|^2)^2}{|z - \bar{a}|^4} dA(z) \leq 2A(z).$$

(Corollary 1, [14]) Let $A(z)$ be the analytic coefficients of (2) and $|A(z)| \leq Q(r)$, $|z| \leq r$ where

$$\int_0^1 Q(r)^2 (1-r)^3 dr < \infty,$$

then all solutions of (2) belong to H^2 . J. Heittokangas [8] studied the equation

$$f^{(k)} + A(z)f = 0, \quad (3)$$

where $A(z)$ is an analytic function in $\mathbb{D}$ and $k \in \mathbb{N}$, and gave the following theorem. [8] Let $q > 0$ There exist constant $\alpha = \alpha(q, n) > 0$ such that, if the analytic coefficients $A(z)$ of (3) satisfy

$$|A(z)|(1-|z|^2)^n \leq \alpha,$$

for all $z \in \mathbb{D}$, then all solutions of (3) belong to H_q^∞ , where

$q = \frac{\alpha}{(k-1)!}$. J. Heittokangas et al [9] studied the linear differential equation where the coefficients $A_0(z), \dots, A_{k-1}(z)$ are analytic in the disc $D_R = \{z \in \mathbb{C} : |z| < R\}$, $0 < R \leq \infty$, where all solution belong to H_q^∞ -space or $\mathcal{D}^p$ -space and proved the following theorem.

[9] Let $q > 0$. There exist constant $\alpha = \alpha(q, k) > 0$ such that, if the coefficients $A_j(z)$ be the analytic coefficients of (1) satisfy

$$\|A_j\|_{H_{k-j}^\infty} = \sup_{z \in \mathbb{D}} |A_j(z)|(1-|z|^2)^{k-j} \leq \alpha, \quad j = 0, \dots, n-1,$$

then all solutions of (1) belong to H_q^∞ . Also, J. Heittokangas et al [10] studied the linear differential equation (1) with coefficients in weighted Bergman or Hardy spaces are studied.

The growth of solutions of the linear differential equation (1) with coefficients A_j in the unit disc has attracted a lot of attention (see [8, 9, 14, 10] and others).

2 Characterization of general weighted $Q_{K,\omega}(p, q)$ spaces

In this section, we list and give results and definitions which characterize $Q_{K,\omega}(p, q)$ spaces.

An important tool in the study of $Q_{K,\omega}(p, q)$ spaces is the auxiliary functions ϕ_K and ψ_ω defined by

$$\phi_K(s) = \sup_{0 < t < 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty,$$

and

$$\psi_\omega(s) = \sup_{0 < t < 1} \frac{\omega(st)}{\omega(t)}, \quad 0 < s < 1.$$

The following conditions have played crucial roles in the study of $Q_{K,\omega}(p, q)$ spaces (see [1, 11]).

$$\int_0^1 \phi_K(s) \frac{ds}{s} < \infty. \quad (4)$$

$$\int_1^\infty \phi_K(s) \frac{ds}{s^2} < \infty. \quad (5)$$

$$\int_1^{\frac{1}{t}} \psi_\omega(s) \frac{ds}{s^2} < \infty. \quad (6)$$

For a subarc $I \subset \partial\mathbb{D}$, where $|I|$ denotes the normalized arc length of I (so that $|\partial\mathbb{D}| = 1$). $S(I)$ is the Carleson box defined by

$$S(I) = \{z \in \mathbb{D} : 1 - |I| < |z| < 1, \frac{z}{|z|} \in I\}.$$

If $|I| \geq 1$, then we set $S(I) = \mathbb{D}$. (see [6]) A positive measure $d\mu$ is said to be a *bounded* K -Carleson measure on $\mathbb{D}$ if

$$\sup_{I \subset \partial\mathbb{D}} \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) d\mu(z) < \infty.$$

Moreover, if

$$\lim_{|I| \rightarrow 0} \int_{S(I)} K \left(\frac{1-|z|}{|I|} \right) d\mu(z) = 0,$$

then μ is a compact K -Carleson measure. [11] Let K satisfy (4). For $0 < p < \infty$, $-2 < q < \infty$, then the following are equivalent

(i) $f \in Q_{K,\omega}(p, q)$,

(ii)

$$\left(\frac{4}{1-|a|^2} \right)^{2-p} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |(f \circ \varphi_a)'(z)|^p (1-|\varphi_a(z)|^2)^q \frac{K(1-|z|^2)}{\omega^p(1-|\varphi_a(z)|)} dA(z) < \infty,$$

(iii) $|f'(z)|^p \frac{(1-|z|^2)^q}{\omega^p(1-|z|)} dA(z)$ is a K -Carleson measure on $\mathbb{D}$.

Let f be an analytic function in $\mathbb{D}$, $K : [0, \infty) \rightarrow [0, \infty)$, $\omega : (0, 1] \rightarrow [0, \infty)$, $0 < p < \infty$, $-2 < q < \infty$. Let $n \in \mathbb{N}$. Denote $d\mu = \frac{|f^{(n)}(z)|^p (1-|z|^2)^{np-p+q}}{\omega^p(1-|\varphi_a(z)|)} dA(z)$.

Then the following quantities are comparable:

$$(1) \|f\|_{Q_{K,\omega}(p,q)}^p + |f(0)|^2,$$

$$(2) \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^p \frac{(1-|z|^2)^{np-p+q}}{\omega^p(1-|z|)} K(1-|\varphi_a(z)|^2) dA(z) + \sum_{j=1}^{n-1} |f^{(j)}(0)|^p,$$

$$(3) \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^p \frac{(1-|z|^2)^{np-p+q}}{\omega^p(1-|z|)} K(g(z, a)) dA(z) + \sum_{j=1}^{n-1} |f^{(j)}(0)|^p,$$

$$(4) \sup_I \frac{\mu(S(I))}{|I|^K} + \sum_{j=1}^{n-1} |f^{(j)}(0)|^p.$$

Proof. The Proof can be obtained by Theorem 3.1 in [5] and [16], so will be omitted. Now, we give following definition. Let $K : [0, \infty) \rightarrow [0, \infty)$ be right-continuous and nondecreasing function. If $0 < p < \infty$, $-2 < q < \infty$, and for given reasonable function $\omega : (0, 1] \rightarrow (0, \infty)$, an analytic function f in $\mathbb{D}$ is said to belong to the space $Q_{K,\omega}(p, q)$ if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f^n(z)|^p \frac{(1-|z|^2)^{np-p+q}}{\omega^p(1-|z|)} K(1-|\varphi_a(z)|^2) dA(z) < \infty. \quad (7)$$

The proof of the following lemma is very similar Lemma 7 in [18]. Let f be an analytic function in $\mathbb{D}$ and let $0 < p < \infty$, $-1 < q < \infty$. Then there exist two positive constant C depending only on p and q , such that

$$\begin{aligned} & \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^q K \left(\log \frac{1}{|z|} \right) dA(z) \\ & \leq C \left(|f(0)|^p + \int_{\mathbb{D}} |f'(z)| (1-|z|^2)^{p+q} K \left(\log \frac{1}{|z|} \right) dA(z) \right). \end{aligned}$$

In this paper, the latter C denotes a positive constant throughout the chapter which may vary at each occurrence.

3 Main results

In this section, we consider the following linear differential equation with an analytic coefficients in the unite disc $\mathbb{D}$.

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \dots + A_1(z)f' + A_0(z)f = 0, \quad (8)$$

our attention is to build up a relationship between the coefficients and solutions of the equation (8), and give the conditions for the coefficients of (8) such that all solutions of (8) belong to $Q_{K,\omega}(p, q)$. We begin with the following

theorem. Let $K : [0, \infty) \rightarrow [0, \infty)$ be right-continuous and nondecreasing function, $0 < p < \infty$, $-2 < q < \infty$, and for given reasonable function $\omega : (0, 1] \rightarrow (0, \infty)$. If K satisfy

$$\int_1^\infty \frac{\varphi_K(s)}{s^{2c}} ds < \infty, \quad (1/2 < c < 1). \quad (9)$$

Then there exist a constant $\alpha = \alpha(n, c, K) > 0$ such that if the coefficient A_j of (8) satisfy

$$\|A_j\|_{H_{n-j}^\infty} = \sup_{z \in \mathbb{D}} |A_j(z)|(1 - |z|^2)^{n-j} < \alpha, \quad j = 1, \dots, n-1. \quad (10)$$

And

$$\|A_0\|_{H_{n-c}^\infty} = \sup_{z \in \mathbb{D}} |A_0(z)|(1 - |z|^2)^{n-c} < \alpha, \quad (11)$$

then all solutions of (8) belong to the $Q_{K,\omega}(p, q)$ spaces.

Proof. Let f be non trivial solution of (8). Let $f_\rho(z) = f(\rho z)$ and $A_{j,\rho}(z) = A_j(\rho z)$, where $1/2 \leq \rho < 1$. Then

$$\begin{aligned} & \int_{\mathbb{D}} |f_\rho^{(n)}(z)|^p \frac{(1 - |z|^2)^{np-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &= \int_{\mathbb{D}} \left| \sum_{j=0}^{n-1} A_j(\rho z) f^{(j)}(\rho z) \right|^p \frac{(1 - |z|^2)^{np-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &\leq C \int_{\mathbb{D}} \sum_{j=0}^{n-1} |A_j(\rho z)|^p |f^{(j)}(\rho z)|^p \frac{(1 - |z|^2)^{np-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &\leq C \alpha^p \int_{\mathbb{D}} \sum_{j=1}^{n-1} |f^{(j)}(\rho z)|^p \frac{(1 - |z|^2)^{jp-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &\quad + C \alpha^p \int_{\mathbb{D}} |f(\rho z)|^p \frac{(1 - |z|^2)^{cp-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &:= C \alpha^p (T_1 + T_2). \end{aligned} \quad (12)$$

Using (7), we have

$$\begin{aligned} T_1 &= \int_{\mathbb{D}} \sum_{j=1}^{n-1} |f^{(j)}(\rho z)|^p \frac{(1 - |z|^2)^{jp-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &\leq (n-1) \|f_\rho\|_{Q_{K,\omega}(p,q)}^p. \end{aligned} \quad (13)$$

On the other hand, fix an r and Lemma 2.13 in [20] yields

$$\begin{aligned} T_2 &= \int_{\mathbb{D}} |f(\rho z)|^p \frac{(1 - |z|^2)^{cp-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &\leq \sum_{i=1}^{\infty} \int_{\mathbb{D}(a_i, r)} |f(\rho z)|^p \frac{(1 - |z|^2)^{cp-p+q}}{\omega^p(1 - |z|)} K(1 - |z|^2) \\ &\quad \times \varphi_K\left(\frac{1 - |a|^2}{|1 - \bar{a}z|^2}\right) dA(z). \end{aligned} \quad (14)$$

From Lemma 2.14 in [20], we have

$$|f_\rho(z)| \leq \frac{C}{|\mathbb{D}(a_i, r)|_A} \int_{\mathbb{D}(a_i, r)} |f(\rho w)|^p dA(w), \quad (15)$$

where $|\mathbb{D}(a_i, r)|_A$ is Euclidean area of the hyperbolic disc $\mathbb{D}(a_i, r)$, and this with the facts (see [20])

$$C^{-1}(1 - |z|^2) \leq |1 - \bar{a}z| \leq C(1 - |z|^2),$$

and

$$|\mathbb{D}(a_i, r)|_A \sim (1 - |z|^2)^2 \sim (1 - |w|^2)^2,$$

we have

$$\begin{aligned} T_2 &= \int_{\mathbb{D}} |f(\rho z)|^p \frac{(1 - |z|^2)^{cp-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &\leq C \sum_{i=1}^{\infty} \int_{\mathbb{D}(a_i, r)} \frac{1}{|\mathbb{D}(a_i, r)|_A} \\ &\quad \times \int_{\mathbb{D}(a_i, r)} |f(\rho w)|^p dA(w) \frac{(1 - |z|^2)^{cp-p+q}}{\omega^p(1 - |z|)} K(1 - |z|^2) \varphi_K\left(\frac{1 - |a|^2}{|1 - \bar{a}z|^2}\right) dA(z) \\ &\leq C \sum_{i=1}^{\infty} \int_{\mathbb{D}(a_i, r)} (1 - |z|^2)^{cp} \varphi_K\left(\frac{1 - |a|^2}{|1 - \bar{a}z|^2}\right) dA(z) \\ &\quad \times \frac{1}{|\mathbb{D}(a_i, r)|_A} \int_{\mathbb{D}(a_i, r)} |f(\rho w)|^p (1 - |w|^2)^{q-p} \frac{K(1 - |w|^2)}{\omega^p(1 - |w|)} dA(w) \\ &\leq C \sum_{i=1}^{\infty} \int_{\mathbb{D}(a_i, r)} (1 - |z|^2)^{cp-2} \varphi_K\left(\frac{1}{1 - |z|^2}\right) dA(z) \\ &\quad \times \int_{\mathbb{D}(a_i, r)} |f(\rho w)|^p (1 - |w|^2)^{q-p} \frac{K(1 - |w|^2)}{\omega^p(1 - |w|)} dA(w). \end{aligned}$$

From Lemma 2.12 in [20], we know that $(1 - |a_i|^2) \sim |1 - \bar{a}z| \sim (1 - |z|^2)$, this gives

$$\begin{aligned} T_2 &= \int_{\mathbb{D}} |f(\rho z)|^p \frac{(1 - |z|^2)^{cp-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &\leq C \sum_{i=1}^{\infty} \int_{\mathbb{D}(a_i, r)} (1 - |z|)^{2c-2} (1 - |z|)^{cp-2c} \varphi_K\left(\frac{1}{1 - |z|^2}\right) dA(z) \\ &\quad \times \int_{\mathbb{D}(a_i, r)} |f(\rho w)|^p (1 - |w|^2)^{q-p} \frac{K(1 - |w|^2)}{\omega^p(1 - |w|)} dA(w). \\ &\leq C \sum_{i=1}^{\infty} \int_{\mathbb{D}(a_i, r)} (1 - |z|^2)^{2c-2} (1 - |a_i|^2)^{cp-2c} \varphi_K\left(\frac{1}{1 - |z|^2}\right) dA(z) \\ &\quad \times \int_{\mathbb{D}(a_i, r)} |f(\rho w)|^p (1 - |w|^2)^{q-p} \frac{K(1 - |w|^2)}{\omega^p(1 - |w|)} dA(w). \\ &\leq C \sum_{i=1}^{\infty} \int_0^1 (1 - r^2)^{2c-2} \varphi_K\left(\frac{1}{1 - r^2}\right) r dr \\ &\quad \times \int_{\mathbb{D}(a_i, r)} |f(\rho w)|^p (1 - |w|^2)^{q-p} \frac{K(1 - |w|^2)}{\omega^p(1 - |w|)} dA(w). \end{aligned}$$

Now, take $s = \frac{1}{1-r^2}$, then $ds = \frac{2r}{(1-r^2)^2} dr$ and using (4), we get

$$\begin{aligned}
T_2 &\leq C \sum_{i=1}^{\infty} \int_1^{\infty} \left(\frac{\varphi_K(s)}{s^{2c}} \right) ds \\
&\quad \times \int_{\mathbb{D}(a_i, r)} |f(\rho w)|^p (1 - |w|^2)^{q-p} \frac{K(1 - |w|^2)}{\omega^p(1 - |w|)} dA(w). \\
&\leq C \sum_{i=1}^{\infty} \int_{\mathbb{D}(a_i, r)} |f(\rho w)|^p (1 - |w|^2)^{q-p} \frac{K(1 - |w|^2)}{\omega^p(1 - |w|)} dA(w). \\
&\leq C \int_{\mathbb{D}} |f(\rho w)|^p (1 - |w|^2)^{q-p} \frac{K(1 - |w|^2)}{\omega^p(1 - |w|)} dA(w).
\end{aligned}$$

Using Lemma 2, we have

$$\begin{aligned}
T_2 &\leq C \int_{\mathbb{D}} |f(\rho w)|^p (1 - |w|^2)^{q-p} \frac{K(1 - |w|^2)}{\omega^p(1 - |w|)} dA(w). \\
&\leq C \left(\int_{\mathbb{D}} |f'(\rho w)|^p (1 - |w|^2)^q \frac{K(1 - |w|^2)}{\omega^p(1 - |w|)} dA(w) + |f(0)|^p \right).
\end{aligned}$$

Then, we have

$$T_2 \leq C \left(\|f_\rho\|_{Q_{K,\omega}(p,q)}^p + |f(0)|^p \right). \quad (16)$$

Combining (12), (14) and (17), we have

$$(1 - C\alpha^p(1 - n + C)) \|f_\rho\|_{Q_{K,\omega}(p,q)}^p \leq C^2 \alpha^p |f(0)|^p, \quad (17)$$

where the constant C depending only on n , K , then $f \in Q_{K,\omega}(p, q)$. Therefor, the proof is complete. (1) Theorem 3 is a generalization of results in [13] and [5].

(2) If we take $p = 2$ and $q = 0$, in Theorem 3, then we have the following corollary. Let K satisfy

$$\int_1^{\infty} \frac{\varphi_K(s)}{s^{2c-1}} ds < \infty, \quad (1 < c < 3/2). \quad (18)$$

Then there exist a constant $\alpha = \alpha(n, c, K) > 0$ such that if the coefficient A_j of (8) satisfy

$$\|A_j\|_{H_{n-j}^\infty} = \sup_{z \in \mathbb{D}} |A_j(z)| (1 - |z|^2)^{n-j} < \alpha, \quad j = 1, \dots, n-1. \quad (19)$$

And

$$\|A_0\|_{H_{n-c}^\infty} = \sup_{z \in \mathbb{D}} |A_0(z)| (1 - |z|^2)^{n-c} < \alpha, \quad (20)$$

then all solutions of (8) belong to the $Q_{K,\omega}$ spaces.

Now, for other condition on K we are able to state a different result comparing to Theorem 3.

Let $K : [0, \infty) \rightarrow [0, \infty)$ be right-continuous and nondecreasing function, $0 < p < \infty$, $-2 < q < \infty$, and for given reasonable function $\omega : (0, 1] \rightarrow (0, \infty)$. If K satisfy

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty. \quad (21)$$

Then, there exist a constant $\beta = \beta(n, K) > 0$ such that if the coefficient A_j of (8) satisfy

$$\|A_j\|_{H_{n-j}^\infty} = \sup_{z \in \mathbb{D}} |A_j(z)| (1 - |z|^2)^{n-j} < \beta, \quad j = 1, \dots, n-1. \quad (22)$$

And

$$\|A_0\|_{H_{n-1}^\infty} = \sup_{z \in \mathbb{D}} |A_0(z)|(1 - |z|^2)^{n-1} < \beta, \quad (23)$$

then all solutions of (8) belong to the $Q_{K,\omega}(p, q)$ spaces.

Proof. Let f be non trivial solution of (8). Let $f_\rho(z) = f(\rho z)$ and $A_{j,\rho}(z) = A_j(\rho z)$, where $1/2 \leq \rho < 1$. Then

$$\begin{aligned} & \int_{\mathbb{D}} |f_\rho^{(n)}(z)|^p \frac{(1 - |z|^2)^{np-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &= \int_{\mathbb{D}} \left| \sum_{j=0}^{n-1} A_j(\rho z) f^{(j)}(\rho z) \right|^p \frac{(1 - |z|^2)^{np-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &\leq C \int_{\mathbb{D}} \sum_{j=0}^{n-1} |A_j(\rho z)|^p |f^{(j)}(\rho z)|^p \frac{(1 - |z|^2)^{np-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &\leq C \beta^p \int_{\mathbb{D}} \sum_{j=1}^{n-1} |f^{(j)}(\rho z)|^p \frac{(1 - |z|^2)^{jp-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &+ C \beta^p \int_{\mathbb{D}} |f(\rho z)|^p \frac{(1 - |z|^2)^q}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &:= C \beta^p (L_1 + L_2). \end{aligned} \quad (24)$$

Then, we have

$$\begin{aligned} L_1 &= \int_{\mathbb{D}} \sum_{j=1}^{n-1} |f^{(j)}(\rho z)|^p \frac{(1 - |z|^2)^{jp-p+q}}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &\leq (n-1) \|f_\rho\|_{Q_{K,\omega}(p,q)}^p. \end{aligned} \quad (25)$$

For a point $z \in \mathbb{D}$ we have

$$|f(z)| \leq |f(0)| + \int_0^{|z|} |f'(\xi)| d|\xi|.$$

Hence,

$$\begin{aligned} L_2 &= \int_{\mathbb{D}} |f(\rho z)|^p \frac{(1 - |z|^2)^q}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &\leq \int_{\mathbb{D}} \left(|f(0)| + \int_0^{|z|} |f'(\xi)| d|\xi| \right)^p \frac{(1 - |z|^2)^q}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &\leq C |f(0)|^p + C \int_0^{|z|} |f'(\xi)|^p d|\xi| \frac{(1 - |z|^2)^q}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z). \end{aligned}$$

Since $Q_{K,\omega}(p, q) \subset \mathcal{B}_{\omega^{\frac{q+2}{p}}}$ (see [15]) and using the same steps which we used in Theorem 2.1 in [11], we deduce

$$\begin{aligned} L_2 &\leq C |f(0)|^p + C \int \int_{|g(z,a)| > 1} |f'(z)|^p \frac{(1 - |z|^2)^q}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z) \\ &+ C \int_{|g(z,a)| \leq 1} |f'(z)|^p \frac{(1 - |z|^2)^q}{\omega^p(1 - |z|)} K(1 - |\varphi_a(z)|^2) dA(z). \end{aligned}$$

Then,

$$L_2 \leq C|f(0)|^p + C(I_1 + I_2) \quad (26)$$

From the prove of Theorem 2.1 in [11], we have

$$I_1 \leq C\|f_\rho\|_{Q_{K,\omega}(p,q)}, \quad (27)$$

and

$$I_2 \leq C \frac{\varphi_K(s)}{s} ds < \infty. \quad (28)$$

Combining (27) and (28) with (26), we have

$$L_2 \leq C \int_{\mathbb{D}} |f(0)|^p + C\|f_\rho\|_{Q_{K,\omega}(p,q)} + C. \quad (29)$$

Thus, from (24), (25) and (29), we see that

$$(1 - C(n - 1 + C)\beta^p)\|f_\rho\|_{Q_{K,\omega}(p,q)} \leq C^2\beta^p(|f(0)|^p + 1). \quad (30)$$

By choosing β sufficiently small and letting $\rho \rightarrow 1^-$. Then the conclusion $f \in Q_{K,\omega}(p,q)$ follows and the proof is complete. If we take $p = 2$ and $q = 0$, in Theorem 3, then we have the following corollary. If K satisfy

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty. \quad (31)$$

Then, there exist a constant $\beta = \beta(n, K) > 0$ such that if the coefficient A_j of (8) satisfy

$$\|A_j\|_{H_{n-j}^\infty} = \sup_{z \in \mathbb{D}} |A_j(z)|(1 - |z|^2)^{n-j} < \beta, \quad j = 1, \dots, n-1. \quad (32)$$

And

$$\|A_0\|_{H_{n-1}^\infty} = \sup_{z \in \mathbb{D}} |A_0(z)|(1 - |z|^2)^{n-1} < \beta, \quad (33)$$

then all solutions of (8) belong to the $Q_{K,\omega}$ spaces.

4 Conclusion

Our contribution is built up a relationship between the coefficients and solutions of the equation. Moreover, sufficient conditions for the analytic coefficients A_n such that all solutions belong to the $Q_{K,\omega}(p,q)$ spaces are given.

References

- [1] A. El-Sayed Ahmed and Alaa Kamal, Generalized composition operators on $Q_K(p,q)$ spaces. Mathematical Sciences 6.1 (2012): 14., DOI:10.1186/2251-7456-6-14.
- [2] A. El-Sayed Ahmed and A. Kamal, Logarithmic order and type on some weighted function spaces. Journal of applied functional analysis 7 (2012), 108-117.
- [3] A. El-Sayed Ahmed and A. Kamal. Generalized composition operators on $Q_K(p,q)$ spaces. J. Frac. Calculus Appl.(Proc. of the 4th. Symb. of Fractional Calculus and Applications) No 18 (2012): 1-9.
- [4] A. El-Sayed Ahmed and Alaa Kamal, Series expansions on some analytic function spaces. J. of Comput. Theo.Nanosci. 12(8) (2015), 1586-1593.

- [5] A. El-Sayed Ahmed and Saleh Omran, Solutions of certain complex linear differential equations, *J. of Comput. Theo.Nanosc.* 12.11 (2015): 4272-4278.
- [6] M. Essen, H. Wulan, and J. Xiao, Several Function-theoretic characterizations of Mobius invariant Q_K spaces, *J. Funct. Anal.* 230 (2006), 78-115.
- [7] G. Frank and H. Wittich, Zur theorie linearer differentialgleichungen im Komplexen. *Ann. Acad. Sci. Fenn. Ser. A I Math.* 130(4)(1973), 363-370.
- [8] J. Heittokangas, On complex differential equations in the unit disc, *Ann. Acad. Sci. Fenn. Math. Diss.*, 122 (2000).
- [9] J. Heittokangas, R. Korhonen and J. Rättyä, Linear differential equations with solutions in the Dirichlet type subspace of the Hardy space, *Nagoya Math. J.* Vol. 187 (2007), 91-113.
- [10] J. Heittokangas, R. Korhonen and J. Rättyä, Linear differential equations with coefficients in weighted Bergman and Hardy spaces, *Trans. American Math. Soc.* 360(2) (2008), 1035-1055.
- [11] A. Kamal, A. El-Sayed Ahmed and T. Yassen, Carleson measures and Hadamard products in some general analytic function spaces, *J. Comput. Theor. Nsc.* 12(2015), 2227-2236.
- [12] J. H. Shapiro, Composition operators and classical function theory, Springer-Verlay, New York, 1993.
- [13] H. Li and H. Wulan, Linear differential equations with solutions in the Q_K spaces, *J Math. Anal. Appl.* 375(2)(2011), 478-489.
- [14] Chr. Pommerenke, On the mean growth of the solutions of complex linear differential equations in the disk, *Complex Variables*, 1 (1982), 23-38.
- [15] R. A. Rashwan, A. El-Sayed Ahmed and Alaa Kamal, Integral characterizations of weighted Bloch spaces and $Q_{K,\omega}(p, q)$ spaces, *Math. tome.* 51(74)(2)(2009), 63-76.
- [16] J. Rättyä, On some Complex function spaces and classes, *Annales Academiae Scientiarum Fennicae. Series A I. Mathematica. Dissertationes.* 124. Helsinki: Suomalainen Tiedeakatemia, (2001), 1-73.
- [17] H. Wulan, and J. Zhou, Q_K type spaces of analytic functions, *J. Fun. Spaces.* 4(1)(2006), 73-84.
- [18] H. Wulan, and J. Zhou, The higher order derivatives of Q_K type spaces, *J. math. anal. appl.* 332 (2007), 1216-1228.
- [19] R. Zhao, On -Bloch functions and $VMOA$, *Acta Math. Sci. Ser.* 16(3) (1996), 349-360.
- [20] K. Zhu, Theory of Bergman spaces, Springer, New York, 2000.