Regional Gradient Exact Enlarge Controllability of the Semilinear Heat Equation

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Abstract. The aim of this paper is to study the enlarge gradient-controllability problem of the heat equation. The purpose is to compute the control u which steers the state gradient of the system between two prescribed levels l_1 and l_2 , only on a subregion ω of the system evolution domain Ω . The obtained results have been proved via two approaches, the sub-differential and Lagrangian multiplier approach.

Keywords: *Distributed systems, parabolic systems, regional controllability, gradient, sub-differential, Lagrangian approach, semilinear systems, minimum energy, heat equation.*

1 Introduction

The regional controllability concept refers to control problems in which the target of interest is not fully specified as a state, but refers only to a region ω of the spatial domain Ω on which the governing partial differential system (1) is considered. In many real-world dynamic systems, this regional idea occurs naturally when it comes to modeling natural phenomenon just in a specific region. For example, when we want to keep the temperature of a place or a device between two bounds T_1 and T_2 . This concept has been widely developed and interesting results have been proved, in particular, the possibility to reach a state only on an internal subregion ω of Ω [1] or on a part of the boundary $\partial\Omega$ of Ω [2].

Later the concept of internal regional gradient controllability was developed [3]. It consists of steering the state gradient of system (1) to a prescribed function only given on ω located in the interior of Ω . Here instead of steering the system to a desired gradient, we are interested in steering its gradient between two prescribed functions given only on a subregion ω in the interior of Ω . This problem is both practical and beneficial to study and this is due to the fact that we may be only interested in the knowledge of the states in a critical subregion of the whole domain. And also it is more realistic and more adapted for system analysis than the classical one since, in general, the dynamic of a physical process isn't really well represented by a mathematical model of a real system, and there can be many reasons for this: the parameters may not be known precisely, the model can be of reduced order, also, in order to simplify the calculations, the model can be a linear approximation of a nonlinear process, and since the calculated control is based only on an approximate model, the final state of the original system will not be exact. Besides in numerical calculations and because of rounding errors, a computed solution can be far from the actual one. Therefore, the solutions of such systems are known approximately as well as their gradients.

There are many reasons motivating the study of this problem which is the enlarge controllability or also called controllability with constraints on the state [4]: first, the mathematical model of a real system is obtained from measurements or from approximation techniques and it is very often affected by perturbations. Consequently, the

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solution of such system is approximately known, and second, in various real problems the target is required to be between two bounds. This is the case, for example, of a nuclear reactor in which the temperature of the water circulating around the reactor should not exceed a certain level. Many works were subject to study the problem in the parabolic case [5] or the hyperbolic for linear systems [6].

This paper is an extension of the previous ones and it deals with the controllability properties of semilinear heat equation where the control u is exerted in an intern subregion ω of the system evolution domain Ω . Thus, let Ω be an open bounded set of $\mathbb{R}^n (n \geq 1)$ with regular boundary $\partial\Omega$. We consider the Banach spaces $\mathcal{H} = L^2(\Omega)$ and $\mathcal{V} = H_0^1(\Omega)$, and the corresponding norms $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{V}}$. For a given $T > 0$, we denote $Q_T = \Omega \times]0, T[$, $\Sigma_T = \partial\Omega \times]0, T[$ and let us consider the following problem : Minimize $J(u) = \frac{\alpha}{2} \|\chi_\omega \nabla y_u(T) - y_d\|_{\mathcal{V}}^2 + \frac{\beta}{2} \|u\|_U^2$

$$\text{subject to: } \begin{cases} \partial_t y(x, t) - \Delta y(x, t) - \mathcal{N}y(x, t) = Bu(t) & Q_T \\ y(x, 0) = y_0(x) & \Omega, \text{ where} \\ y(\xi, t) = 0 & \Sigma_T, \end{cases}$$

- α and β are two positive constants in $\mathbb{N}$,
- χ_ω the restriction operator defined by:

$$\begin{aligned} \chi_\omega : (\mathcal{H})^n &\longrightarrow (L^2(\omega))^n \\ y_i &\longmapsto \chi_\omega y_i = y_i|_\omega, \end{aligned}$$

while $\omega \in \Omega$ is of Lebesgue positive measure.

- ∇ is nabla operator given by the formula:

$$\begin{aligned} \nabla : \mathcal{V} &\longrightarrow (\mathcal{H})^n \\ y &\longmapsto \nabla y = \left(\frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n} \right), \end{aligned}$$

- $y_u(T)$ is the mild solution of (1) at the time T ($y_u(T) \in \mathcal{H}$ [7]), y_d is the desired state in $\mathcal{V}$ and u is a the control function in the control space $U = L^2(0, T; \mathbb{R}^m)$ (where m is the number of actuators),
- Δ is the Laplacian operator with dense domain and it is an infinitesimal generator of a C_0 -semigroup $(S(t))_{t \geq 0}$ on $\mathcal{H}$ which we consider compact,
- $\mathcal{N} : L^2(0, T; \mathcal{H}) \rightarrow \mathcal{H}$ is a non linear k -Lipschitz operator [8, 9],
- $B \in \mathcal{L}(\mathbb{R}^m, \mathcal{H})$ and y_0 is the initial datum in $\mathcal{H}$.

The problem (1) is well-posed and has a unique solution.

The remainder contents of this paper are structured as follows. Some definitions, properties and some notations are introduced in the next section and in section 3, the proof of the existence and the uniqueness of the solution of problem (1) is given. Two approaches to compute the control u are used in section 4.

2 Gradient enlarge controllability

Without loss of generality, we denote by $y_u(\cdot)$ the mild solution of (1) when it is excited by a control u , it is obtained using the idea of Duhamel's principle. The final solution at time T could be written as follows:

$$y_u(T) = S(T)y_0 + \int_0^T S(T-t)\mathcal{N}y(t) dt + \int_0^T S(T-t)Bu(t) dt.$$

We first introduce the closed sub-vectorial space F of $\mathcal{H}$. The problem of gradient exact enlarge controllability (depending on F) is defined as follows [10]:

Definition 2.1. *Given $T > 0$. We say that there is gradient exact enlarge controllability, if, for every y_0 (in a suitable functional space), we can find a control u such that*

$$\chi_\omega \nabla y(\cdot, T; u) \in F.$$

Remark 2.2. 1. This notion depends of course on the choice of the functional space F .

2. If $F = \{0\}$, we retrieve the classical notion of the exact controllability.

For the particular case, we will study the gradient enlarge controllability just in $[a(\cdot), b(\cdot)]$. For that let us consider $(a(\cdot))_i$ and $(b(\cdot))_i$ be two given real functions in $(L^2(\omega))^n$ such that $(a(\cdot))_i \leq (b(\cdot))_i$ for $i = 1, \dots, n$ a.e in ω , and we set:

$$\gamma = [a(\cdot), b(\cdot)] = \{(y_1, \dots, y_n) \in (L^2(\omega))^n \mid a_i(\cdot) \leq y_i(\cdot) \leq b_i(\cdot) \text{ a.e on } \omega \text{ for all } i \in \{1, \dots, n\}\}.$$

so the definition of the gradient enlarge controllability in γ is the following:

Definition 2.3. We say that (1) is γ -gradient controllable in ω at time T if there exists a command $u \in U$ such that:

$$\chi_\omega \nabla y_u(T) \in \gamma.$$

Let us define some operators:

- The Duhamel operator H from U to $\mathcal{V}$ such that for $u \in U$:

$$Hu = \int_0^T S(T-s)Bu(s)ds,$$

- and the operator G_γ given by:

$$\begin{aligned} G_\gamma : L^2(0, T; \mathcal{V}) &\longrightarrow \mathcal{V} \\ y(\cdot) &\longmapsto \int_0^T S(T-\tau)\mathcal{N}y(\tau)d\tau. \end{aligned} \quad (1)$$

Finally, the solution of the system (1) at the time T could be written as follow:

$$y_u(T) = S(T)y_0 + G_\gamma y + Hu.$$

We have the following proposition:

Proposition 2.4. We say that system (1) is γ -gradient controllable in ω at time T if and only if:

$$(Im \chi_\omega \nabla G_\gamma + Im \chi_\omega \nabla H) \cap \gamma \neq \emptyset.$$

Proof. We suppose that the system (1) is γ -gradient controllable in ω at time T which is equivalent to write: $\chi_\omega \nabla y_u(T) \in \gamma$.

We know that:

$$\begin{aligned} \chi_\omega \nabla y_u(T) &= \chi_\omega \nabla (S(T)y_0 + G_\gamma y + Hu) \\ &= \chi_\omega \nabla S(T)y_0 + \chi_\omega \nabla G_\gamma y + \chi_\omega \nabla Hu. \end{aligned}$$

We have $\nabla S(T)y_0 = 0$, we denote by: $z_1 = \chi_\omega \nabla G_\gamma y$, $z_2 = \chi_\omega \nabla Hu$ and $z = z_1 + z_2 = \chi_\omega \nabla y_u(T)$. This leads to: $z_1 \in Im(\chi_\omega \nabla G_\gamma)$ and $z_2 \in Im(\chi_\omega \nabla H)$. Thus, $z \in Im(\chi_\omega \nabla G_\gamma) + Im(\chi_\omega \nabla H)$ and $z \in \gamma$ which gives:

$$(Im \chi_\omega \nabla G_\gamma + Im \chi_\omega \nabla H) \cap \gamma \neq \emptyset.$$

Conversely, we suppose that the following expression is true:

$$(Im \chi_\omega \nabla G_\gamma + Im \chi_\omega \nabla H) \cap \gamma \neq \emptyset,$$

then, there exists $z \in \gamma$ such that $z \in (Im \chi_\omega \nabla G_\gamma + Im \chi_\omega \nabla H)$. So $z = z_1 + z_2$ where $z_1 = \chi_\omega \nabla G_\gamma y$ with $y \in L^2(0, T; (H)^n)$ and $\exists u \in \mathcal{U}$ such that $z_2 = \chi_\omega \nabla Hu$. Then $z = \chi_\omega \nabla G_\gamma y + \chi_\omega \nabla Hu$.

While $\chi_\omega \nabla S(T)y_0 = 0$ we could be write:

$$\begin{aligned} z &= \chi_\omega \nabla S(T)y_0 + \chi_\omega \nabla G_\gamma y + \chi_\omega \nabla Hu \\ &= \chi_\omega \nabla (S(T)y_0 + G_\gamma y + Hu) \\ &= \chi_\omega \nabla y_u(T). \end{aligned}$$

Finally, $z = \chi_\omega \nabla y_u(T) \in \gamma$, and we prove the equivalence. ■

- Remark 2.5.** 1. The above definition means that we are interested in the transfer of the system (1) to an unknown state just in γ .
2. If $\gamma = \{0, 0\}$ or $a(\cdot) = b(\cdot)$ we retrieve the regional exact controllability. So, for $a(\cdot) \neq b(\cdot)$ the γ -gradient controllability constitutes an extension of regional controllability.

We can also characterize the enlarge controllability by using the notion of strategic actuators. Hence, we recall that an actuator is conventionally defined by a couple (D, f) , where D is a nonempty closed part of $\bar{\Omega}$ representing the geometric support of the actuator, and $f \in L^2(D)$ define the spatial distribution of the action on the support D .

Definition 2.6. The actuator (D, f) is said to be γ -strategic on ω if the excited system is γ -gradient controllable in ω .

In the case of a pointwise actuator (internal or boundary) $D = \{b\}$ and $f = \delta(b - \cdot)$, where δ is the Dirac mass concentrated in b , and the actuator is then denoted by (b, δ_b) . For definitions and properties of strategic actuators we refer to [11, 12].

3 Existence and uniqueness

In this section we prove the existence and uniqueness of the solution of the problem (1), for that we consider the following problem:

$$\inf_{u \in \mathcal{U}_{ad}} J(u), \quad (2)$$

where $\mathcal{U}_{ad} = \{u \in U \mid \chi_\omega \nabla y_u(T) = y_d \in \gamma\}$ is the set of admissible directions. Hence the problem (1) becomes

$$\begin{cases} \text{Minimize} & J(u) = \frac{\beta}{2} \|u\|_U^2 \\ \text{subject to:} & \begin{cases} \partial_t y(x, t) - \Delta y(x, t) - \mathcal{N}y(x, t) = Bu(t) & Q_T, \\ y(x, 0) = y_0(x) & \Omega, \\ y(\xi, t) = 0 & \Sigma_T. \end{cases} \end{cases} \quad (3)$$

The following result prove the existence of the solution for problem (3).

Proposition 3.1. There exists $u^* \in \mathcal{U}_{ad}$ solution of problem (3).

Proof. The set $\mathcal{K} = \{J(u) \mid u \in \mathcal{U}_{ad}\}$ is nonempty and nonnegative then it has a nonnegative infimum. Let $(u_n)_{n \in \mathbb{N}}$ be a minimizing sequence in $\mathcal{U}_{ad}$. From the problem (1) we can see that:

$$\|u_n\|_U^2 \leq \frac{2}{\beta} J(u_n),$$

then $(u_n)_{n \in \mathbb{N}}$ is bounded.

Thus, there exists a subsequence of $(u_n)_{n \in \mathbb{N}}$ denoted also $(u_n)_{n \in \mathbb{N}}$ which converges weakly to u^* . The set $\mathcal{U}_{ad}$ is closed and convex, so $u^* \in \mathcal{U}_{ad}$. And we have,

$$\begin{aligned} y_{u_n}(t) - y_{u^*}(t) &= \int_0^t S(t-s) \mathcal{N}y_{u_n}(s) ds + \int_0^t S(t-s) Bu_n(s) ds \\ &\quad - \int_0^t S(t-s) \mathcal{N}y_{u^*}(s) ds - \int_0^t S(t-s) Bu^*(s) ds \\ &= \int_0^t S(t-s) \mathcal{N}(y_{u_n}(s) - y_{u^*}(s)) ds \\ &\quad + \int_0^t S(t-s) B(u_n(s) - u^*(s)) ds. \end{aligned}$$

So

$$|y_{u_n}(t) - y_{u^*}(t)| \leq \int_0^t \|S(t-s)\|_k |y_{u_n}(s) - y_{u^*}(s)| ds + \left| \int_0^t S(t-s) B(u_n(s) - u^*(s)) ds \right|.$$

$\mathcal{N}$ is a k -Lipschitz non linear operator ($|\mathcal{N}(y_{u_n}(t) - y_{u^*}(t))| \leq k|y_{u_n}(t) - y_{u^*}(t)|$), and using the Gronwall lemma, we have:

$$|y_{u_n}(t) - y_{u^*}(t)| \leq e^{\int_0^t k\|S(t-s)\|ds} \left| \int_0^t S(t-s)B(u_n(s) - u^*(s))ds \right|.$$

Using the property of semi-group [8], there exists $M \geq 1$, $\rho \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\rho t}$, then we obtain:

$$|y_{u_n}(t) - y_{u^*}(t)| \leq e^{kMe^{\rho t}t^{1/2}} \left| \int_0^t S(t-s)B(u_n(s) - u^*(s))ds \right|. \quad (4)$$

By theorem (3.9) in [13], the weak convergence $u_n \rightharpoonup u^*$ implies $Bu_n \rightharpoonup Bu^*$ weakly in $L^2(0, T; \mathcal{H})$ with $n > 1$. We know that the semi-group $(S(t))_{t \geq 0}$ is compact, using the corollary (3.3) in chapter 3 in [14] we have:

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^t S(t-s)B(u_n(s) - u^*(s))ds \right| = 0.$$

It follows from (4) that $y_{u_n} \rightarrow y_{u^*}$ strongly in $L^2(0, T; \mathcal{H})$.

Using the uniform continuity of $(y_{u_n})_n$ and also the continuity of the restriction operator we obtain:

$$\lim_{n \rightarrow \infty} \|\chi_\Gamma \gamma_0 y_{u_n}(T) - y_d\|^2 = \|\chi_\Gamma \gamma_0 y_{u^*}(T) - y_d\|^2.$$

Then $J(u^*) \leq \liminf_{n \rightarrow \infty} J(u_n)$, because $\|u^*\| \leq \liminf_{n \rightarrow \infty} \|u_n\|$ and $(u_n)_n$ is weakly convergent. Thus, we obtain:
 $J(u^*) = \inf_{u \in \mathcal{U}_{ad}} J(u_n).$ ■

For the uniqueness property we have the following proposition:

Proposition 3.2. *The system (1) is γ -gradient controllable in ω , then the problem (3) have a unique solution $u^* \in \mathcal{U}_{ad}$.*

Proof. If the system (1) is γ -gradient controllable in ω , we have $\chi_\omega \nabla y_u(T) \in \gamma$, which implies that $\mathcal{U}_{ad} \neq \emptyset$. If $u \in \mathcal{U}_{ad}$ then $J(u) = \frac{\beta}{2}\|u\|^2$ and since the mapping $f : u \mapsto \frac{1}{2}\|u\|^2$ is strictly convex, coercive ($\lim_{\|u\| \rightarrow +\infty} f(u) = +\infty$ for $u \in U$), proper and lower semi-continuous in U (which is reflexive), we verify that $\mathcal{U}_{ad}$ is closed convex subset of U .

It is obvious that $\mathcal{U}_{ad}$ is convex, indeed, for $(u, v) \in \mathcal{U}_{ad}^2$ and $t \in [0, 1]$:

$$\chi_\omega \nabla y_{(1-t)u+tv}(T) \in \gamma.$$

To prove that $\mathcal{U}_{ad}$ is closed, we consider a sequence $(u_n)_n$ in $\mathcal{U}_{ad}$ such that $u_n \rightarrow u$ strongly in U . Since $\chi_\omega \nabla H$ and $\chi_\omega \nabla G_\gamma$ are both continuous then: $\chi_\omega \nabla H u_n \rightarrow \chi_\omega \nabla H u$ strongly in $(L^2(\omega))^n$.

Hence $\chi_\omega \nabla y_u(T) \in \gamma$ where γ is a closed subset.

Then $\chi_\omega \nabla y_u(T) = \chi_\omega \nabla (S(T)y_0 + G_\gamma y + H u_n) \in \gamma$. So $\chi_\omega \nabla y_u(T) \in \gamma$, which means that $u \in \mathcal{U}_{ad}$. Thus we prove that $\mathcal{U}_{ad}$ is closed. Then (2) admits a unique solution u^* . ■

4 Computation of the control

In order to compute the control subject to our problem, we will use the sub-differential analysis. For that, we consider the problem (1) with the constant $\beta = 1$.

4.1 Sub-differential approach

Let f be a nontrivial, lower semi-continuous, proper and convex function from a Hilbert space W to $\tilde{R} =]-\infty, +\infty[$. We denote by $\mathcal{F}(W)$ the set of the functions f , $\|\cdot\|$ is the Hilbert norm of W .

- For $f \in \mathcal{F}(W)$, let $\text{dom}(f) = \{u \in W \mid f(u) < \infty\}$ and f^* is the polar function of f .

- For $f \in \mathcal{F}(W)$ and λ a positive parameter, there exists a unique solution J_λ [15] of the problem: $f_\lambda(x) := \inf_{y \in W} \left\{ f(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}$, such that:

$$f_\lambda(x) = \left\{ f(J_\lambda x) + \frac{1}{2\lambda} \|J_\lambda x - x\|^2 \right\}.$$

This solution could be characterized by the following variational inequality:

$$\forall y \in W, \quad \frac{1}{\lambda} \langle J_\lambda x - x, J_\lambda x - y \rangle + f(J_\lambda x) - f(y) \leq 0.$$

- For $v_0 \in \text{dom}(f)$, the set

$$\partial f(v_0) = \{u^* \in W \mid f(u) \geq f(v_0) + \langle u^*, u - v_0 \rangle \quad \forall u \in W\},$$

denotes the sub-differential of f at v_0 .

With all these notations, the problem (2) is equivalent to:

$$\begin{cases} \inf (\sigma + \Psi_{\mathcal{U}_{ad}})(u) \\ u \in \mathcal{U}_{ad}, \end{cases} \quad (5)$$

where:

- $\Psi_K(u) = \begin{cases} 0 & \text{if } u \in K \\ +\infty & \text{otherwise} \end{cases}$

denotes the indicator functional of K (K a non empty subset of W).

- $\sigma(u) = \frac{1}{2} \|u\|^2$.

Hence the solution of (5) is characterized by the following result.

Proposition 4.1. u^* is a solution of (2) if and only if the system (1) is γ -gradient controllable in ω and:

$$u^* \in \mathcal{U}_{ad} \quad \text{and} \quad \Psi_{\mathcal{U}_{ad}}^*(-u^*) = -\|u^*\|^2. \quad (6)$$

Proof. System (1) is γ -gradient controllable in ω , then $\mathcal{U}_{ad} \neq \emptyset$. Using Fermat's rule, we have u^* is a minimum of (2) if and only if $0 \in \partial(\sigma + \Psi_{\mathcal{U}_{ad}})(u^*)$. Although, $\mathcal{U}_{ad}$ is convex and $\sigma \in \mathcal{F}(U)$ so since $\mathcal{U}_{ad}$ is closed, convex and non empty, we have $\Psi_{\mathcal{U}_{ad}} \in \mathcal{F}(U)$.

Moreover, $\text{dom } \sigma \cap \text{dom } \Psi_{\mathcal{U}_{ad}} \neq \emptyset$ because the system (1) is γ -gradient controllable in ω .

Furthermore, $\partial(\sigma + \Psi_{\mathcal{U}_{ad}})(u^*) = \partial\sigma(u^*) + \partial\Psi_{\mathcal{U}_{ad}}(u^*)$ (σ is continuous). It follows that u^* is a solution of (2) if and only if

$$0 \in \partial\sigma(u^*) + \partial\Psi_{\mathcal{U}_{ad}}(u^*).$$

But we have σ is Frechet-differentiable, so $\partial\sigma(u^*) = \{u^*\}$. Then u^* is the solution of (2) if and only if $-u^* \in \partial\Psi_{\mathcal{U}_{ad}}(u^*)$ which is equivalent to $\Psi_{\mathcal{U}_{ad}}(u^*) + \Psi_{\mathcal{U}_{ad}}^*(-u^*) = -\|u^*\|^2$. We conclude that $u^* \in \mathcal{U}_{ad}$ and $\Psi_{\mathcal{U}_{ad}}^*(-u^*) = -\|u^*\|^2$. ■

We consider: $\tilde{\gamma} = [\tilde{a}(\cdot), \tilde{b}(\cdot)]$ where:

$$\begin{aligned} \tilde{a}(\cdot) &= a(\cdot) - \chi_\omega \nabla S(T) y_0 \\ \text{and } \tilde{b}(\cdot) &= b(\cdot) - \chi_\omega \nabla S(T) y_0 \end{aligned}$$

Then, $\mathcal{U}_{ad} = \{u \in \mathcal{U} \mid \chi_\omega \nabla H u \in \tilde{\gamma}\}$. We have the following result:

Proposition 4.2. u^* is the solution of (2) if and only if

$$\inf_{y \in \tilde{\gamma}} \left\{ u^*, (\chi_\omega \nabla H)^\dagger y \right\} = \|u^*\|^2 \quad (7)$$

where $(\chi_\omega \nabla H)^\dagger = (\chi_\omega \nabla H)^* ((\chi_\omega \nabla H)(\chi_\omega \nabla H)^*)^{-1}$ is the generalized inverse of $(\chi_\omega \nabla H)$. Moreover, the system (1) is γ -gradient controllable in ω .

Proof. We have $\mathcal{U}_{ad} = (\chi_\omega \nabla H)^\dagger \tilde{\gamma}$, from the previous proposition (4.1), u^* is the solution of the problem (2) if and only if $u^* \in \mathcal{U}_{ad}$ and $\Psi_{\mathcal{U}_{ad}}^*(-u^*) = -\|u^*\|^2$. Then, we have

$$\begin{aligned} \Psi_{\mathcal{U}_{ad}}^*(v^*) &= \sup_{v \in \mathcal{U}_{ad}} \langle v^*, v \rangle \quad \forall v^* \in \mathcal{U} \\ &= \sup_{v \in (\chi_\omega \nabla H)^\dagger \tilde{\gamma}} \langle v^*, v \rangle \\ &= \sup_{y \in \tilde{\gamma}} \langle v^*, (\chi_\omega \nabla H)^\dagger y \rangle \\ &= \sup_{\lambda \in [0,1]} \left\langle \left((\chi_\omega \nabla H)^\dagger \right)^* v^*, \lambda \tilde{a}(\cdot) + (1-\lambda) \tilde{b}(\cdot) \right\rangle \end{aligned}$$

Since the mapping $\lambda \mapsto \left\langle \left((\chi_\omega \nabla H)^\dagger \right)^* v^*, \lambda \tilde{a}(\cdot) + (1-\lambda) \tilde{b}(\cdot) \right\rangle$ is convex and continuous from $[0, 1]$ to $\mathbb{R}$, we obtain according to Krein-Milman theorem [16]:

$$\Psi_{\mathcal{U}_{ad}}^*(v^*) = \sup_{\lambda \in [0,1]} \left\langle \left((\chi_\omega \nabla H)^\dagger \right)^* v^*, \lambda \tilde{a}(\cdot) + (1-\lambda) \tilde{b}(\cdot) \right\rangle.$$

It follows that:

$$\Psi_{\mathcal{U}_{ad}}^*(v^*) = \max \left\{ \left\langle v^*, (\chi_\omega \nabla H)^\dagger \tilde{a}(\cdot) \right\rangle, \left\langle v^*, (\chi_\omega \nabla H)^\dagger \tilde{b}(\cdot) \right\rangle \right\}.$$

So we obtain (7) from (6). ■

Remark 4.3. If $a(\cdot) = b(\cdot) = \{y_d\}$, then

$$\tilde{a}(\cdot) = \tilde{b}(\cdot) = y_d - \chi_\omega \nabla S(T)y_0 - \chi_\omega \nabla G_\gamma y,$$

and

$$\mathcal{U}_{ad} = (\chi_\omega \nabla H)^\dagger (y_d - \chi_\omega \nabla S(T)y_0 - \chi_\omega \nabla G_\gamma y).$$

This implies that

$$u^*(t) = (\chi_\omega \nabla H)^\dagger (y_d - \chi_\omega \nabla S(T)y_0 - \chi_\omega \nabla G_\gamma y),$$

if it exists.

4.2 Lagrangian approach

We consider the problem (2) when the system (1) is excited by one actuator. The following result gives a useful characterization of the solution of the problem.

Theorem 4.4. If the system (1) is γ -gradient controllable in ω then the solution of (2) is given by:

$$u^* = -(\chi_\omega \nabla H)^* \lambda^*, \tag{8}$$

where $\lambda^* \in L^2(\omega)$ satisfies:

$$\begin{cases} z^* = P_\gamma(\rho \lambda^* + z^*) \\ R_\gamma \lambda^* + z^* = \chi_\omega \nabla [S(T)y_0 + G_\gamma y], \end{cases} \tag{9}$$

while $P_\gamma : (L^2(\omega))^n \rightarrow \gamma$ denotes the projection operator, $\rho > 0$ and $R_\gamma = (\chi_\omega \nabla H)(\chi_\omega \nabla H)^*$.

Proof. If the system (1) is γ -gradient controllable in ω then $\mathcal{U}_{ad} \neq \emptyset$ and the problem (2) has a unique solution. Problem (2) is equivalent to the saddle point problem:

$$\begin{cases} \inf \frac{1}{2} \|u\|^2 \\ (u, z) \in Z, \end{cases} \tag{10}$$

where $Z = \{(u, z) \in \mathcal{U} \times \gamma \mid \chi_\omega \nabla y_u(T) - z = 0\}$.

To study this constraint, we will use a Lagrangian functional and steer the problem (10) to a saddle point problem. We associate to the problem (10) the Lagrangian functional defined by

$$\forall (u, z, \lambda) \in \mathcal{U} \times \gamma \times (L^2(\omega))^n, L(u, z, \lambda) = \frac{1}{2} \|u\|^2 + \langle \lambda, \chi_\omega \nabla y_u(T) - z \rangle. \quad (11)$$

▷ We prove that L admits a saddle point.

The set $\mathcal{U} \times \gamma$ is non empty, closed and convex. The functional L satisfies conditions:

- $(u, z) \mapsto L(u, z, \lambda)$ is convex and lower semi-continuous for all $\lambda \in (L^2(\omega))^n$.
 - $\lambda \mapsto L(u, z, \lambda)$ is concave and upper semi-continuous for all $(u, z) \in \mathcal{U} \times \gamma$.
- Moreover, there exists $\lambda_0 \in (L^2(\omega))^n$ such that

$$\lim_{\|(u, z)\| \rightarrow +\infty} L(u, z, \lambda_0) = +\infty, \quad (12)$$

and there exists $(u_0, z_0) \in \mathcal{U} \times \gamma$ such that

$$\lim_{\|\lambda\| \rightarrow +\infty} L(u_0, z_0, \lambda) = -\infty. \quad (13)$$

Using the theorem 1 in [17] the functional L admits a saddle point.

▷ We prove then that u^* is a solution of (2) and it is the minimum one:

Let (u^*, z^*, λ^*) be a saddle point of L . Hence, we have:

$$L(u^*, z^*, \lambda) \leq L(u^*, z^*, \lambda^*) \leq L(u, z, \lambda^*) \quad \forall (u, z, \lambda) \in \mathcal{U} \times \gamma \times (L^2(\omega))^n. \quad (14)$$

From the first inequality of (14) we have:

$$\langle \lambda, \chi_\omega \nabla y_{u^*}(T) - z^* \rangle \leq \langle \lambda^*, \chi_\omega \nabla y_{u^*}(T) - z^* \rangle \quad \forall \lambda \in (L^2(\omega))^n,$$

which implies $\chi_\omega \nabla y_{u^*}(T) = z^*$ and hence $\chi_\gamma \nabla y_{u^*}(T) \in \gamma$.

The second inequality of (14) means that for all $u \in \mathcal{U}$ and $z \in \gamma$, we have:

$$\frac{1}{2} \|u^*\|^2 + \langle \lambda^*, \chi_\omega \nabla y_{u^*}(T) - z^* \rangle \leq \frac{1}{2} \|u\|^2 + \langle \lambda^*, \chi_\omega \nabla y_u(T) - z \rangle \quad \forall (u, z) \in \mathcal{U} \times \gamma.$$

Since $\chi_\omega \nabla y_{u^*}(T) = z^*$, it follows that:

$$\frac{1}{2} \|u^*\|^2 \leq \frac{1}{2} \|u\|^2 + \langle \lambda^*, \chi_\omega \nabla y_u(T) - z \rangle \quad \forall (u, z) \in \mathcal{U} \times \gamma.$$

Taking $z = \chi_\omega \nabla y_u(T) \in \gamma$, we obtain:

$$\frac{1}{2} \|u^*\|^2 \leq \frac{1}{2} \|u\|^2,$$

which implies that u^* is of minimum energy.

The following assumptions hold, if (u^*, z^*, λ^*) is a saddle point of L :

$$\langle u^*, u - u^* \rangle + \langle \lambda^*, \chi_\omega \nabla H(u - u^*) \rangle = 0 \quad \forall u \in \mathcal{U}, \quad (15)$$

$$-\langle \lambda^*, z - z^* \rangle \geq 0 \quad \forall z \in \gamma, \quad (16)$$

$$\langle \lambda - \lambda^*, \chi_\omega \nabla y_{u^*}(T) - z^* \rangle = 0 \quad \forall \lambda \in (L^2(\omega))^n. \quad (17)$$

For more details about the saddle point and its theory, we refer to [18, 19, 20].

From (15) we deduce (8).

The equation (17) is equivalent to:

$$\chi_\omega \nabla y_{u^*}(T) = z^*.$$

Since $y_{u^*}(T) = S(T)y_0 + G_\gamma y(\cdot) + Hu^*$, we have:

$$\chi_\omega \nabla [S(T)y_0 + G_\gamma y(\cdot) + Hu^*] - z^* = 0.$$

Then $\chi_\omega \nabla [S(T)y_0 + G_\gamma y(\cdot)] + \chi_\omega \nabla Hu^* = z^*$.

With (8), we have:

$$\chi_\omega \nabla [S(T)y_0 + G_\gamma y(\cdot)] - (\chi_\omega \nabla H) (\chi_\omega \nabla H)^* \lambda^* = z^*,$$

with $R_\gamma = (\chi_\omega \nabla H) (\chi_\omega \nabla H)^*$, we obtain the first equation of (9).

From inequality (16), we obtain:

$$\langle (\rho \lambda^* + z^*) - z^*, z - z^* \rangle_{H^{1/2}(\Gamma)} \leq 0 \quad \forall z \in \gamma, \rho > 0,$$

which is equivalent to the second equation of (9). ■

Corollary 4.5. *If the system (1) is exactly gradient controllable in ω and ρ suitably chosen, then the system (9) has only one solution (λ^*, z^*) .*

Proof.

The gradient exact controllability in ω implies that $(\chi_\omega \nabla H)^*$ and R_γ are bijective. So if (u^*, z^*, λ^*) is a saddle point of L then the system (9) is equivalent to

$$\begin{cases} R_\gamma \lambda^* + z^* = \chi_\omega \nabla [S(T)y_0 + G_\gamma y(\cdot)] \\ z^* = P_\gamma \left[\left(-\rho R_\gamma^{-1} z^* + \rho R_\gamma^{-1} \chi_\omega \nabla (S(T)y_0 + G_\gamma y(\cdot)) \right) + z^* \right]. \end{cases} \quad (18)$$

It follows that z^* is a fixed point of the function

$$\begin{aligned} F_\rho : \quad \gamma &\longrightarrow \gamma \\ y &\longmapsto P_\gamma \left[\left(-\rho R_\gamma^{-1} y + \rho R_\gamma^{-1} \chi_\omega \nabla (S(T)y_0 + G_\gamma y(\cdot)) \right) + y \right]. \end{aligned} \quad (19)$$

The operator R_γ is coercive, i.e.

$$\exists m > 0 \quad \text{such that} \quad \langle R_\gamma^{-1} y, y \rangle \geq m \|y\|^2 \quad \forall y \in (L^2(\omega))^n.$$

It follows that

$$\begin{aligned} \|F_\rho(y) - F_\rho(z)\| &\leq \| -\rho R_\gamma^{-1} (y - z) + (y - z) \|^2 \\ &\leq \rho^2 \|R_\gamma^{-1}\|^2 \|y - z\|^2 + \|y - z\|^2 - 2\rho \langle R_\gamma^{-1} (y - z), (y - z) \rangle \\ &\leq \rho^2 \|R_\gamma^{-1}\|^2 \|y - z\|^2 + \|y - z\|^2 - 2\rho m \|y - z\|^2 \\ &\leq (1 + \rho^2 \|R_\gamma^{-1}\|^2 - 2\rho m) \|y - z\|^2 \quad \forall y, z \in \gamma \end{aligned}$$

and we deduce that if $0 < \rho < \frac{2m}{\|R_\gamma^{-1}\|^2}$, then F_ρ is a contraction mapping. This implies the uniqueness of z^* and λ^* . ■

Remark 4.6.

If $a(\cdot) = b(\cdot)$, we obtain the notion of exact gradient controllability in ω , and the solution of (2) is given by:

$$u^* = (\chi_\omega \nabla H)^* R_\gamma^{-1} [b(\cdot) - \chi_\omega \nabla (S(T)y_0 + R_\gamma y(\cdot))].$$

5 Conclusion

We developed an extension of the regional controllability to a situation encountered in many real problems where we are interested in the knowledge of the states in a critical subregion of the whole domain. Various open questions are still under consideration. The problem of constrained control as well as the case of fractional-order distributed parameter dynamic systems are of great interest.

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