Existence of Solution of Neutral Fractional Impulsive Differential Equations with Infinite Delay

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Abstract. In this work we define the solution of the fractional neutral impulsive differential equation with infinite delay. The results of existence are obtained by using the Banach contraction and Schaefer fixed point. Some properties of probability density functions and semi-group theory are also used.

Keywords: Fixed point theorem, Fractional impulsive differential equations, Neutral equations, infinite delay.

1 Introduction

Differential equation of fractional order occur more frequently different research areas and engineering , such as physics, biology, chemistry, economics, control of dynamical systems, etc. Numerous books and articles focused in this field, see the monographs Podlubny [11], Miller and Ross [10], Kilbas [6] and papers [12], [8], [9], [5], [7]. The class of impulsive differential functional equations have attracted much attention where the impulsion is instantaneous. It has modeled many physical phenomena processes in the real world, it is often subjected to abrupt changes such as shocks, and short time perturbations during the evolution of processes, so it attracted many authors [3], [2], [5].

In [3] the authors studied the existence of solutions for an abstract impulsive fractional differential equations where the impulses are not instantaneous is that the subject of some dynamics of evolution processes in Pharmacotherapy. In fact this situation of impulsives actions starts at an arbitrary fixed point and stays active on a finite time interval, and the model equations reads

$$\begin{cases} D^\alpha x(t) = Ax(t) + f(t, x(t)), t \in (s_i, t_{i+1}], i = 0, 1, 2, \dots, m, \\ x(t) = I_i(x(t_i)) + g_i(t, x(t)), t \in (t_i, s_i], i = 1, 2, \dots, m, \\ x(0) = x_0, \end{cases}$$

where $A : D(A) \subset X \rightarrow X$ is the generator of C_0 -semigroup of bounded linear operators $\{T(t)\}_{t \geq 0}$ on Banach space $(X, \|\cdot\|)$, $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 \leq \dots \leq s_m < t_{m+1} = T$ are fixed numbers, $g_i \in C((t_i, s_i] \times X, X)$ and $I_i : X \rightarrow X$, for $(i = 1, 2, \dots, m)$. At the point s_i , the function x is continuous. The term $I_i(x(t_i))$ means that the impulses are related to the value of $x(t_i) = x(t_i^-)$.

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Motivated by [1] where the authors studied the following problem

$$\begin{cases} D^\alpha x(t) = f(t, x_t), \text{ for each } t \in J = [0, T], 0 < \alpha < 1, \\ x(t) = \phi(t), \quad t \in (-\infty, 0], \end{cases}$$

where $f : J \times \mathfrak{B} \rightarrow \mathbb{R}$, $\mathfrak{B}$ the phase space. For any function x defined on $(-\infty, T]$ and $t \in J$, we denote by x_t the element of $\mathfrak{B}$ defined by

$$x_t(\theta) = x(t + \theta), \quad \theta \in (-\infty, 0].$$

We shall study the following problem:

$$(1) \begin{cases} D^\alpha(x(t) - h(t, x_t)) = A(x(t) - h(t, x_t)) + f(t, x_t), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \dots, m, \\ x(t) = I_i(x(t_i)) + g_i(t, x(t)), t \in (t_i, s_i] \quad i = 1, 2, \dots, m, \\ x_0 = \phi \in \mathfrak{B}, \end{cases}$$

where A, g_i, I_i which are already given, and $f : J \times \mathfrak{B} \rightarrow X$ and $h : J \times \mathfrak{B} \rightarrow X$ are appropriate functions to be specified later, and D^α is the Caputo fractional derivative with $0 < \alpha < 1$.

The outline of this paper goes as follows. In section 2 we introduce some preliminaries about fractional calculus, and about construction of solution involving probability density function. In section 3 we establish the existence of solution to the problem (1) by Banach's contraction and schaffer's fixed point.

2 Preliminary Notes

In this section, we introduce definitions and results which will be used throughout this paper.

Definition 2.1. [12] Let $x \in C^n[0, \infty)$ and $n - 1 < \alpha < n$, the Caputo's derivative of order α for function $x : [0, \infty) \rightarrow \mathbb{R}$ can be written as

$$D^\alpha x(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n - \alpha - 1} x^{(n)}(s) ds.$$

Definition 2.2. [12] The fractional integral of order α is defined as

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} x(s) ds$$

which is called the Riemann-Liouville integral.

To study the fractional differential equation with infinite delay, we assume that the space $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ is a semi-normed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and satisfying the following fundamental axioms introduced by Hale and Kato in [4].

(A) If $x : (-\infty, T] \rightarrow \mathbb{R}$, and $y_0 \in \mathfrak{B}$, then for every $t \in [0, T]$ the following conditions hold:

- i) x_t is in $\mathfrak{B}$,
 - ii) $\|x_t\|_{\mathfrak{B}} \leq C_1(t) \sup \{|x(s)| : 0 \leq s \leq t\} + C_2(t)\|x_0\|_{\mathfrak{B}}$, where $C_1 : [0, T] \rightarrow [0, \infty)$ is continuous, $C_2 : [0, \infty) \rightarrow [0, \infty)$ is locally bounded and C_1, C_2 are independent of $x(\cdot)$.
- (A - 1) For the function $x(\cdot)$ in (A), x_t is a $\mathfrak{B}$ -valued continuous function on $[0, T]$.
- (A - 2) The space $\mathfrak{B}$ is complete.

We consider the space

$\mathfrak{B}' = \{x : (-\infty, T] \rightarrow X \text{ such that } x \in C(J_i, X) \text{ and there exist } x(t_i^-) \text{ and } x(t_i^+),$
 $i = 1, 2, \dots, m \text{ with } x(t_k) = x(t_k^-), x_0 = \phi \in \mathfrak{B}\}$, where $J = [0, T]$, and $J_i = (t_i, t_{i+1}]$, for $i = 1, \dots, m$, and we define the semi-norm in $\mathfrak{B}'$ by

$$\|x\|_{\mathfrak{B}'} = \sup_{t \in J} \|x(t)\| + \|\phi\|_{\mathfrak{B}}, \quad x \in \mathfrak{B}'.$$

Lemma 2.3. [2] Let $v : [0, T] \rightarrow [0, \infty)$ be a real function, ω a nonnegative and locally integrable function on $[0, T]$, and there are constant $a > 0$, and $0 < \alpha < 1$ such that

$$v(t) \leq \omega(t) + a \int_0^t \frac{v(s)}{(t-s)^\alpha} ds,$$

then there exists a constant $K(\alpha)$ such that

$$v(t) \leq w(t) + aK(\alpha) \int_0^t \frac{w(s)}{(t-s)^\alpha} ds.$$

Lemma 2.4. [2] Let P be a continuous and compact mapping on Banach space X into itself, such that the set $\{x \in X : x = \lambda Px, \text{ for some } 0 < \lambda < 1\}$ is bounded, then P has a fixed point.

Consider the following problem

$$(2) \begin{cases} D^\alpha(x(t) - h(t, x_t)) = A(x(t) - h(t, x_t)) + f(t), & t \in (s_i, t_{i+1}], i = 0, 1, 2, \dots, m, \\ x(t) = I_i(x(t_i)) + g_i(t, x(t)), & t \in (t_i, s_i] \quad i = 1, 2, \dots, m, \\ x_0 = \phi \in \mathfrak{B}. \end{cases}$$

To construct the solution for this problem we follow these steps :

- *Step 1:* We apply the operator I^α to (the first equation of the) problem (2) we obtain

$$\left\{ \begin{array}{l} x(t) - h(t, x_t) = (\phi(0) - h(0, x_0)) \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(x(s) - h(s, x_s)) + f(s) ds, t \in [0, t_1], \\ x(t) = I_1(x(t_1)) + g_1(t, x(t)), t \in (t_1, s_1], \\ x(t) - h(t, x_t) = d_i + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(x(s) - h(s, x_s)) + f(s) ds, t \in [s_1, t_2], \\ \vdots \\ x(t) = I_i(x(t_i)) + g_i(t, x(t)), t \in (t_i, s_i], \\ x(t) - h(t, x_t) = d_i + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(x(s) - h(s, x_s)) + f(s) ds, t \in [s_i, t_{i+1}], \end{array} \right.$$

for $i = 1, 2, \dots, m$, and $d_i \in X$. The continuity property of the function x at points s_i allows us to write

$$x(t) - h(t, x_t) = d_i \chi_{[s_i, t_{i+1})} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(x(s) - h(s, x_s)) + f(s) ds, \quad t \in (s_i, t_{i+1}),$$

for $i = 1, 2, \dots, m$, and where $\chi_{[s_i, t_{i+1})}(t)$ is the characteristic function of $[s_i, t_{i+1})$.

- *Step 2:* We apply the Laplace transformations and we follow the idea used in [13] to obtain

$$u(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} e^{-\lambda s_i} d_i - \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} e^{-\lambda t_{i+1}} d_i + (\lambda^\alpha I - A)^{-1} v(\lambda),$$

where u and v are Laplace transformation of the function $x - h$ and f and I is the identity operator defined in X .

We follow the same computation in [12],[13] and using inverse Laplace transformation and the property that

$$\int_0^\infty e^{-\lambda s} \omega_\alpha(s) ds = e^{-\lambda^\alpha},$$

is Laplace transformation of ω_α given in [12], [13] by

$$\omega_\alpha(s) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} s^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad s \in (0, \infty),$$

We conclude that

$$x(t) - h(t, x_t) = P_\alpha(t - s_i) d_i \chi_{[s_i, t_{i+1})} + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s) ds, \quad t \in (s_i, t_{i+1}],$$

where

$$P_\alpha(t) = \int_0^\infty \xi_\alpha(s) T(t^\alpha s) ds, \quad (1)$$

and

$$Q_\alpha(t) = \alpha \int_0^\infty s \xi_\alpha(s) T(t^\alpha s) ds, \quad (2)$$

with

$$\xi_\alpha(s) = \frac{1}{\alpha} s^{-1-\frac{1}{\alpha}} \omega_\alpha(s^{-\frac{1}{\alpha}}), \quad (3)$$

$$\int_0^\infty \xi_\alpha(s) ds = 1,$$

$$\int_0^\infty s \xi_\alpha(s) ds = \frac{1}{\Gamma(\alpha+1)},$$

where the relation (1), (2) and (3) are given in [12], [13].

• *Step 3:* We determine the values of d_i using the fact that x is continuous at the points s_i . We get

$$d_i = I_i(x(t_i)) - h(s_i, x_{s_i}) + g_i(s_i, x(s_i)) - \int_0^{s_i} (s_i - s)^{\alpha-1} Q_\alpha(s_i - s) f(s) ds.$$

Now we are in a position to define our solution.

Definition 2.5. The function $x \in \mathfrak{B}'$ is mild solution of problem (1) if it satisfies

$$(7) \quad x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ P_\alpha(t)(\phi(0) - h(0, x_0)) + h(t, x_t) + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, x_s) ds, & t \in [0, t_1], \\ I_i(x(t_i)) + g_i(t, x(t)), & t \in (t_i, s_i] \\ \vdots \\ P_\alpha(t) d_i + h(t, x_t) + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, x_s) ds, & t \in [s_i, t_{i+1}]. \end{cases}$$

We make the following assumption on A in the whole paper.

$H(A)$ The operator A is generator of a strongly continuous semigroup $\{T(t) : t \geq 0\}$ in X , and $\sup_{t \in [0, \infty)} \|T(t)\|_{L(X)} \leq M_A$ with $M_A \geq 1$. For $t > 0$, $T(t)$ is compact.

Lemma 2.6. [3] Let $H(A)$ hold then the operators P_α and Q_α have the following properties:

1. For any fixed $t \geq 0$ $P_\alpha(t), Q_\alpha(t)$ are linear and bounded operators, and for any $x \in X$

$$\|P_\alpha(t)x\| \leq M_A \|x\|, \|Q_\alpha(t)x\| \leq \frac{\alpha M_A}{\Gamma(\alpha + 1)} \|x\|.$$

2. $\{P_\alpha(t), t \geq 0\}$ and $\{Q_\alpha(t), t \geq 0\}$ are strongly continuous.
3. for every $t > 0$ $P_\alpha(t)$ and $Q_\alpha(t)$ are compact operators.

3 Main results

This section will be devoted to the study of the existence of the solution of problem (1). In order to do this, we make the following assumptions:

(H1) $f : [0, T] \times \mathfrak{B} \rightarrow X$ is continuous, and there exists L_f such that

$$\|f(t, \varphi) - f(t, \psi)\| \leq L_f \|\varphi - \psi\|_{\mathfrak{B}}. \quad \text{for } t \in J, \varphi, \psi \in \mathfrak{B},$$

$h : [0, T] \times \mathfrak{B} \rightarrow X$ is continuous, and there exists L_h such that

$$\|h(t_1, \varphi) - h(t_2, \psi)\| \leq L_h(|t_1 - t_2| + \|\varphi - \psi\|_{\mathfrak{B}}).$$

(H2) $C_1 = \sup_{t \in [0, T]} C_1(t), \quad C_2 = \sup_{t \in [0, T]} C_2(t),$

(H3) There exists L_I , such that $\|I_i(x) - I_i(y)\| \leq L_I \|x - y\|, i = 1, 2, \dots, m$, for $x, y \in X$.
There exists L_g such that $\|g_i(t, x) - g_i(t, y)\| \leq L_g \|x - y\|, i = 1, 2, \dots, m$, for $x, y \in X$.

The first result is based on Banach contraction.

Theorem 3.1. Assume that (A), (H1), (H2) and (H3) are satisfied. Furthermore

$$\left(M_A(L_I + L_g + C_1 L_h + \frac{M_A C_1 L_f}{\Gamma(\alpha + 1)} T^\alpha) + \frac{C_1 M_A}{\Gamma(\alpha + 1)} T^\alpha C_1 L_f + C_1 L_h \right) < 1.$$

Then, the problem (1) has a unique solution.

Proof. Let the operator $N : \mathfrak{B}' \rightarrow \mathfrak{B}'$ defined by

$$N(x)(t) = \begin{cases} \phi(t), t \in (-\infty, 0], \\ P_\alpha(t)(\phi(0) - h(0, x_0)) + h(t, x_t) \\ \quad + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, x_s) ds, t \in [0, t_1], \\ I_1(x(t_1)) + g_1(t, x(t)), t \in (t_1, s_1], \\ \vdots \\ I_i(x(t_i)) + g_i(t, x(t)), t \in (t_i, s_i], \\ P_\alpha(t)d_i + h(t, x_t) + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, x_s) ds, t \in [s_i, t_{i+1}]. \end{cases}$$

Let $y(\cdot) : (-\infty, T] \rightarrow X$ be the function defined by

$$y(t) = \begin{cases} 0 & \text{if } t \in [0, T], \\ \phi(t) & \text{if } t \in (-\infty, 0]. \end{cases}$$

Then, $y_0 = \phi$. For each $z \in C([0, T], X)$ with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t) & \text{if } t \in [0, T] \\ 0 & \text{if } t \in (-\infty, 0]. \end{cases}$$

So we decompose $x(\cdot)$ as $x(t) = y(t) + \bar{z}(t)$ for $t \in [0, T]$, which implies $x_t = y_t + \bar{z}_t$ for $t \in [0, T]$ where $x(\cdot)$ satisfies the relation (7), the function $z(\cdot)$ is given by

$$z(t) = \begin{cases} P_\alpha(t)(\phi(0) - h(0, \phi)) + h(t, y_t + \bar{z}_t) \\ \quad + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, y_t + \bar{z}_t) ds, t \in [0, t_1], \\ I_1(x(t_1)) + g_1(t, x(t)), t \in (t_1, s_1], \\ \vdots \\ I_i(x(t_i)) + g_i(t, x(t)), t \in (t_i, s_i], \\ P_\alpha(t)d_i + h(t, y_t + \bar{z}_t) + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, y_t + \bar{z}_t) ds, t \in (s_i, t_{i+1}]. \end{cases}$$

Set

$$\mathfrak{B}'' = \{z \in \mathfrak{B}' \text{ such that } z_0 = 0\}$$

and let $\|\cdot\|_{\mathfrak{B}''}$ be the seminorm in $\mathfrak{B}''$ defined by

$$\|z\|_{\mathfrak{B}''} = \sup_{t \in J} \|z(t)\| + \|z_0\|_{\mathfrak{B}} = \sup_{t \in J} \|z(t)\|, \quad z \in \mathfrak{B}'',$$

$\mathfrak{B}''$ is a Banach space with norm $\|\cdot\|_{\mathfrak{B}''}$.

We define the operator $P : \mathfrak{B}'' \longrightarrow \mathfrak{B}''$ by

$$P(z)(t) = \begin{cases} P_\alpha(t)(\phi(0) - h(0, \phi)) + h(t, y_t + \bar{z}_t) \\ \quad + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, y_t + \bar{z}_t) ds, \quad t \in [0, t_1], \\ I_1(x(t_1)) + g_1(t, x(t)), t \in (t_1, s_1], \\ \vdots \\ I_i(x(t_i)) + g_i(t, x(t)), t \in (t_i, s_i], \\ P_\alpha(t)d_i + h(t, y_t + \bar{z}_t) + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, y_t + \bar{z}_t) ds, t \in (s_i, t_{i+1}]. \end{cases}$$

Note that the operator N has a unique fixed point is equivalent to P has a unique fixed point too. Let us prove that

P has a fixed point. Consider $z, z^* \in \mathfrak{B}''$, then we have for $t \in [0, t_1]$

$$\begin{aligned}
\|P(z)(t) - P(z^*)(t)\| &\leq \|h(t, y_t + \bar{z}_t) + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, y_t + \bar{z}_t) ds \\
&\quad - h(t, y_t + \bar{z}_t^*) - \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, y_t + \bar{z}_t^*) ds\| \\
&\leq \|h(t, y_t + \bar{z}_t) - h(t, y_t + \bar{z}_t^*)\| \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|Q_\alpha(t-s)(f(s, y_t + \bar{z}_t) - f(s, y_t + \bar{z}_t^*))\| ds \\
&\leq L_h \|z - z^*\|_{\mathfrak{B}} + \frac{\alpha M_A}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} L_f \|z - z^*\|_{\mathfrak{B}} ds \\
&\leq (C_1 L_h + M_A C_1 \frac{T^\alpha L_f}{\Gamma(\alpha+1)}) \|z - z^*\|_{\mathfrak{B}''}.
\end{aligned}$$

For $t \in [t_i, s_i]$; $i = 1, 2, \dots, m$

$$\|P(z)(t) - P(z^*)(t)\| \leq (L_I + L_g) \|z - z^*\|_{\mathfrak{B}''},$$

and for $t \in [s_i, t_{i+1}]$; $i = 1, 2, \dots, m$

$$\begin{aligned}
\|P(z)(t) - P(z^*)(t)\| &\leq \|P_\alpha(t) d_i + h(t, y_t + \bar{z}_t) + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, y_t + \bar{z}_t) ds \\
&\quad - P_\alpha(t) d_i^* + h(t, y_t + \bar{z}_t^*) - \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, y_t + \bar{z}_t^*) ds\| \\
&\leq \|h(t, y_t + \bar{z}_t) - h(t, y_t + \bar{z}_t^*)\| + \|P_\alpha(t) d_i - P_\alpha(t) d_i^*\| \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|Q_\alpha(t-s)(f(s, y_t + \bar{z}_t) - f(s, y_t + \bar{z}_t^*))\| ds \\
&\leq \left(M_A(L_I + L_g + C_1 L_h + \frac{M_A C_1 L_f}{\Gamma(\alpha+1)} T^\alpha) + \frac{C_1 M_A}{\Gamma(\alpha+1)} T^\alpha C_1 L_f + C_1 L_h \right) \|z - z^*\|_{\mathfrak{B}''}
\end{aligned}$$

where

$$\begin{aligned}
d_i^* &= I_i(y(t_i) + \bar{z}^*(t_i)) - h(s_i, y_{s_i} + \bar{z}_{s_i}^*) + g_i(s_i, y(s_i) + \bar{z}^*(s_i)) \\
&\quad - \int_0^{s_i} (s_i - s)^{\alpha-1} Q_\alpha(s_i - s) f(s, y_s + \bar{z}_s^*) ds.
\end{aligned}$$

From what is mentioned above, for all $t \in [0, T]$, we deduce that

$$\begin{aligned}
&\|P(z)(t) - P(z^*)(t)\|_{\mathfrak{B}''} \\
&\leq \left(M_A(L_I + L_g + C_1 L_h + \frac{M_A C_1 L_f}{\Gamma(\alpha+1)} T^\alpha) + \frac{C_1 M_A}{\Gamma(\alpha+1)} T^\alpha C_1 L_f + C_1 L_h \right) \|z - z^*\|_{\mathfrak{B}''}.
\end{aligned}$$

The operator P is a contraction map on the space $\mathfrak{B}''$. Therefore it has a unique fixed point in $\mathfrak{B}''$, which is a mild solution of problem (1). □

In the second part we use Lemma 2.2 to prove the existence of solution, but before that, we make the following hypothesis:

- (H5) The functions $I_i : X \rightarrow X$ are continuous and there exists a constant M_I such that $\|I_i(x)\| \leq M_I, i = 1, 2, \dots, m$, for $x \in X$.
The functions $g_i : (t_i, s_i] \times X \rightarrow X$ are continuous and there exists a constant M_g such that $\|g_i(t, x)\| \leq M_g, i = 1, 2, \dots, m$, for $x \in X$.
- (H6) f and h are continuous functions, and there exists a constant M_h such that $\|h(t, x)\| \leq M_h$ for $t \in J$ and $x \in \mathfrak{B}$.

- (H7) There exist $p, q \in C(J, \mathbb{R}^+)$, such that $\|f(t, x)\| \leq p(t) + q(t)\|x\|_{\mathfrak{B}}$ for $t \in J$ and $x \in \mathfrak{B}$.

Theorem 3.2. *Assume that the assumptions, (A), (H2), and (H5) – (H7) are satisfied. Then, the problem (1) has at least one mild solution in $(-\infty, T]$.*

Proof. We define again the operator P as in Theorem 3.1, and we divide the proof into four steps:

- *Step 1:* P is continuous.

Let $(z_n)_n$ be a sequence in $\mathfrak{B}''$, such that $z_n \rightarrow z$ in $\mathfrak{B}''$, then we have for $t \in [0, t_1]$

$$\begin{aligned} \|P(z_n)(t) - P(z)(t)\| &\leq \|h(t, \bar{z}_{n_t} + y_t) - h(t, \bar{z}_t + y_t)\| \\ &\quad + \frac{\alpha M_A}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} \|f(s, \bar{z}_{n_s} + y_s) - f(s, \bar{z}_s + y_s)\| ds. \end{aligned}$$

Since f and h are continuous functions, we have $\|P(z_n)(t) - P(z)(t)\|_{\mathfrak{B}''} \rightarrow 0$ as $n \rightarrow \infty$.

For $t \in [t_i, s_i]$, $i = 1, 2, \dots, m$

$$\|P(z_n)(t) - P(z)(t)\| \leq \|I_i(z_n(t_i)) - I_i(z(t_i))\| + \|g_i(z_n(t)) - g_i(z(t))\|,$$

Since the functions I_i and g_i are continuous for $i = 1, \dots, m$, we get

$$\|P(z_n)(t) - P(z)(t)\|_{\mathfrak{B}''} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

For $t \in [s_i, t_{i+1}]$; for $i = 1, 2, \dots, m$

$$\begin{aligned} \|P(z_n)(t) - P(z)(t)\| &\leq \|h(t, \bar{z}_{n_t} + y_t) - h(t, \bar{z}_t + y_t)\| \\ &\quad + \frac{\alpha M_A}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} \|f(s, \bar{z}_{n_s} + y_s) - f(s, \bar{z}_s + y_s)\| ds \\ &\quad + M_A (\|h(s_i, \bar{z}_{n_{s_i}} + y_{s_i}) - h(s_i, \bar{z}_{s_i} + y_{s_i})\| \\ &\quad + \|I_i(z_n(t_i)) - I_i(z(t_i))\| + \|g_i(z_n(s_i)) - g_i(z(s_i))\| \\ &\quad + \frac{\alpha M_A}{\Gamma(\alpha + 1)} \int_0^{s_i} (s_i-s)^{\alpha-1} \|f(s, \bar{z}_{n_s} + y_s) - f(s, \bar{z}_s + y_s)\| ds). \end{aligned}$$

So

$$\|P(z_n)(t) - P(z)(t)\|_{\mathfrak{B}''} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This implies that P is continuous.

- *Step 2:* P maps bounded sets into bounded sets in $\mathfrak{B}''$.

Indeed, we must prove that for $r > 0$, there exists a positive l such that for each $z \in B_r = \{z \in \mathfrak{B}'', \|z\|_{\mathfrak{B}''} \leq r\}$ we have $\|Pz\|_{\mathfrak{B}''} \leq l$. Let $z \in B_r$,

for $t \in [0, t_1]$ we have

$$\begin{aligned}
\|P(z)(t)\| &= \|P_\alpha(t)(\phi(0) - h(0, \phi)) + h(t, y_t + \bar{z}_t) + \\
&\quad \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, y_s + \bar{z}_s) ds\| \\
&\leq \|P_\alpha(t)(\phi(0) - h(0, \phi))\| + \|h(t, y_t + \bar{z}_t)\| + \frac{\alpha M_A}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \|f(s, y_s + \bar{z}_s)\| ds \\
&\leq M_A(\|\phi(0)\| + M_h) + M_h + \frac{T^\alpha M_A}{\Gamma(\alpha+1)} (\|p\|_\infty + \|q\|_\infty \|y_s + \bar{z}_s\|_{\mathfrak{B}}) \\
&\leq M_A(\|\phi(0)\| + M_h) + M_h + \frac{T^\alpha M_A}{\Gamma(\alpha+1)} (\|p\|_\infty + \|q\|_\infty (\|y_s\|_{\mathfrak{B}} + \|\bar{z}_s\|_{\mathfrak{B}})) \\
&\leq M_A(\|\phi(0)\| + M_h) + M_h + \frac{T^\alpha M_A}{\Gamma(\alpha+1)} (\|p\|_\infty + \|q\|_\infty (C_2 \|\phi\|_{\mathfrak{B}} + C_1 r)) \\
&\leq M_A(\|\phi(0)\| + M_h) + \frac{T^\alpha M_A}{\Gamma(\alpha+1)} (\|p\|_\infty + \|q\|_\infty r^*) + M_h,
\end{aligned}$$

with $r^* = C_2 \|\phi\|_{\mathfrak{B}} + C_1 r$.

For $t \in [t_i, s_i]$, $i = 1, 2, \dots, m$

$$\begin{aligned}
\|P(z)(t)\| &\leq \|I_i(z(t_i))\| + \|g_i(t, z(t))\| \\
&\leq M_I + M_g = l_2.
\end{aligned}$$

For $t \in [s_i, t_{i+1}]$; $i = 1, 2, \dots, m$

$$\begin{aligned}
\|P(z)(t)\| &= \|P_\alpha(t)d_i + h(t, y_t + \bar{z}_t) + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, y_s + \bar{z}_s) ds\| \\
&\leq \|P_\alpha(t)d_i\| + M_h + \frac{T^\alpha M_A}{\Gamma(\alpha+1)} (\|p\|_\infty + \|q\|_\infty r^*) \\
&\leq M_A \left(M_I + M_g + M_h + M_A \left(\frac{T^\alpha}{\Gamma(\alpha+1)} (\|p\|_\infty + \|q\|_\infty r^*) \right) \right. \\
&\quad \left. + M_h + \frac{T^\alpha M_A}{\Gamma(\alpha+1)} (\|p\|_\infty + \|q\|_\infty r^*) \right) \\
&\leq M_A \left(M_I + M_g + M_h + (1 + M_A) \frac{T^\alpha}{\Gamma(\alpha+1)} (\|p\|_\infty + \|q\|_\infty r^*) \right) + M_h = l_3.
\end{aligned}$$

This implies that $\|Pz\|_{\mathfrak{B}''} \leq \max(l_1, l_2, l_3) = l$, for $t \in [0, T]$.

- *Step 3:* P maps bounded sets into equi-continuous sets of $\mathcal{B}''$.

Let B_r be the set of step 2. Then, for every $t', t'' \in [0, t_1]$, with $t' < t''$ and $z \in B_r$, we have

$$\begin{aligned}
\|P(z)(t'') - P(z)(t')\| &\leq \|(P_\alpha(t'') - P_\alpha(t'))(\phi(0) + h(0, \phi))\| \\
&\quad + \|h(t'', y_{t''} + \bar{z}_{t''}) - h(t', y_{t'} + \bar{z}_{t'})\| + E_1 + E_2 + E_3,
\end{aligned}$$

where

$$\begin{aligned}
E_1 &= \left\| \int_{t'}^{t''} (t'' - s)^{\alpha-1} Q_\alpha(t'' - s) f(s, y_s + \bar{z}_s) ds \right\|, \\
E_2 &= \left\| \int_0^{t'} ((t'' - s)^{\alpha-1} - (t' - s)^{\alpha-1}) Q_\alpha(t'' - s) f(s, y_s + \bar{z}_s) ds \right\|, \\
E_3 &= \left\| \int_0^{t'} (t' - s)^{\alpha-1} (Q_\alpha(t'' - s) - Q_\alpha(t' - s)) f(s, y_s + \bar{z}_s) ds \right\|.
\end{aligned}$$

Following the idea in [13],[14], we can prove that E_1, E_2, E_3 tend to zero as $t' \rightarrow t''$. Since $P_\alpha(t)$ and $P_\alpha(t)$ ($t > 0$) are continuous in t , in the uniform operator topology,

we deduce that $\|P(z)(t'') - P(z)(t')\| \rightarrow 0$ as $t' \rightarrow t''$.

For every $t', t'' \in [t_i, s_i]$, with $t' < t''$, we have

$$\|P(z)(t'') - P(z)(t')\| \leq \|g_i(t'', y(t'') + \bar{z}(t'')) - g_i(t', y(t') + \bar{z}(t'))\|.$$

Since the functions g_i are continuous, for $i = 1, 2, \dots, m$, then

$$\|P(z)(t'') - P(z)(t')\| \rightarrow 0 \text{ as } t'' \rightarrow t'.$$

And for every $t', t'' \in [t_i, s_i]$, such that $t' < t''$ then

$$\begin{aligned} \|P(z)(t'') - P(z)(t')\| &\leq \|(P_\alpha(t'') - P_\alpha(t'))d_i\| \\ &+ \|h(t', y_{t'} + \bar{z}_{t'}) - h(t'', y_{t''} + \bar{z}_{t''})\| + E_1 + E_2 + E_3. \end{aligned}$$

Similarly to the case where $t \in [0, t_1]$, we prove that $\|P(z)(t'') - P(z)(t')\| \rightarrow 0$ as $t'' \rightarrow t'$.

- *Step 4:* We show that the set $\Omega = \{z \in \mathcal{B}'' : z = \lambda Pz\}$ is bounded.

Let $z \in \Omega$, then $z(t) = \lambda Pz(t)$, with $0 < \lambda < 1$. For $t \in [0, t_1]$, we have

$$\begin{aligned} \|z(t)\| &\leq \lambda (\|(P_\alpha(t)(\phi(0) - h(0, \phi)) + h(t, y_t + \bar{z}_t) \\ &+ \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, y_s + \bar{z}_s) ds\|) \\ &\leq \lambda \left(M_A(\|\phi\| + M_h) + M_h + \frac{\alpha M_A}{\Gamma(\alpha+1)} \left(\int_0^t (t-s)^{\alpha-1} \|f(s, y_s + \bar{z}_s)\| ds \right) \right). \end{aligned}$$

For $t \in [t_i, s_i]$, we have for $i = 1, 2, \dots, m$,

$$\begin{aligned} \|z(t)\| &\leq \|\lambda I_i(y(t_i) + \bar{z}(t_i))\| + \|\lambda g_i(t, y(t) + \bar{z}(t))\| \\ &\leq \lambda(M_I + M_g). \end{aligned}$$

For $t \in [s_i, t_{i+1}]$, for $i = 1, 2, \dots, m$, we have

$$\begin{aligned} \|z(t)\| &= \|P_\alpha(t)d_i + h(t, y_t + \bar{z}_t) + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) f(s, y_s + \bar{z}_s) ds\| \\ &\leq M_A\|d_i\| + \|h\| + \frac{\alpha M_A}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \|f(s, y_s + \bar{z}_s)\| ds, \end{aligned}$$

then for $t \in [0, T]$, we have

$$\begin{aligned} \|z(t)\| &\leq M_A D + M_h + \frac{\alpha M_A}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \|f(s, y_s + \bar{z}_s)\| ds \\ &\leq M_A D + M_h + \frac{\alpha M_A}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} (p(s) + q(s) \|y_s + \bar{z}_s\|) ds \\ &\leq M_A D + M_h + \frac{T^\alpha M_A (P_1 + q_1 C_2 \|\phi\|)}{\Gamma(\alpha+1)} \\ &\quad + \frac{\alpha C_1 q_2 M_A}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \sup_{0 \leq \tau \leq s} \|z(\tau)\| ds \\ &\leq W_1 + W_2 \int_0^t (t-s)^{\alpha-1} \sup_{0 \leq \tau \leq s} \|z(\tau)\| ds \end{aligned}$$

where

$$D = \max_{1 \leq i \leq m} d_i, \quad W_1 = M_A \|d_i\| + M_h + \frac{T^\alpha M_A (P_1 + q_1 C_2 \|\phi\|)}{\Gamma(\alpha + 1)},$$

$$W_2 = \frac{\alpha C_1 q_2 M_A}{\Gamma(\alpha + 1)}, \quad p_1 = \sup_{t \in [0, T]} p(t), \quad q_1 = \sup_{t \in [0, T]} q(t).$$

Let $\tau^* \in [0, s]$ be such that $\sup_{0 \leq \tau \leq s} \|z(\tau)\| = \|z(\tau^*)\|$, $0 \leq s \leq t$, and we use Lemma 2.3, then there exists a constant $K(\alpha)$ such that

$$\begin{aligned} \|z(t)\| &\leq W_1 + K(\alpha) W_1 \int_0^t (t-s)^{\alpha-1} ds \\ &\leq W_1 \left(1 + \frac{K(\alpha) T^\alpha}{\alpha}\right). \end{aligned}$$

Using lemma 2.4, we deduce that P has at least a fixed point on $(-\infty, T]$. This completes the proof of theorem. $\square$

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