Fuzzy α -semigroups of operators

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Abstract. In this paper, we generalize the definition of a fuzzy strongly continuous semigroup and its generator. We define a conformable fractional fuzzy semigroups of operators and its generator. We establish some of their properties and some results about this concept.

Keywords: *Conformable fractional derivatives, fuzzy sets, semigroups of operators.*

1 Introduction

The theory of semigroups, as well as linear [1] and nonlinear [2], is a well-studied topic in functional analysis. The semigroup generated by linear operators of a fuzzy-valued function was introduced by Gal and Gal [3]. Fractional semigroups are related to the problem of fractional powers of operators initiated first by Bochner (see [4]). Balakrishnan (see [5]), studied the problem of fractional powers of closed operators and the semigroups generated by them.

In [6] Mohammed AL Horani Roshdi Khalil and Thabet Abdeljawad studied, with more details, the fractional semigroups of operators associated with the conformable fractional derivative and the fractional abstract Cauchy problem according to the conformable fractional derivative which was introduced in [7].

The goal of this paper is to develop the theory of the semigroups of operators on spaces of fuzzy-number-valued functions: To introduce the fractional fuzzy semigroups of operators associated with the conformable fractional fuzzy derivative.

2 Preliminary Notes

Let $\mathcal{P}_K(\mathbb{R}^n)$ denotes the family of all nonempty compact convex subsets of $\mathbb{R}^n$ and define the addition and scalar multiplication in $\mathcal{P}_K(\mathbb{R}^n)$ as usual. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorff metric

$$d(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}$$

where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

Then it is clear that $(\mathcal{P}_K(\mathbb{R}^n), d)$ becomes a complete and separable metric space (see[8]).

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{t \in \mathbb{R}^n / u(t) \geq \alpha\}$.

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Definition 2.1. A fuzzy set A of a classical set X is called convex if $[A]^\alpha$ is a convex subset of X for all $\alpha \in [0, 1]$.

Denote

$$E^n = \{u : \mathbb{R}^n \longrightarrow [0, 1] \mid u \text{ satisfies (i)-(iv) below} \},$$

where

- (i) u is normal i.e there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,
- (ii) u is fuzzy convex,
- (iii) u is upper semicontinuous,
- (iv) $[u]^0 = cl\{x \in \mathbb{R}^n / u(x) > 0\}$ is compact.

From (i)-(iv), it follows that the α -level set $[u]^\alpha \in \mathcal{P}_K(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows :

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha$$

where $u, v \in E^n$, $k \in \mathbb{R}^n$ and $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \rightarrow \mathbb{R}^+$ by the equation

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha)$$

where d is the Hausdorff metric for non-empty compact sets in $\mathbb{R}^n$.

Then it is easy to see that D is a metric in E^n .

We know that :

- (1) (E^n, D) is a complete metric space;
- (2) $D(u + w, v + w) = D(u, v)$ for all $u, v, w \in E^n$;
- (3) $D(ku, kv) = |k| D(u, v)$ for all $u, v \in E^n$ and $k \in \mathbb{R}^n$.

For details see [8].

On E^n , we can define the subtraction $\ominus$, called the H -difference (see [9]) as follows: $u \ominus v$ has sense if there exists $w \in E^n$ such that $u = v + w$.

Denote $\mathcal{C}_a = \mathcal{C}([0, a], E^n) = \{f : [0, a] \longrightarrow E^n; f \text{ is continuous on } [0, a]\}$, endowed with the metric

$$H(u, v) = \sup_{t \in [0, a]} D(u(t), v(t)).$$

Then $(\mathcal{C}_a, H)$ is a complete metric space (see [3]).

Let $T = [c, d] \subset \mathbb{R}$ be a compact interval.

Definition 2.2. A mapping $F : T \rightarrow E^n$ is Hukuhara differentiable at $t_0 \in T$ if there exists a $F'(t_0) \in E^n$ such that the following limits

$$\lim_{h \rightarrow 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h}$$

exist and equal to $F'(t_0)$.

2.1 Operator theory

We give here a definition of linear operator, which is similar to that given by C. G. Gal and S. G. Gal in [3].

Definition 2.3. $A : E^n \longrightarrow E^n$ is a linear operator if

$$\begin{cases} A(x + y) = Ax + Ay \\ A(\lambda x) = \lambda A(x) \end{cases}$$

for all $x, y \in E^n$, $\lambda \in \mathbb{R}$.

2.2 Fuzzy strongly continuous semigroups.

We cite here a generalization of the classical Radstrom embedding theorem in E^1 .

Theorem 2.4. *There exists a real Banach space X such that E^n can be embedded as a convex cone C with vertex 0 in X . Furthermore the following conditions hold true:*

- (i) *the embedding j is isometric,*
- (ii) *the addition in X induces the addition in E^n ,*
- (iii) *the multiplication by a nonnegative real number in X induces the corresponding operation in E^n ,*
- (iv) *$C - C = \{a - b \mid a, b \in E^n\}$ is dense in X ,*
- (v) *C is closed.*

For more details on properties of this embedding see [10, 11].

We give here a definition of a fuzzy semigroups, which is similar to that given in [12, 13, 14].

Definition 2.5. *A family $\{T(t), t \geq 0\}$ of operators from E^n into itself is a fuzzy strongly continuous semigroup if*

- (i) *$T(0) = i$, the identity mapping on E^n ,*
- (ii) *$T(t + s) = T(t)T(s)$ for all $t, s \geq 0$,*
- (iii) *the function $g : [0, \infty[\rightarrow E^n$, defined by $g(t) = T(t)x$ is continuous at $t = 0$ for all $x \in E^n$ i.e*

$$\lim_{t \rightarrow 0^+} T(t)x = x$$

- (iv) *There exist two constants $M > 0$ and ω such that*

$$D(T(t)x, T(t)y) \leq M e^{\omega t} D(x, y), \quad \text{for } t \geq 0, x, y \in E^n$$

In particular if $M = 1$ and $\omega = 0$, we say that $\{T(t), t \geq 0\}$ is a contraction fuzzy semigroup.

Remark 2.6. *The condition (iii) implies that the function $t \rightarrow T(t)x$ is continuous on $[0, \infty[$ for all $x \in E^n$.*

Definition 2.7. *Let $\{T(t), t \geq 0\}$ be a fuzzy strongly continuous semigroup on E^n and $x \in E^n$. If for $h > 0$ sufficiently small, the Hukuhara difference $T(h)x \ominus x$ exists, we define*

$$Ax = \lim_{h \rightarrow 0^+} \frac{T(h)x \ominus x}{h}$$

whenever this limit exists in the metric space (E^n, D) . Then the operator A defined on

$$D(A) = \left\{ x \in E^n : \lim_{h \rightarrow 0^+} \frac{T(h)x \ominus x}{h} \text{ exists} \right\} \subset E^n$$

is called the infinitesimal generator of the fuzzy semigroup $\{T(t), t \geq 0\}$.

Lemma 2.8. *Let A be the generator of a fuzzy semigroup $\{T(t), t \geq 0\}$ on E^n , then for all $x \in E^n$ such that $T(t)x \in D(A)$ for all $t \geq 0$, the mapping $t \rightarrow T(t)x$ is differentiable and*

$$\frac{d}{dt} (T(t)x) = AT(t)x, \quad \forall t \geq 0.$$

2.3 Conformable fractional semigroups of operators

Now, we will give the basic concepts and results on conformable fractional strongly continuous semigroups, which is similar to that given in [6] and which will be used in the rest of the paper.

Definition 2.9. Let $\alpha \in (0, a]$ for any $a > 0$. For a Banach space X , A family $\{T(t), t \geq 0\}$ is called a fractional strongly continuous α -semigroup (or $\mathcal{C}_0$ - α -semigroup) of operators if:

1. $T(0) = I$.
2. $T(t+s)^{\frac{1}{\alpha}} = T(t^{\frac{1}{\alpha}})T(s^{\frac{1}{\alpha}})$, for all $t, s \geq 0$.
3. For each fixed $x \in X$,

$$T(t)x \longrightarrow x \text{ as } x \longrightarrow 0^+.$$

Definition 2.10. Let $\{T(t), t \geq 0\}$ be a strongly continuous α -semigroup on X , the operator $A : x \mapsto Ax$ defined by $Ax = \lim_{t \rightarrow 0^+} T^{(\alpha)}(t)x$ on

$$D(A) = \left\{ x \in X : \lim_{t \rightarrow 0^+} T^{(\alpha)}(t)x \text{ exists} \right\},$$

where

$$T^{(\alpha)}(t)x = \lim_{\epsilon \rightarrow 0} \frac{T(t + \epsilon t^{1-\alpha})x - T(t)x}{\epsilon}.$$

is called the α -infinitesimal generator of the α -semigroup $T(t)$.

3 Main Results

Let us start by defining what we mean by a fuzzy conformable fractional derivative function.

Definition 3.1. A mapping $f : [0, +\infty) \longrightarrow E^1$ is fuzzy conformable fractional derivative at t of order α if there exists $F_\alpha(f)(t) \in E^1$ such that the following limit

$$\lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) \ominus f(t)}{\epsilon},$$

exists and equal to $F_\alpha(f)(t)$.

f is fuzzy conformable fractional derivative of order α , if f is fuzzy conformable fractional derivative of order α for all $t > 0$.

The limit is taken in the metric space (E^1, D) .

Let $f^{(\alpha)}(t)$ stands for $F_\alpha(f)(t)$. Hence

$$f^{(\alpha)}(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) \ominus f(t)}{\epsilon}.$$

If $f^{(\alpha)}(t)$ exists on $(0, b)$, $b > 0$ and $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t) = \lim_{t \rightarrow 0^+} F_\alpha(f)(t)$ exists, then let

$$f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t).$$

Definition 3.2. Let $\alpha \in (0, a]$ for any $a > 0$. A family $\{T(t), t \geq 0\}$ on E^1 is called a fuzzy fractional strongly continuous α -semigroup (or fuzzy α -semigroup) of operators if

1. $T(0) = i$, the identity mapping on E^1 .
2. $T(t+s)^{\frac{1}{\alpha}} = T(t^{\frac{1}{\alpha}})T(s^{\frac{1}{\alpha}})$, for all $t, s \geq 0$.

3. For each fixed $x \in E^1$,

$$T(t)x \longrightarrow x \text{ as } x \longrightarrow 0^+.$$

4. There exist two constants $M > 0$ and ω such that

$$D\left(T(t)^{\frac{1}{\alpha}}x, T(t)^{\frac{1}{\alpha}}y\right) \leq Me^{\omega t^{\frac{1}{\alpha}}} D(x, y), \text{ for } t \geq 0, x, y \in E^1.$$

In particular if $M = 1$ and $\omega = 0$, we say that $\{T(t), t \geq 0\}$ is a contraction fuzzy α -semigroup.

Remark 3.3. Clearly, if $\alpha = 1$, then the fuzzy 1-semigroups are just the usual fuzzy strongly continuous semigroups.

Example 3.4. Let $f \in \mathcal{C}([0, +\infty), E^1)$, define the linear operator $T(t)$ by

$$(T(t)f)(s) = f(s + 2\sqrt{t}).$$

$\{T(t), t \geq 0\}$ is a fuzzy $\frac{1}{2}$ -semigroup. Indeed:

We have

1. $(T(0)f)(s) = f(s)$, for all $f \in \mathcal{C}([0, +\infty), E^1)$.
2. For $t_1, t_2 \geq 0$, $f \in \mathcal{C}([0, +\infty), E^1)$

$$\begin{aligned} (T(t_1 + t_2)^2 f)(s) &= f\left(s + 2\sqrt{(t_1 + t_2)^2}\right) \\ &= f(s + 2(t_1 + t_2)), \end{aligned}$$

and

$$\begin{aligned} (T(t_1^2)T(t_2^2)f)(s) &= \left(T(t_1^2)f\left(s + 2\sqrt{t_2^2}\right)\right) \\ &= (T(t_1^2)f)(s + 2t_2) \\ &= f\left(s + 2t_2 + 2\sqrt{t_1^2}\right) \\ &= f(s + 2(t_1 + t_2)) \end{aligned}$$

Therefore

$$T(t_1 + t_2)^2 f = T(t_1^2)T(t_2^2)f.$$

3. For $t \geq 0$, $f \in \mathcal{C}([0, +\infty), E^1)$, we have

$$D((T(t)f)(s), f(s)) = D(f(s + 2\sqrt{t}), f(s)) \longrightarrow 0 \text{ as } t \longrightarrow 0^+.$$

4. For $t \geq 0$, $f, g \in \mathcal{C}([0, +\infty), E^1)$, we have

$$\begin{aligned} D((T(t)f)(s), (T(t)g)(s)) &= D\left(f(s + 2\sqrt{t}), g(s + 2\sqrt{t})\right) \\ &\leq H(f, g), \quad \forall s \geq 0. \end{aligned}$$

Then

$$H(T(t)f, T(t)g) \leq H(f, g).$$

Consequently, $\{T(t), t \geq 0\}$ is a fuzzy $\frac{1}{2}$ -semigroup.

Definition 3.5. Let $\{T(t), t \geq 0\}$ be a fuzzy α -semigroup on E^1 and $x \in E^1$. If for $t > 0$ sufficiently small,

$$T^{(\alpha)}(t)x = \lim_{\epsilon \rightarrow 0} \frac{T(t + \epsilon t^{1-\alpha})x \odot T(t)x}{\epsilon},$$

exists, we define

$$Ax = \lim_{t \rightarrow 0^+} T^{(\alpha)}(t)x.$$

Then the operator $A : E^1 \rightarrow E^1$, $x \mapsto Ax$ defined on

$$D(A) = \left\{ x \in E^1 : \lim_{t \rightarrow 0^+} T^{(\alpha)}(t)x \text{ exists} \right\} \subset E^1,$$

is called fuzzy α -infinitesimal generator of $\{T(t), t \geq 0\}$.

Example 3.6. We define on E^1 the family of operator $\{T(t), t \geq 0\}$ by

$$T(t)x = e^{\lambda \sqrt{t}}x, \quad \lambda \in \mathbb{R}.$$

For $\lambda \geq 0$, $\{T(t), t \geq 0\}$ is a fuzzy $\frac{1}{2}$ -semigroup, and the operator A defined by $Ax = \lambda x$ is the fuzzy $\frac{1}{2}$ -infinitesimal generator of this fuzzy $\frac{1}{2}$ -semigroup.

Due to properties of the embedding j reported in theorem 2.4, we have the following result:

Proposition 3.7. Let $A : E^1 \rightarrow E^1$ and $A_1 = jAj^{-1} : C \rightarrow C$ two operators. A is the fuzzy α -infinitesimal generator of a fuzzy α -semigroup $\{T(t), t \geq 0\}$ on E^1 if and only if A_1 is the α -infinitesimal generator of the α -semigroup $\{T_1(t), t \geq 0\}$ defined on the convex closed set C by $T_1(t) = jT(t)j^{-1}$, for $t \geq 0$.

Proof. It is clear to say that $\{T(t), t \geq 0\}$ is a fuzzy α -semigroup on E^1 if and only if $\{T_1(t), t \geq 0\}$ is an α -semigroup on C .

We assume that A is a fuzzy α -infinitesimal generator of a fuzzy α -semigroup $\{T(t), t \geq 0\}$ on E^1 .

Let $x \in j^{-1}(D(A))$, we have

$$\begin{aligned} T_1^{(\alpha)}(t)x &= \lim_{\epsilon \rightarrow 0} \frac{T_1(t + \epsilon t^{1-\alpha})x - T_1(t)x}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{jT(t + \epsilon t^{1-\alpha})j^{-1}x - jT(t)j^{-1}x}{\epsilon} \\ &= j \left(\lim_{\epsilon \rightarrow 0} \frac{T(t + \epsilon t^{1-\alpha})j^{-1}x \odot T(t)j^{-1}x}{\epsilon} \right) \\ &= jT^{(\alpha)}(t)j^{-1}x. \end{aligned}$$

Then

$$\lim_{t \rightarrow 0^+} T_1^{(\alpha)}(t)x = \lim_{t \rightarrow 0^+} jT^{(\alpha)}(t)j^{-1}x = j \lim_{t \rightarrow 0^+} T^{(\alpha)}(t)j^{-1}x = jAj^{-1}x = A_1x.$$

Conversely, if A_1 is the generator of an α -semigroup $\{T_1(t), t \geq 0\}$ on C , then for all $x \in D(A)$

$$\begin{aligned} T^{(\alpha)}(t)x &= \lim_{\epsilon \rightarrow 0} \frac{T(t + \epsilon t^{1-\alpha})x \odot T(t)x}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{j^{-1}T_1(t + \epsilon t^{1-\alpha})jx \odot j^{-1}T_1(t)jx}{\epsilon} \\ &= j^{-1} \left(\lim_{\epsilon \rightarrow 0} \frac{T_1(t + \epsilon t^{1-\alpha})jx - T_1(t)jx}{\epsilon} \right) \\ &= j^{-1}T_1^{(\alpha)}(t)jx. \end{aligned}$$

Then

$$\lim_{t \rightarrow 0^+} T^{(\alpha)}(t)x = \lim_{t \rightarrow 0^+} j^{-1}T_1^{(\alpha)}(t)jx = j^{-1} \lim_{t \rightarrow 0^+} T_1^{(\alpha)}(t)jx = j^{-1}A_1jx = Ax.$$

□

Example 3.8. Consider $T(t)$ and A defined in the example 3.6.

$\{T_1(t) = jT(t)j^{-1}, t \geq 0\}$ is a $\frac{1}{2}$ -semigroup, and the operator A_1 defined by $A_1 = jAj^{-1}$ is the $\frac{1}{2}$ -infinitesimal generator of this $\frac{1}{2}$ -semigroup.

Theorem 3.9. Let A be the fuzzy α -infinitesimal generator of a fuzzy α -semigroup $\{T(t), t \geq 0\}$ on E^1 , then for all $x \in D(A)$ such that $T(t)x \in D(A)$ for all $t \geq 0$, the mapping $t \rightarrow g(t) = T(t)x$ is fuzzy conformable fractional derivative at t of order α and

$$g^{(\alpha)}(t) = T^{(\alpha)}(t)x = AT(t)x.$$

Proof. For $t \geq 0$, $x \in E^1$, we have

$$\begin{aligned} g^{(\alpha)}(t) = T^{(\alpha)}(t)x &= \lim_{\epsilon \rightarrow 0} \frac{g(t + \epsilon t^{1-\alpha}) \ominus g(t)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{T(t + \epsilon t^{1-\alpha})x \ominus T(t)x}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{T((t + \epsilon t^{1-\alpha})^\alpha)^{\frac{1}{\alpha}} x \ominus T(t)x}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{T(t^\alpha + (t + \epsilon t^{1-\alpha})^\alpha - t^\alpha)^{\frac{1}{\alpha}} x \ominus T(t)x}{\epsilon} \end{aligned}$$

Since $T(t)$ is a fuzzy α -semigroup of operators, then $T(a + b)^{\frac{1}{\alpha}} = T\left(a^{\frac{1}{\alpha}}\right) T\left(b^{\frac{1}{\alpha}}\right)$. Hence

$$\begin{aligned} T^{(\alpha)}(t)x &= \lim_{\epsilon \rightarrow 0} \frac{T((t + \epsilon t^{1-\alpha})^\alpha - t^\alpha)^{\frac{1}{\alpha}} T(t^\alpha)^{\frac{1}{\alpha}} x \ominus T(t)x}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{j^{-1}T_1((t + \epsilon t^{1-\alpha})^\alpha - t^\alpha)^{\frac{1}{\alpha}} jj^{-1}T_1(t^\alpha)^{\frac{1}{\alpha}} jx \ominus j^{-1}T_1(t)jx}{\epsilon} \\ &= j^{-1} \left(\lim_{\epsilon \rightarrow 0} \frac{T_1((t + \epsilon t^{1-\alpha})^\alpha - t^\alpha)^{\frac{1}{\alpha}} T_1(t^\alpha)^{\frac{1}{\alpha}} jx - T_1(t)jx}{\epsilon} \right) \\ &= j^{-1} \left(\lim_{\epsilon \rightarrow 0} \frac{T_1((t + \epsilon t^{1-\alpha})^\alpha - t^\alpha)^{\frac{1}{\alpha}} T_1(t)jx - T_1(t)jx}{\epsilon} \right). \end{aligned}$$

Now, using the Mean Value Theorem for conformable fractional derivative (see [7]), we get

$$\frac{T_1((t + \epsilon t^{1-\alpha})^\alpha - t^\alpha)^{\frac{1}{\alpha}} T_1(t)jx - T_1(t)jx}{\epsilon} = T_1^{(\alpha)}(c)T_1(t)jx \frac{[(t + \epsilon t^{1-\alpha})^\alpha - t^\alpha]}{\alpha\epsilon}$$

for some $0 < c < (t + \epsilon t^{1-\alpha})^\alpha - t^\alpha$.

If $\epsilon \rightarrow 0$, then $c \rightarrow 0$, and $\lim_{\epsilon \rightarrow 0} T_1^{(\alpha)}(c)T_1(t)jx = T_1^{(\alpha)}(0)T_1(t)jx = A_1T_1(t)jx$. Consequently,

$$\begin{aligned} T^{(\alpha)}(t)x &= j^{-1} \left(\lim_{\epsilon \rightarrow 0} \frac{T_1((t + \epsilon t^{1-\alpha})^\alpha - t^\alpha)^{\frac{1}{\alpha}} T_1(t)jx - T_1(t)jx}{\epsilon} \right) \\ &= j^{-1} \left(A_1T_1(t)jx \lim_{\epsilon \rightarrow 0} \frac{[(t + \epsilon t^{1-\alpha})^\alpha - t^\alpha]}{\alpha\epsilon} \right). \end{aligned}$$

Using Hospital's rule, we get

$$\lim_{\epsilon \rightarrow 0} \frac{[(t + \epsilon t^{1-\alpha})^\alpha - t^\alpha]}{\alpha\epsilon} = 1.$$

Hence $g^{(\alpha)}(t) = T^{(\alpha)}(t)x = j^{-1}A_1T_1(t)jx = j^{-1}A_1jj^{-1}T_1(t)jx = AT(t)x$ □

Remark 3.10. *In the linear case, we have*

$$g^{(\alpha)}(t) = T^{(\alpha)}(t)x = AT(t)x = T(t)Ax, \quad \forall t \geq 0.$$

In the nonlinear case, there is no commutation between operators, it depends on linearity of semigroups.

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