RESEARCH ARTICLE

Computational contact mechanics for a medium consisting of functionally graded material coating and orthotropic substrate

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Abstract

This paper presents a semi-analytical method to investigate the frictionless contact mechanics between a functionally graded material (FGM) coating and an orthotropic substrate when the system is indented by a rigid flat punch. From the bottom, the orthotropic substrate is completely bonded to the rigid foundation. The body force of the orthotropic substrate is ignored in the solution, while the body force of the FGM coating is considered. An exponential function is used to define the smooth variation of the shear modulus and density of the FGM coating, and the variation of Poisson's ratio is assumed to be negligible. The partial differential equation system for the FGM coating and the orthotropic substrate is solved analytically through Fourier transformations. After applying boundary and interface continuity conditions to the mixed boundary value problem, the contact problem is reduced to a singular integral equation. The Gauss–Chebyshev integration method is then used to convert the singular integral equation into a system of linear equations, which are solved using an appropriate iterative algorithm to calculate the contact stress under the rigid flat punch. The parametric analyses presented here demonstrate the effects of normalized punch length, material inhomogeneity, dimensionless press force, and orthotropic material type on contact stresses at interfaces, critical load factor, and initial separation distance between FGM coating and orthotropic substrate. The developed solution procedures are verified through the comparisons made to the results available in the literature. The solution methodology and numerical results presented in this paper can provide some useful guidelines for improving the design of multibody indentation systems using FGMs and anisotropic materials.

Keywords

Critical load factor; FGM coating; Flat punch; Initial separation distance; Orthotropic substrate

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1. Introduction

The field of contact mechanics, concerned with the deformation of solid materials that are placed in contact with one another, has a long history, dating back to the pioneering works of Hertz [1] and Boussinesq [2]. Contact mechanics issues can be classified based on a variety of factors. For example, the type of foundational material law (elastic/viscoelastic,

homogeneous/inhomogeneous, degree of isotropy, etc.), the geometry of the applied load (normal contact, tangential contact, etc.), the friction and adhesion regime (frictionless, no-slip, etc.), or the shape of contacting bodies are all systematization categories [3]. The contact mechanics of homogeneous isotropic materials was a prominent issue when studies on contact mechanics were first investigated. Regarding the literature on the

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theoretical investigations of the contact mechanics, problems involving isotropic materials are well investigated by [4–9]. The reader should consult the studies [10–17] for sliding contact, partial slip contact, rolling, and rolling tractive contact. Also, for frictional, elastodynamic, and thermoelastic contact problems, [18–20] papers should be read. However, in parallel with the progress of science and technology, there has been a tendency toward new innovative materials in studies in contact mechanics. As a result, studies on the contact mechanics of functionally graded materials (FGMs), a new type of composite material, was quickly introduced to the literature in the years that followed. FGMs are non-homogeneous composites that have been enhanced through the use of special deposition techniques. The primary advantage of graded materials over laminated composites is that the transition from one constituent to another can be smoothly tailored, thereby avoiding potential mismatch issues. Contact and indentation of FGMs have been studied to investigate the response of elastic graded coatings or interfacial layers using different material distributions, geometrical and loading conditions, and solution techniques. Frictionless and frictional contact problems of FGMs have been studied using semi-analytical solutions in [21–25]. In recent years, there has also been an increase in research interest in the contact of anisotropic materials.

Willis [26] attacked Hertz's problem for generally anisotropic bodies by a Fourier transform method. A solution to the frictionless contact of rigid flat indenters on arbitrarily layered, anisotropic half-planes using Fourier transforms and the local-global stiffness matrix technique is presented by Urquhart and Pindera [27]. Lin and Ovaert [28] investigated a method for solving the two-dimensional isothermal rough surface contact problem with friction in general anisotropic materials. Kahya et al. [29] investigated the frictionless receding contact problem between an anisotropic elastic layer and an anisotropic elastic half-plane when the two bodies are pressed together with a rigid circular stamp. A mathematical model which describes the frictional contact between the

piezoelectric body and an electrically conductive foundation is considered by Migórski et al. [30]. Bagault et al. [31] developed a semi-analytical method for the contact problem of anisotropic elastic materials with an anisotropic coating. Li et al. [32] present the fundamental contact solutions of a magneto-electro-elastic half-space indented by a smooth and rigid half-infinite punch. Zhou and Lee [33] investigated the frictional sliding contact problem in monoclinic piezoelectric materials subjected to a rigid punch. The sliding frictional contact between a monoclinic coating/isotropic substrate system and a rigid cylindrical punch is examined by Yilmaz et al. [34]. A double frictional receding contact mechanics problem for a homogeneous orthotropic/isotropic system is investigated by Yildirim et al. [35]. Çömez and Yilmaz [36] considered the frictional contact problem of a half-plane that has monoclinic material property. Song et al. [37] investigate the two-dimensional frictionless contact problem of the piezoelectric half-plane under rigid flat and cylindrical punches. Cao et al. [38] present the frictional receding contact between a functionally graded elastic layer and an orthotropic substrate under the indentation of a rigid flat-ended stamp. The nonlinear partial slip contact problem between a monoclinic half-plane and a rigid punch of an arbitrary profile subjected to a normal load was solved by Çömez et al. [39]. Öner [40] investigated the receding contact problem of a homogeneous orthotropic coating resting on a homogeneous isotropic substrate without any interfacial defects. Investigation of the literature indicates that there is no previous study related to the continuous contact mechanics between an FGM coating and an orthotropic substrate. These contact mechanics studies have great importance for the design of components in the field of tribology, automotive and aerospace industries. To fill this gap, we investigate the frictionless continuous contact problem of an FGM-coated orthotropic substrate indented by a normally loaded rigid flat punch using generalized plane strain conditions for the elasticity equations. The two-dimensional Navier's equations of equilibrium and boundary conditions

Kolosov's constant which can be calculated depending on Poisson's ratio ν for plane strain state by

$$\kappa_1 = 3 - 4\nu_1 \quad (5)$$

Define the Fourier's sine and cosine transforms as

$$u_{1h}(x, z) = \frac{2}{\pi} \int_0^\infty \phi_{1h}(\alpha, z) \sin \alpha x d\alpha \quad (6)$$

$$w_{1h}(x, z) = \frac{2}{\pi} \int_0^\infty \psi_{1h}(\alpha, z) \cos(\alpha x) d\alpha \quad (7)$$

where α is the transform variable. Above Fourier's sine and cosine transforms are used to solve the partial differential equation system (3–4) reducing them to a system of ordinary differential equations. Employing the Fourier transforms to Eqs. (3–4) produces

$$\begin{aligned} & -(\kappa_1 + 1)\alpha^2 \phi_{1h} + (\kappa_1 - 1)\phi_{1h,zz} - 2\alpha\psi_{1h,z} \\ & + \beta(\kappa_1 - 1)[\phi_{1h,z} - \alpha\psi_{1h}] = 0 \end{aligned} \quad (8)$$

$$\begin{aligned} & -(\kappa_1 - 1)\alpha^2 \psi_{1h} + (\kappa_1 + 1)\psi_{1h,zz} + 2\alpha\phi_{1h,z} \\ & + \beta[(3 - \kappa_1)\alpha\phi_{1h} + (\kappa_1 + 1)\psi_{1h,z}] = 0 \end{aligned} \quad (9)$$

After solving Eqs. (8–9) and using the corresponding inverse Fourier transforms, we find the displacement and stress fields for FGM coating in the form

$$u_{1h}(x, z) = \frac{2}{\pi} \int_0^\infty \sum_{j=1}^4 A_j e^{\tilde{\lambda}_j z} \sin(\alpha x) d\alpha \quad (10)$$

$$w_{1h}(x, z) = \frac{2}{\pi} \int_0^\infty \sum_{j=1}^4 A_j m_j e^{\tilde{\lambda}_j z} \cos(\alpha x) d\alpha \quad (11)$$

$$\sigma_{1xh} = \frac{2}{\pi} \int_0^\infty \sum_{j=1}^4 \frac{\mu_0}{(\kappa_1 - 1)} A_j \chi_j e^{(\tilde{\lambda}_j + \beta)z} \cos(\alpha x) d\alpha \quad (12)$$

$$\sigma_{1zh} = \frac{2}{\pi} \int_0^\infty \sum_{j=1}^4 \frac{\mu_0}{(\kappa_1 - 1)} A_j \zeta_j e^{(\tilde{\lambda}_j + \beta)z} \cos(\alpha x) d\alpha \quad (13)$$

$$\tau_{1xz} = \frac{2}{\pi} \int_0^\infty \sum_{j=1}^4 \mu_0 A_j \gamma_j e^{(\tilde{\lambda}_j + \beta)z} \sin(\alpha x) d\alpha \quad (14)$$

where

$$\tilde{\lambda}_1 = \frac{1}{2} \left(-\beta - \sqrt{\frac{(4\alpha^2 + \beta^2)(1 + \kappa_1) - 4\sqrt{\alpha^2 \beta^2 (-3 - 2\kappa_1 + \kappa_1^2)}}{1 + \kappa_1}} \right) \quad (15a)$$

$$\tilde{\lambda}_2 = \frac{1}{2} \left(-\beta + \sqrt{\frac{(4\alpha^2 + \beta^2)(1 + \kappa_1) - 4\sqrt{\alpha^2 \beta^2 (-3 - 2\kappa_1 + \kappa_1^2)}}{1 + \kappa_1}} \right) \quad (15b)$$

$$\tilde{\lambda}_3 = \frac{1}{2} \left(-\beta - \sqrt{\frac{(4\alpha^2 + \beta^2)(1 + \kappa_1) + 4\sqrt{\alpha^2 \beta^2 (-3 - 2\kappa_1 + \kappa_1^2)}}{1 + \kappa_1}} \right) \quad (15c)$$

$$\tilde{\lambda}_4 = \frac{1}{2} \left(-\beta + \sqrt{\frac{(4\alpha^2 + \beta^2)(1 + \kappa_1) + 4\sqrt{\alpha^2 \beta^2 (-3 - 2\kappa_1 + \kappa_1^2)}}{1 + \kappa_1}} \right) \quad (15d)$$

$$m_j = \frac{(3\beta + 2\tilde{\lambda}_j - \beta\kappa_1)(\tilde{\lambda}_j(\beta + n_j)(\kappa_1 + 1) - \alpha^2(\kappa_1 + 3))}{\alpha(4\alpha^2 - \beta^2(\kappa_1 - 3)(\kappa_1 + 1))} \quad (16a)$$

$$\chi_j = (3 - \kappa_1)m_j\tilde{\lambda}_j + \alpha(\kappa_1 + 1) \quad (16b)$$

$$\zeta_j = (\kappa_1 + 1)m_j\tilde{\lambda}_j + \alpha(3 - \kappa_1) \quad (16c)$$

$$\gamma_j = \tilde{\lambda}_j - \alpha m_j \quad (j = 1, \dots, 4) \quad (16d)$$

Using Hooke's law and the equilibrium equations in the existing body forces, the following partial differential equations in terms of the displacement components u_{1p} and w_{1p} for are obtained:

$$(\kappa_1 + 1)u_{1p,xx} + \beta(\kappa_1 - 1)u_{1p,z} + (\kappa_1 - 1)u_{1p,zz} + \beta(\kappa_1 - 1)w_{1p,x} + 2w_{1p,xz} = 0 \quad (17)$$

$$(\kappa_1 + 1)w_{1p,zz} + \beta(3 - \kappa_1)u_{1p,x} + (\kappa_1 - 1)w_{1p,xx} + \beta(\kappa_1 + 1)w_{1p,z} + 2u_{1p,xz} = \rho_p e^{(\Gamma - \beta)z} \quad (18)$$

where subscript p represents the particular solution related to existing body forces. Also, ρ_p in Eq. (18) is defined as follows:

$$\rho_p = \frac{(\kappa_1 - 1)}{\mu_0} \rho_0 g \quad (19)$$

$$\sigma_{1xp} = \frac{\mu_0 \left(8e^{z\beta} \Gamma K_1 (-1 + \kappa_1) - (-3 + \kappa_1) \left(\Gamma L_1 (1 + \kappa_1) + e^{z\Gamma} \rho_p \right) \right)}{\Gamma (-1 + \kappa_1^2)} \quad (24)$$

$$\sigma_{1zp} = \frac{\mu_0 \left(\Gamma L_1 (1 + \kappa_1) + e^{z\Gamma} \rho_p \right)}{\Gamma (-1 + \kappa_1)} \quad (25)$$

$$\tau_{1xzp} = 0 \quad (26)$$

K_1 , K_2 , L_1 , L_2 in Eqs. (22–25) are the unknowns and they will be assigned applying the boundary conditions of the given below.

$$u_{1p}(0) = 0 \quad (27a)$$

$$w_{1p}(0) = 0 \quad (27b)$$

For the particular solution, assuming displacement components $u_{1p} = u_{1p}(x)$ and $w_{1p} = w_{1p}(z)$, Eqs. (17) and (18) yield

$$(\kappa_1 + 1)u_{1p,xx} = 0 \quad (20)$$

$$(\kappa_1 + 1)w_{1p,zz} + \beta(3 - \kappa_1)u_{1p,x} + \beta(\kappa_1 + 1)w_{1p,z} = \rho_p e^{(\Gamma - \beta)z} \quad (21)$$

The solutions of Eqs. (20–21) are

$$u_{1p}(x) = K_1 x + K_2 \quad (22)$$

$$w_{1p}(z) = \frac{(\kappa_1 - 3)z}{(\kappa_1 + 1)} K_1 - \frac{e^{-\beta z}}{\beta} L_1 + L_2 - \frac{e^{(\Gamma - \beta)z} \rho_p}{(\kappa_1 + 1)(\beta - \Gamma)\Gamma} \quad (23)$$

When the displacement expressions u_{1p} and w_{1p} determined above are substituted in the constitutive equations known from the theory of elasticity, the stress components are obtained as follows:

$$\int_0^{h_1} \sigma_{1xp} dz = 0 \quad (27c)$$

$$\sigma_{1zp} = \int_z^{h_1} -\rho_0 g e^{\Gamma z} dz \quad (27d)$$

The coefficients obtained after the application of the above boundary conditions are given in the Appendix. The total displacement and stress components for FGM coating will be the

superposition of two cases for the presence and absence of body forces.

$$u_1(x, z) = u_{1h} + u_{1p} \quad (28)$$

$$u_1(x, z) = u_{1h} + u_{1p} \quad (29)$$

$$\sigma_{1x}(x, z) = \sigma_{1xh} + \sigma_{1xp} \quad (30)$$

$$\sigma_{1z}(x, z) = \sigma_{1zh} + \sigma_{1zp} \quad (31)$$

$$\tau_{1xz}(x, z) = \tau_{1xzh} + \tau_{1xzp} \quad (32)$$

The constitutive relations for the plane strain orthotropy of a substrate can be reduced to the following matrix form:

$$\begin{Bmatrix} \sigma_{2x} \\ \sigma_{2y} \\ \sigma_{2z} \\ \tau_{2yz} \\ \tau_{2xz} \\ \tau_{2xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_2}{\partial x} \\ 0 \\ \frac{\partial w_2}{\partial z} \\ 0 \\ \frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x} \\ 0 \end{Bmatrix} \quad (33)$$

where σ_x , σ_y , σ_z , τ_{yz} , τ_{xz} and τ_{xy} are the stress components; u and w are the displacement components. C_{ij} denotes the stiffness coefficients for the orthotropic material. The subscript 2 refers to the orthotropic substrate.

When the constitutive relations are substituted into the stress equilibrium equations, two partial differential equations (PDEs) in terms of displacements are obtained, yielding the governing equations describing the elastic behavior of orthotropic materials.

$$C_{11}u_{2,xx} + C_{55}u_{2,zz} + (C_{13} + C_{55})w_{2,xz} = 0 \quad (34)$$

$$C_{55}w_{2,xx} + C_{33}w_{2,zz} + (C_{13} + C_{55})u_{2,xz} = 0 \quad (35)$$

Similar to the middle tiers for FGM coating, applying Fourier integral transform to Eqs. (34–35), we have

$$-\alpha^2 C_{11}\phi_2 + C_{55}\phi_{2,zz} - \alpha(C_{13} + C_{55})\psi_{2,z} = 0 \quad (36)$$

$$-\alpha^2 C_{55}\psi_2 + C_{33}\psi_{2,zz} + \alpha(C_{13} + C_{55})\phi_{2,z} = 0 \quad (37)$$

Solving Eqs. (36–37) gives the displacement and stress fields for the orthotropic substrate as follows:

$$u_2(x, z) = \frac{2}{\pi} \int_0^\infty \sum_{i=1}^4 B_i e^{\eta_i \alpha z} \sin(\alpha x) d\alpha \quad (38)$$

$$w_2(x, z) = \frac{2}{\pi} \int_0^\infty \sum_{i=1}^4 B_i \bar{\mathfrak{O}}_i e^{\eta_i \alpha z} \cos(\alpha x) d\alpha \quad (39)$$

$$\sigma_{2x}(x, z) = \frac{2}{\pi} \int_0^\infty \alpha \sum_{i=1}^4 B_i (C_{11} + C_{13} \bar{\mathfrak{O}}_i \eta_i) e^{\eta_i \alpha z} \cos(\alpha x) d\alpha \quad (40)$$

$$\sigma_{2z}(x, z) = \frac{2}{\pi} \int_0^\infty \alpha \sum_{i=1}^4 B_i (C_{13} + C_{33} \bar{\mathfrak{O}}_i \eta_i) e^{\eta_i \alpha z} \cos(\alpha x) d\alpha \quad (41)$$

$$\tau_{2xz}(x, z) = \frac{2}{\pi} \int_0^\infty \alpha \sum_{i=1}^4 B_i C_{55} (\eta_i - \bar{\mathfrak{O}}_i) e^{\eta_i \alpha z} \sin(\alpha x) d\alpha \quad (42)$$

where

$$\eta_1 = -\sqrt{\frac{C_{13}^2 - C_{11}C_{33} + 2C_{13}C_{55} + \sqrt{(C_{13}^2 - C_{11}C_{33})(C_{13}^2 - C_{11}C_{33} + 4C_{13}C_{55} + 4C_{55}^2)}}{2C_{33}C_{55}}} \quad (43a)$$

$$\eta_2 = \sqrt{\frac{C_{13}^2 - C_{11}C_{33} + 2C_{13}C_{55} + \sqrt{(C_{13}^2 - C_{11}C_{33})(C_{13}^2 - C_{11}C_{33} + 4C_{13}C_{55} + 4C_{55}^2)}}{2C_{33}C_{55}}} \quad (43b)$$

$$\eta_3 = -\sqrt{\frac{-C_{13}^2 + C_{11}C_{33} - 2C_{13}C_{55} + \sqrt{(C_{13}^2 - C_{11}C_{33})(C_{13}^2 - C_{11}C_{33} + 4C_{13}C_{55} + 4C_{55}^2)}}{2C_{33}C_{55}}} \quad (43c)$$

$$\eta_4 = \sqrt{\frac{-C_{13}^2 + C_{11}C_{33} - 2C_{13}C_{55} + \sqrt{(C_{13}^2 - C_{11}C_{33})(C_{13}^2 - C_{11}C_{33} + 4C_{13}C_{55} + 4C_{55}^2)}}{2C_{33}C_{55}}} \quad (43d)$$

$$\overline{\sigma}_i = \frac{\eta_i (C_{13}^2 - C_{11}C_{33} + 2C_{13}C_{55} + C_{55}^2 + C_{33}C_{55}\eta_i^2)}{C_{55}(C_{13} + C_{55})} \quad (43e)$$

The coefficients A_i and B_i , ($i = 1, \dots, 4$) in Eqs. (10–14) and (28–32) are the unknowns, which will be determined from the boundary conditions of the contact problem which solution has to satisfy.

$$\sigma_{1z}(x, h_1) = \begin{cases} -p(x) & -a < x < a \\ 0 & x \leq -a, \quad x \geq a \end{cases} \quad (44a)$$

$$\tau_{1xz}(x, h_1) = 0 \quad (44b)$$

$$\tau_{1xz}(x, 0) = 0 \quad (44c)$$

$$\tau_{2xz}(x, 0) = 0 \quad (44d)$$

$$\sigma_{1z}(x, 0) = \sigma_{2z}(x, 0) \quad (44e)$$

$$\frac{\partial}{\partial x} [w_1(x, 0) - w_2(x, 0)] = 0 \quad (44f)$$

$$u_2(x, -h_2) = 0 \quad (44g)$$

$$w_2(x, -h_2) = 0 \quad (44h)$$

where $p(x)$ is the unknown contact stress between the rigid flat punch and FGM coating on the contact boundary $(-a, +a)$.

3. Deriving SIE, the normalization and numerical solution procedures of SIE

Taking the Fourier transforms of the boundary conditions given by Eqs. (44) produces an algebraic equation system. Solving the equation system yields eight unknowns A_i and B_i ($i = 1, \dots, 4$) that appear in the stress and displacement fields. These unknowns may be determined in terms of the unknown contact stress. Notably, A_i and B_i are fairly long, and therefore, not presented here.

The problem is reduced to a SIE by using the punch profile displacement derivative condition (mixed condition), which is expressed for flat punch as follows:

$$\frac{\partial w_1(x, h_1)}{\partial x} = 0, \quad -a < x < +a \quad (45)$$

By substituting the previously determined A_i and B_i coefficients into equation (45) and after some algebraic manipulations, the SIE is derived in the following form:

$$\frac{1}{\pi} \int_{-a}^a p(t) dt \left[\frac{1}{t-x} + \varpi(x, t) \right] = 0, \quad -a < x < +a \quad (46)$$

where

$$\varpi(x, t) = \frac{1}{\Lambda} \int_0^{\infty} (k_1(\alpha) - \Lambda) \sin \alpha(t - x) d\alpha, \quad (47a)$$

$$k_1(\alpha) = -\alpha \sum_{i=1}^4 A_i m_i e^{\tilde{\lambda}_i h_1} \quad (47b)$$

$$\Lambda = \lim_{\alpha \rightarrow \infty} k_1(\alpha) \quad (47c)$$

Note that equilibrium of the rigid punch requires that

$$\int_{-a}^{+a} p(t) dt = P \quad (48)$$

Note that when the value of press force exceeds P_{cr} , the interface separation takes place between the FGM coating and the orthotropic substrate. The continuous contact problem can only be solved if the pressing force is less than P_{cr} , which has yet to be determined. To determine the critical load factor and initial separation distance, the contact stress on the interface $\sigma_z(x, 0)$ should be evaluated first. Substituting the values of A_j ($j = 1, \dots, 4$) as evaluated in terms of $p(x)$ into (31), after some algebra the contact stress between the FGM coating and the orthotropic substrate can be found as

$$\sigma_{1z}(x, 0) = \frac{1}{\pi} \int_{-a}^{+a} p(t) \varphi(x, t) dt + \frac{(1 - e^{\Gamma h_1}) \rho_0 g}{\Gamma} \quad (49)$$

where

$$\varphi(x, t) = \int_0^{\infty} k_2(\alpha) \cos \alpha(t - x) d\alpha, \quad (50a)$$

$$k_2(\alpha) = \sum_{j=1}^4 \frac{\mu_0}{(\kappa_1 - 1)} A_j \zeta_j \quad (50b)$$

To solve the SIE, the following normalizations are defined:

$$s = \frac{x}{a}, \quad r = \frac{t}{a}, \quad Y(r) = \frac{p(r)}{P/h_1} \quad (51)$$

Making use of the normalizations, the integral equation is rewritten as

$$\frac{1}{\pi} \int_{-1}^1 Y(r) \left[\frac{1}{r - s} + \varpi(s, r) \right] dr = 0 \quad (52)$$

Equilibrium condition for the flat punch and the contact stress between FGM coating and orthotropic substrate are expressed in the form

$$\frac{a}{h_1} \int_{-1}^1 Y(r) dr = 1 \quad (53)$$

$$\frac{\sigma_{1z}(x, 0)}{P/h_1} = \frac{1}{\pi} \int_{-1}^1 Y(r) \varphi(s, r) dr + \frac{(1 - e^{\Gamma h_1})}{\Gamma h_1} \frac{1}{\lambda} \quad (54)$$

where λ is the load factor and is defined as:

$$\lambda = \frac{P}{\rho_0 g h_1^2} \quad (55)$$

Through the use of Gauss–Chebyshev integration formulas, Erdogan and Gupta [41] successfully tackled two kinds of SIEs. Due to the existence of singularities of the unknown stress function at both limits of the integral, the singularity index of the integral equation (52) is +1, and the numerical solution of the singular integral equation can be found as follows

$$Y(r) = \mathcal{G}(r) (1 - r^2)^{-1/2} \quad (56)$$

Based on these observations, we have for an integral equation, equilibrium condition, and contact stress between FGM coating and orthotropic substrate, respectively as follows:

$$\sum_{i=1}^n W_i \left[\frac{1}{r_i - s_j} + \varpi(s_j, r_i) \right] \mathcal{G}(r_i) = 0, \quad (j = 1, \dots, n-1) \quad (57)$$

$$\frac{a}{h_1} \sum_{i=1}^n W_i \mathcal{G}(r_i) = 1 \quad (58)$$

$$\frac{\sigma_{1z}(x,0)}{P/h_1} = \frac{a}{h_1} \sum_{i=1}^n W_i \varphi(s_j, r_i) \mathcal{G}(r_i) + \frac{(1 - e^{\Gamma h_1})}{\Gamma h_1} \frac{1}{\lambda} \quad (59)$$

where

$$W_1 = W_n = \frac{\pi}{2(n-1)}, \quad (60a)$$

$$W_i = \frac{\pi}{n-1} \quad (i = 2, \dots, n-1)$$

$$r_i = \cos\left(\frac{i-1}{n-1}\pi\right) \quad (i = 1, \dots, n) \quad (60b)$$

$$s_j = \cos\left(\frac{2j-1}{2n-2}\pi\right) \quad (j = 1, \dots, n-1) \quad (60c)$$

The algebraic system (57–58) contains n equations for the determination of n unknowns, i.e., the contact stress $\mathcal{G}(r_i)$. The unknowns $\mathcal{G}(r_i)$, ($i = 1, \dots, n$) are determined from this algebraic system. By using (56), substituting the results into (59), the contact stress is evaluated. It should be noted that the integral equation (52) is valid if the contact stress obtained from (54) is compressive everywhere. By varying the load factor λ and evaluating the contact stress for a given value of a/h_1 , one can easily determine both the location x_{cr} at which the interface separation begins and the corresponding critical load factor λ_{cr} . This factor is related to the initial separation load P_{cr} , by

$$\lambda_{cr} = \frac{P_{cr}}{\rho_0 g h_1^2} \quad (61)$$

4. Results and discussion

In this section, the numerical results of the continuous contact problem based on the above analysis will be discussed in detail. The materials used for the orthotropic substrate are given in Table 1.

First of all, it is aimed to verify the results by comparing the present study and some special cases of the present study available in the literature. Tables 2–5 contain some numerical values related to this comparing. The FGM coating was transformed into a homogeneous isotropic layer in this context by bringing the material inhomogeneity parameter value closer to zero (by giving a value very close to zero, that is, $\beta h_1 \rightarrow 0$). The h_2 height was set to be very large ($h_2 \rightarrow \infty$), and the substrate was transformed into a half-plane. If the h_{2re} height is set too low ($h_2 \rightarrow 0$), a transition to the monolayer problem supported on a rigid foundation is provided. If the a/h_1 normalized punch length is chosen too small ($a/h_1 \rightarrow 0$), the special case that is pressed with a single load is provided. To transform the orthotropic substrate into an isotropic one, the elastic constants were determined using the Eqs. (62).

$$C_{11} = C_{33} = \frac{(1 + \kappa_2)\mu_2}{\kappa_2 - 1} \quad (62a)$$

$$C_{13} = \frac{(3 - \kappa_2)\mu_2}{\kappa_2 - 1} \quad (62b)$$

$$C_{55} = \mu_2 \quad (62c)$$

Table 1. Mechanical properties of the orthotropic materials used for the substrate in the study [42–44]

Mechanical Property	Orthotropic Material		
	Material 1 (Carbon/Epoxy)	Material 2 (Glass/Vinylester)	Material 3 (Boron/Aluminum)
E_x (GPa)	157.9	19.268	227.5
E_y (GPa)	9.584	13.210	137.9
E_z (GPa)	9.584	13.210	137.9
μ_{xy} (GPa)	5.930	4.572	55.152
μ_{xz} (GPa)	5.930	4.572	55.152
μ_{yz} (GPa)	3.227	3.7	49.244
ν_{xy}	0.32	0.31	0.24
ν_{xz}	0.32	0.31	0.24
ν_{yz}	0.49	0.35	0.40

Table 2. Comparison of the results of the present study with the results in Çakıroğlu [45], which is a special case of the present study ($(1 + \kappa_2)\mu_1 / (1 + \kappa_1)\mu_2 = 1$, $h_2 \rightarrow \infty$)

a/h_1	Present Study		Çakıroğlu [45]	
	λ_{cr}	x_{cr}/h_1	λ_{cr}	x_{cr}/h_1
0.5	158.352	3.07	158.305	3.074
1	209.868	3.51	209.738	3.506
2	325.994	4.53	325.881	4.532

Table 3. Comparison of the results of the present study with the results in Geçit [46], which is a special case of the present study ($a/h_1 \rightarrow 0$, $h_2 \rightarrow \infty$)

$\frac{(1 + \kappa_2)\mu_1}{(1 + \kappa_1)\mu_2}$	Present Study		Geçit [46]	
	λ_{cr}	x_{cr}/h_1	λ_{cr}	x_{cr}/h_1
0.1	55.0893	1.89	56.42	1.86
1	139.042	2.90	140.39	2.94

Table 4. Comparison of the results of the present study with the results in Civelek and Erdogan [47], which is a special case of the present study ($h_2 \rightarrow 0$)

a/h_1	Present Study		Civelek and Erdogan [47]	
	λ_{cr}	x_{cr}/h_1	λ_{cr}	x_{cr}/h_1
$\rightarrow 0$	44.136	1.77	44.13924	1.77055

Table 5. Comparison of the results of the present study with the results in Civelek et al. [48], which is a special case of the present study ($h_2 \rightarrow 0$)

a/h_1	Present Study	Civelek et al. [48]
	λ_{cr}	λ_{cr}
0.5	58.8567	58.88
1	92.3691	92.40
2	171.052	169.57

First, by performing the relevant transformations mentioned above, the results of the continuous contact of the homogeneous isotropic layer and the homogeneous isotropic half-plane pressed with the rigid flat punch, which is a special case of this study and previously studied by Çakıroğlu [45], were obtained. The obtained results were compared to Çakıroğlu [45], and the accuracy of the results was verified. As a result of the comparison, it was determined that the results were quite compatible with one another (see Table 2). This validated the accuracy of the current study's findings.

The second comparison is the comparison made with Geçit [46], who worked on this problem before, by transitioning to the contact problem between the homogeneous layer loaded with a single load and the homogeneous half-plane, which

is also a special case of this study. The comparison results are given in Table 3 and the compatibility of the results is once again demonstrated.

The results of another special case of this problem, the continuous contact problem of the homogeneous isotropic layer resting on a rigid foundation and loaded with a single load, were obtained using the above-mentioned transformations and compared with Civelek and Erdogan [47]. (see Table 4). This comparison also showed the accuracy of the results of the current study as a result of the agreement between the two studies.

The final comparison made to verify the accuracy of the results of the study is given in Table 5. The FGM coating was converted into a homogeneous isotropic layer, and the orthotropic substrate's height was chosen to be very small, and

the problem was matched with the continuous contact problem of the homogeneous isotropic layer resting on a rigid foundation and pressed with a rigid flat punch. The obtained results were compared with the previously discussed study by Civelek et al. [48], and it was discovered that the results were quite close to each other. Thus, verification of the present analytical procedure is provided by the comparison.

In Figs. 2–5, the changes of the normalized contact stress distributions occurring at the punch-FGM coating interface, depending on the normalized punch length, inhomogeneity parameter, material type of orthotropic substrate, and dimensionless press force are given, respectively. The normalized contact stresses, as shown in the figures, go to infinity under the sharp edges of the punch and take minimum values on the axis of symmetry or near the axis of symmetry. When the integral equation is examined, the sharp edges of the punch are expected to be singular points for stresses.

Distributions of the normalized contact stress along the punch-FGM coating interface are illustrated in Fig. 2 considering different values of the normalized punch length. As shown in the figure, as the punch length increases, the force spreads over a larger area and the normalized contact stresses at this interface decrease.

In Fig. 3, the effect of the inhomogeneity parameter βh_1 on the normalized contact stress at the punch-FGM coating interface is presented. Using Eq. (1), it is possible to conclude that changing h_1 changes the stiffness of the FGM coating in the z directions. The stiffness of the upper surface of the FGM coating is greater than that of the lower surface at positive values of h_1 . When h_1 is negative, the stiffness of the FGM coating on the lower surface is greater than on the upper surface.

For instance, if βh_1 is equal to 0.6931, the stiffness of the top surface of the FGM coating is approximately two times greater than the stiffness of the lower surface of the FGM coating. If βh_1 equals -1.6094 , the stiffness of the upper surface of

the FGM coating is approximately five times that of the lower surface of the FGM coating.

When Fig. 3 is examined, as βh_1 values increase, or in other words, as the stiffness on the upper surface of the FGM coating is increased compared to the stiffness on the lower surface, an increase is observed in the stresses at the corners of the punch, while the normalized contact stress values decrease toward the middle of the punch.

Fig. 4 depicts the normalized contact stress distributions along with the punch-FGM coating interface for different material types used for the orthotropic substrate. The direct effect of stiffness in the direction of the pressing force is striking when orthotropic materials are examined. When the material types are compared, it is clear that the stiffness increases from Material 1 to Material 3 in the direction of the pressing force (loading direction). When using an orthotropic substrate that is more rigid in the direction of loading, the normalized contact stress values under the punch increase toward the punch's center and decrease toward the punch's edges.

Fig. 2. The effect of normalized punch length on normalized contact stress distribution under punch ($\beta h_1 = 0.6931$, $\kappa_1 = 1.84$, $\mu_0 = 17$, Material 1, $h_1/h_2 = 1$)

Fig. 3. The effect of inhomogeneity parameter on normalized contact stress distribution under punch ($a/h_1 = 1$, $\kappa_1 = 1.84$, $\mu_0 = 17$, Material 2, $h_1/h_2 = 1$)

Fig. 4. The effect of orthotropic material type on normalized contact stress distribution under punch ($a/h_1 = 0.8$, $\beta h_1 = 0.4055$, $\kappa_1 = 1.84$, $\mu_0 = 17$, $h_1/h_2 = 1$)

Fig. 3. The effect of inhomogeneity parameter on normalized contact stress distribution under punch ($a/h_1 = 1$, $\kappa_1 = 1.84$, $\mu_0 = 17$, Material 2, $h_1/h_2 = 1$)

Fig. 5. The effect of dimensionless press force on normalized contact stress distribution under punch ($a/h_1 = 0.5$, $\beta h_1 = 0.6931$, Material 2, $h_1/h_2 = 1$)

The effect of the dimensionless press force on the normalized contact stress distributions under punch is depicted in Fig. 5. Note that $P/(\mu_0 h_1)$ varies between 0.01 and 0.1. As $P/(\mu_0 h_1)$ increases, the values of the normalized contact stress increase because of the increase in the net pressure on the contact surface.

In Figs. 6–9, the effects of variation of normalized punch length, inhomogeneity parameters, and orthotropic substrate material type on the normalized contact stress distributions at the interface of FGM coating and orthotropic substrate are presented. In the case of continuous contact, two regions emerge, as shown in the figures.

Fig. 6. Normalized contact stress distributions on the contact surface between the FGM coating and the orthotropic substrate for various values of normalized punch length ($\beta h_1 = 0.6931$, $\Gamma h_1 = 0.4055$, $\kappa_1 = 1.84$, $\mu_0 = 17$, Material 2, $h_1/h_2 = 1$)

Fig. 7. Normalized contact stress distributions on the contact surface between the FGM coating and the orthotropic substrate for various values of inhomogeneity parameter βh_1 ($a/h_1 = 0.4$, $\Gamma h_1 = 0.6931$, $\kappa_1 = 1.84$, $\mu_0 = 17$, Material 3, $h_1/h_2 = 1$)

Fig. 8. Normalized contact stress distributions on the contact surface between the FGM coating and the orthotropic substrate for various orthotropic materials ($a/h_1 = 1$, $\beta h_1 = 0.6931$, $\Gamma h_1 = 0.6931$, $\kappa_1 = 1.84$, $\mu_0 = 17$, $h_1/h_2 = 1$)

Fig. 9. Normalized contact stress distributions on the contact surface between the FGM coating and the orthotropic substrate for various values of inhomogeneity parameter Γh_1 ($a/h_1 = 0.5$, $\beta h_1 = 0.6931$, $\kappa_1 = 1.84$, $\mu_0 = 17$, Material 1, $h_1/h_2 = 1$)

The first is the contact zone where the pressing force has a large effect, and the second is the contact zone where the press force's effect decreases and disappears. The normalized contact stresses reach their maximum value in the contact region where the pressing force is affected, and these points form the contact stress peaks. Only the effect of the FGM

coating's body force emerges in the contact region where the effect of the pressing force disappears, and the contact stresses in this region are equivalent to the body forces. Fig. 6 shows the variation of critical load factor and initial separation distance depending on the normalized punch length. As seen in the figure, as the normalized punch length

increases, the critical load factor increases. Separations between the FGM coating and the orthotropic substrate will be more difficult as a result of this. Another conclusion from the graph is that increasing the normalized punch length shifts the initial separation distance away from the symmetry axis.

In Fig. 7, the effect of inhomogeneity parameter βh_1 change on the critical load factor and initial separation distance between FGM coating and the orthotropic substrate is illustrated. The figure shows that as the inhomogeneity parameter value increases, that is, as the rigidity of the upper surface of the FGM coating increases in comparison to the lower surface, the critical load factor value decreases while the initial separation distance increases. This result indicates that the increase in the relevant parameter value makes the separations easier, and the position of the initial separation moves away from the symmetry axis.

Fig. 8 depicts the effect of the material type used for the orthotropic substrate on the critical load factor and initial separation distance at the interface of FGM coating and orthotropic substrate. As given in Table 1, Young's modulus in $-z$ direction which is the loading direction is the highest for Material 3. The lowest critical load factor value can be seen in the figure when this orthotropic material is used. FGM coating is separated from the orthotropic substrate with less load effect, that is, more easily, as the orthotropic material becomes more rigid in the direction of loading. Another conclusion to be drawn from the graph is that using a more rigid orthotropic substrate in the loading direction brings the initial separation distance closer to the axis of symmetry.

The variations of the critical load factor and the initial separation distance for different values of the inhomogeneity parameter Γh_1 are displayed in Fig. 9. An increase in the value of this parameter with a positive sign indicates that the density of the FGM coating's upper surface is greater than that of the lower surface. Increasing the aforementioned parameter's absolute value with a negative sign represents the inverse of this situation. As the density of the FGM coating increases from the

lower surface to the upper surface, the FGM coating becomes heavier, which, as expected, makes the separation at the interface of the FGM coating and the orthotropic substrate more difficult. An interesting result that emerges from the figure is that the inhomogeneity parameter Γh_1 has no effect on the initial separation distance at the mentioned interface.

5. Conclusions

In this study, elastostatic contact mechanics analysis between the FGM coating which is pressed by a rigid flat punch, and the orthotropic substrate is performed. The formulation of the contact problem is carried out considering the general theory of elasticity. Constitutive relations of the plane elasticity are utilized under the plane strain deformation case. The governing PDEs for the elastostatic contact problem are derived. The Fourier sine and cosine transformations are used to solve these PDEs analytically. The unknown functions that appear in the displacement fields for the FGM coating and orthotropic substrate are determined by applying boundary conditions, and the mixed boundary value problem is reduced to a SIE. The SIE is solved numerically using Gauss–Chebyshev integration formulas. Some essential conclusions drawn from the study are provided as follows:

- While increasing the normalized punch length reduces the normalized contact stresses under the punch, it increases the critical load factor and initial separation distance at the interface of FGM coating and orthotropic substrate.
- The increase in the dimensionless press force causes an increase in the normalized contact stress values under the punch.
- As the upper surface of the FGM coating becomes rigid relative to the lower surface, the minimum value of the contact stress under the punch decreases. This change in FGM coating stiffness facilitates separation at the contact surface with the orthotropic substrate and shifts the first separation point away from the symmetry axis. As the value of the inhomogeneity parameter Γh_1 increases, that is,

the heavier the FGM coating, the greater press force is required for the separation at the interface with the orthotropic substrate. The change of this parameter does not affect the initial separation distance.

- For the orthotropic substrate, the value of Young's modulus in the loading direction is important. The normalized contact stress values under the punch tend to increase toward the center of the punch and decrease toward the edges of the punch as Young's modulus value in this direction increases, that is, as the stiffness of the substrate increases in the direction of loading.
- The most difficult separation that will occur at the interface of the orthotropic substrate and the FGM coating occurs when Material 1 is used.
- The present work provided the theoretical basis for the design of the FGM coating–orthotropic substrate system.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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Appendix

$$K_1 = \frac{\beta(-3 + \kappa_1) \left((-1 + e^{\Gamma h_1}) \mu_0 \rho_p + \Gamma h_1 \left((e^{z\Gamma} - e^{\Gamma h_1}) g(-1 + \kappa_1) \rho_0 - e^{z\Gamma} \mu_0 \rho_p \right) \right)}{8(-1 + e^{\beta h_1}) \Gamma^2 (-1 + \kappa_1) \mu_0} \quad (A1)$$

$$K_2 = 0 \quad (A2)$$

$$L_1 = \frac{(e^{z\Gamma} - e^{\Gamma h_1}) g(-1 + \kappa_1) \rho_0 - e^{z\Gamma} \mu_0 \rho_p}{\Gamma(1 + \kappa_1) \mu_0} \quad (A3)$$

$$L_2 = \frac{-(e^{z\Gamma} - e^{\Gamma h_1}) g(-\beta + \Gamma)(-1 + \kappa_1) \rho_0 + (\beta - e^{z\Gamma} \beta + e^{z\Gamma} \Gamma) \mu_0 \rho_p}{\beta(\beta - \Gamma) \Gamma(1 + \kappa_1) \mu_0} \quad (A4)$$