RESEARCH ARTICLE

Transfer matrix formulation for dynamic response of Timoshenko beams resting on two-parameter elastic foundation subjected to moving load

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Abstract

In this study, the dynamic response of beams resting on two-parameter elastic foundation subjected to moving load is investigated by using the transfer matrix method (TMM). The Timoshenko beam theory (TBT) which considers shear deformation and rotational inertia is used to model the beam. The two-parameter elastic foundation model is selected as Pasternak foundation that takes into account a shear layer at the end of linear springs of Winkler foundation. The TMM which uses the relation between analytically obtained state vectors of each end of the beam is applied to solve the free vibration problem. After performing the free vibration analysis, the mathematical model is simplified into an equivalent single degree of freedom (SDOF) system by using the exact mode shapes to obtain dynamic responses. The generalized displacement is calculated for each mode by using the Runge-Kutta algorithm. A numerical case study is presented for a simply-supported Timoshenko beam on the Pasternak foundation subjected to a concentrated load. The natural frequencies obtained from finite element method (FEM) results of SAP2000 are presented with the results of TMM for comparison purposes using the Winkler foundation. The effects of shear layer on the natural frequencies of the model are revealed. The mode shapes are plotted. The proposed approach for calculating dynamic responses is validated by using the results of FEM for Winkler foundation model. Then, the effects of Winkler springs and shear layer of the foundation model on the dynamic responses are presented in figures. The effects of modal damping are discussed. Finally, the critical velocities for the model are calculated for various elastic foundation scenarios and the effects of elastic foundation parameters on the dynamic response of beam model subjected to moving load with high velocity are observed.

Keywords

Dynamic response; Free vibration; Moving load; Timoshenko beam; Transfer matrix; Two-parameter elastic foundation

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1. Introduction

The investigation of dynamic response of beams on elastic foundation subjected to moving load has of great importance for the engineering structures such as railways, concrete pavement etc. In general, the open literature focusing on this topic can be

classified into two parts according to the foundation model as Winkler foundation and two-parameter elastic foundations. The Winkler foundation model which is also known as one-parameter model is the most common basic foundation representation based on distributed unconnected linear elastic

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springs that behave independently. Two-parameter foundation models were developed to reduce the error due to assumption of ignoring continuity of soil. One of the important two-parameter foundation models is the Pasternak foundation which uses a shear layer connected to the end of Winkler springs to reflect the continuity [1].

Finite element analyses were performed for beams on Winkler foundation subjected to moving load [2-5]. It is a known issue that analytical based solutions provide theoretically more accurate results when compared to results of classical FEM. Due to complicated analytical formulations of moving load problem for beam on elastic foundation models, limited studies have been performed on this topic. Chen et al. [6] applied the dynamic stiffness method (DSM) to investigate the dynamic response of infinite Timoshenko beams on viscoelastic foundation subjected to moving load. Kim [7] studied on the Euler-Bernoulli beams on Winkler foundation under moving harmonic loads by using analytical formulations and Fourier transform. Mallik et al. [8] investigated steady-state response of infinite Euler-Bernoulli beams on one-parameter and two-parameter foundations by using analytical method. Basu and Rao [9] presented the analytical solutions for Euler-Bernoulli beams on viscoelastic foundation under the effects of moving load. When it comes to analytical solutions of the moving load problem for the Timoshenko beams on two-parameter elastic foundation, the research is limited in open literature. Kargarnovin and Younesian [10] used Fourier transform and convolution integral theorem to obtain dynamic response of infinite Timoshenko beams on Pasternak foundation. Yang et al. [11] applied Galerkin's method for forced vibrations of beams resting on Pasternak foundation subjected to moving load by using TBT. Ghafarian and Ariaei [12] investigated the dynamic response of Timoshenko beams featuring bending-torsion on Pasternak foundation by using differential transformation and different time functions (DTF). Ding et al. [13] obtained nonlinear response of Timoshenko beams resting on Pasternak foundation subjected to moving harmonic force by using

Adomian polynomials. Ding et al. [14] investigated the coupled nonlinear vibration of vehicle-pavement system by considering the mathematical model as a simply-supported Timoshenko beam on Pasternak foundation subjected to moving oscillator.

It is important to choose the appropriate technique for analytical solution of the governing equations of motions of complicated systems. The transfer matrix formulation which is based on the relation between state vectors of structural members has been effectively used for free vibrations of beams on elastic foundation models [15-17].

In this study, the dynamic responses of a simply-supported Timoshenko beam resting on Pasternak foundation subjected to a concentrated moving load are calculated by using exact mode shapes obtained from analytical based formulations of TMM. Firstly, the governing equations of motion of a beam on Pasternak foundation are solved analytically according to TBT. Then, transfer matrix of the mathematical model is obtained and free vibration analysis is performed for various elastic foundation scenarios. The natural frequencies that obtained from TMM are validated by using the results of FEM for the case of Winkler foundation. The mode shapes are plotted. For the dynamic response analysis, the simply-supported Timoshenko beam on Pasternak foundation model is simplified into the equivalent SDOF system by using exact mode shapes. The generalized displacement is calculated by solving the uncoupled modal equation via Runge-Kutta algorithm. Finally, time-history responses are obtained by sum of contribution of each mode. The time-history responses obtained from proposed approach considering Winkler foundation are verified according to results of FEM. The effects of spring stiffness, shear layer of elastic foundation and modal damping on the time-history response of beam model are discussed.

2. Mathematical model and formulations

The mathematical model of simply-supported Timoshenko beam on Pasternak foundation

subjected to a concentrated moving load is presented in Fig. 1 where x and y represent the coordinate system, L is length of the beam, P is magnitude of the moving load, v is velocity of the moving load, k_w and G_p are spring stiffness and modulus of shear layer, respectively.

The following assumptions are considered in this study: i) The behavior of the beam is linear and elastic ii) the material of the beam is homogenous and isotropic, and iii) the cross-section of the beam is constant.

The governing equations of motion of a Timoshenko beam resting on Pasternak foundation can be written as [18]:

$$AG\bar{k}\left(\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{\partial \varphi(x,t)}{\partial x}\right) - \rho A \frac{\partial^2 y(x,t)}{\partial t^2} - k_w y(x,t) + G_p \frac{\partial^2 y(x,t)}{\partial x^2} = 0 \quad (1)$$

$$EI \frac{\partial^2 \varphi(x,t)}{\partial x^2} + AG\bar{k}\left(\frac{\partial y(x,t)}{\partial x} - \varphi(x,t)\right) - \rho I \frac{\partial^2 \varphi(x,t)}{\partial t^2} = 0 \quad (2)$$

where A is area of cross-section, G is shear modulus, $\bar{k}$ is shear correction factor, ρ is density, E is modulus of elasticity, I is area moment of inertia, $y(x,t)$ and $\varphi(x,t)$ are the function of transverse displacement and the function of bending rotation, respectively.

Eqs. (1)-(2) can be rewritten by considering the harmonic motion of transverse displacement and

bending rotation as $y(x,t) = W(x)e^{i\omega t}$ and $\varphi(x,t) = \theta(x)e^{i\omega t}$, where i is imaginary unit, ω and t are natural frequency and time, respectively:

$$\left(AG\bar{k} + G_p\right) \frac{d^2 W(x)}{dx^2} - AG\bar{k} \frac{d\theta(x)}{dx} + (\rho A \omega^2 - k_w) W(x) = 0 \quad (3)$$

$$EI \frac{d^2 \theta(x)}{dx^2} + AG\bar{k} \frac{dW(x)}{dx} + (\rho I \omega^2 - AG\bar{k}) \theta(x) = 0 \quad (4)$$

Eqs. (5)-(6) are obtained by substituting $W(x) = \{c_n\} e^{is_n x}$ and $\theta(x) = \{d_n\} e^{is_n x}$ into Eqs. (3)-(4).

$$\left(\rho A \omega^2 - k_w - (AG\bar{k} + G_p) s_n^2\right) \{c_n\} - AG\bar{k} i s_n \{d_n\} = 0 \quad (5)$$

$$AG\bar{k} i s_n \{c_n\} + (\rho I \omega^2 - AG\bar{k} - EI s_n^2) \{d_n\} = 0 \quad (6)$$

where s_n and n are the characteristic roots and the order of the equation system in Eqs. (3)-(4), $\{c_n\}$ and $\{d_n\}$ are integration constants for the $W(x)$ and $\theta(x)$, respectively.

Eqs. (7)-(8) represent the solution of $W(x)$ and $\theta(x)$ as

$$W(x) = \sum_{n=1}^4 \{c_n\} e^{is_n x} \quad (7)$$

$$\theta(x) = \sum_{n=1}^4 \{c_n\} J_n e^{is_n x} \quad (8)$$

where $J_n = -\frac{AG\bar{k} i s_n}{\rho I \omega^2 - AG\bar{k} - EI s_n^2}$, $n = 1, \dots, 4$

Fig.1. Mathematical model of simply-supported Timoshenko beam resting on Pasternak foundation subjected to moving load

The shear force $Q(x)$ and bending moment $M(x)$ functions of beam are presented in Eqs. (9)-(10), respectively [18].

$$Q(x) = AG\bar{k} \frac{dW(x)}{dx} - AG\bar{k}\theta(x) \quad (9)$$

$$M(x) = EI \frac{d\theta(x)}{dx} \quad (10)$$

Eqs. (11)-(12) are obtained as in explicit form by substituting Eqs. (7)-(8) into Eqs. (9)-(10).

$$Q(x) = \sum_{n=1}^4 c_n (AG\bar{k}i s_n - AG\bar{k}J_n) e^{i s_n x} \quad (11)$$

$$M(x) = \sum_{n=1}^4 c_n (EI i s_n J_n) e^{i s_n x} \quad (12)$$

3. Free vibration analysis using the TMM

The TMM is based on constructing the relation between the state vectors of each end of the beam. The state vector for the left end of the model ($\mathbf{Z}_0$) can be written as

$$\mathbf{Z}_0 = \mathbf{C}_0 \mathbf{c} \quad (13)$$

where $\mathbf{Z}_0 = \{W(x=0) \theta(x=0) Q(x=0) M(x=0)\}^T$, $\mathbf{c} = \{c_1 \ c_2 \ c_3 \ c_4\}^T$ and

$$\chi_n = AG\bar{k}i s_n - AG\bar{k}J_n, \quad \lambda_n = EI i s_n J_n,$$

$$\mathbf{C}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ J_1 & J_2 & J_3 & J_4 \\ \chi_1 & \chi_2 & \chi_3 & \chi_4 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}$$

The state vector for the right end of the model ($\mathbf{Z}_L$) can be written as

$$\mathbf{Z}_L = \mathbf{C}_L \mathbf{c} \quad (14)$$

where $\mathbf{Z}_L = \{W(x=L) \theta(x=L) Q(x=L) M(x=L)\}^T$ and

$$\mathbf{C}_L = \begin{bmatrix} e^{i s_1 L} & e^{i s_2 L} & e^{i s_3 L} & e^{i s_4 L} \\ e^{i s_1 L} J_1 & e^{i s_2 L} J_2 & e^{i s_3 L} J_3 & e^{i s_4 L} J_4 \\ e^{i s_1 L} \chi_1 & e^{i s_2 L} \chi_2 & e^{i s_3 L} \chi_3 & e^{i s_4 L} \chi_4 \\ e^{i s_1 L} \lambda_1 & e^{i s_2 L} \lambda_2 & e^{i s_3 L} \lambda_3 & e^{i s_4 L} \lambda_4 \end{bmatrix}$$

The relation between the $\mathbf{Z}_0$ and $\mathbf{Z}_L$ are obtained as:

$$\mathbf{Z}_L = \mathbf{TM} \mathbf{Z}_0 \quad (15)$$

where $\mathbf{TM}$ represents the transfer matrix and $\mathbf{TM} = \mathbf{C}_L \mathbf{C}_0^{-1}$

A reduction procedure is applied to $\mathbf{TM}$ according to boundary conditions to eliminate inactive rows and columns. For simple-simple boundary condition, the reduced transfer matrix is presented in Eq. (16).

$$\mathbf{TM}_R = \begin{bmatrix} \mathbf{TM}(1,2) & \mathbf{TM}(1,3) \\ \mathbf{TM}(4,2) & \mathbf{TM}(4,3) \end{bmatrix} \quad (16)$$

where $\mathbf{TM}_R$ is the reduced transfer matrix of the vibrating system.

The natural frequencies of the system can be calculated by using Eq. (17).

$$\det(\mathbf{TM}_R) = 0 \quad (17)$$

A root finding algorithm based on bisection method that uses a trial and error on interpolation can be used for calculation of roots from Eq. (17). If there is change of sign between two intervals, there must be root. After some iterations, the natural frequencies are calculated with desired accuracy. The mode shapes can be obtained easily after calculation of natural frequencies by equating a non-zero state vector member to unity. Then, mode shapes are plotted for each mode by usual manner.

4. Dynamic response analysis

The series expansion of displacement fields in terms of generalized displacement and shape functions can be written as follows [19]:

$$y(x,t) = \sum_{j=1}^N W(x) q_j(t) \quad (18)$$

$$\varphi(x,t) = \sum_{j=1}^N \theta(x) q_j(t) \quad (19)$$

where $q_j(t)$ are the sets of generalized displacements, j is the number of mode and N is the total number of modes considered.

For the time-history responses, the generalized damping can be introduced in the formulations. The modal equation in Eq. (20) can be written by assuming the classical damping [19].

$$\ddot{q}_j(t) + 2\xi\omega_j\dot{q}_j(t) + \omega_j^2 q_j(t) = \frac{\bar{P}_j}{\bar{m}_j} \quad (20)$$

where ξ is damping ratio, $\bar{P}_j$ and $\bar{m}_j$ are generalized force and mass of j th mode of equivalent SDOF system. For the undamped time-history analysis, ξ is equated to zero.

The generalized force according to a concentrated moving load is presented in Eq. (21) [20].

$$\bar{P}_j = \int_0^L P\delta(x - vt)W_j(x)dx = PW_j(S(t)) \quad (21)$$

where δ and $S(t)$ are Dirac delta and current position of the load, respectively.

The generalized mass components of beam model are presented in Eqs. (22)-(23) for bending deformation and shearing deformation, respectively [19]:

$$\bar{m}_{bj} = \int_0^L \rho I \theta_j^2(x) dx \quad (22)$$

$$\bar{m}_{sj} = \int_0^L \rho A W_j^2(x) dx \quad (23)$$

where $\bar{m}_{bj}$ and $\bar{m}_{sj}$ are generalized mass components of Timoshenko beam resting on Pasternak foundation due to bending deformation and shearing deformation for the j th mode, respectively.

The generalized mass of the vibrating system is obtained as Eq. (24)

$$\bar{m}_j = \bar{m}_{bj} + \bar{m}_{sj} \quad (24)$$

Eq. (20) can be solved by ode45 solver of MATLAB which is based on Runge-Kutta algorithm. The ode45 is fast and straightforward for

this problem. Once the $q(t)$ is calculated, the time-history response of any point of the model can be obtained by using Eqs. (18)-(19). The total time-history responses are calculated by sum of contributions of desired number of modes.

5. Numerical case study

The following material and geometric properties are considered for the numerical case study: $L = 8$ m, cross-sectional height (h) = 0.2 m, cross-sectional width (b) = 1 m, $A = 0.2$ m², $I = 6.67 \times 10^{-4}$ m⁴, $\rho = 2500$ kg/m³, $E = 30000$ MPa, $G = 12500$ MPa, $\bar{k} = 0.833$. The magnitude of concentrated moving load P is considered as 9.807 kN.

The foundation parameters k_w and G_p are calculated by using Eqs. (25)-(26), respectively. [21].

$$k_w = \frac{E_s(1 - \nu_s)}{2(1 + \nu_s)(1 - 2\nu_s)} \quad (25)$$

$$G_p = \frac{E_s b^2}{3(1 + \nu_s)} \quad (26)$$

where E_s and ν_s are elastic modulus and Poisson's ratio of elastic foundation and ν_s is assumed as 0.3 for the numerical case study.

For validation purposes, the first four natural frequencies of the simply-supported Timoshenko beam on Winkler foundation are presented in Table 1 for TMM and FEM comparatively. It is seen from Table 1 that the agreement between TMM and FEM is perfect for various E_s values. The accuracy of TMM is also verified for no foundation case implemented by $E_s = 0$. Table 1 shows that increasing E_s significantly increases natural frequencies of simply-supported Timoshenko beam on Winkler foundation. According to Table 1, for the fundamental mode, the natural frequency is increased by 107% and 174% when E_s is changed from 10 MPa to 45 MPa and 80 MPa, respectively. It is observed from Table 1 that effects of spring stiffness on the natural frequencies are decreased for higher modes.

Table 1. First four natural frequencies (rad/s) of simply-supported Timoshenko beam on Winkler foundation ($G_p = 0$)

Method	E_s (MPa)	1st mode	2nd mode	3rd mode	4th mode
TMM	0	30.818	122.903	275.180	485.918
FEM		30.820	123.003	275.708	487.493
TMM	10	120.037	168.949	298.546	499.477
FEM		120.047	169.089	299.126	501.109
TMM	45	248.023	274.916	368.848	544.280
FEM		248.044	275.146	369.582	546.099
TMM	80	329.579	350.163	427.748	585.666
FEM		329.606	350.457	428.609	587.654

Table 2 represents the first four natural frequencies of simply-supported Timoshenko beam on Pasternak foundation to show the effects of shear layer in addition to spring stiffness of Winkler foundation. Table 2 shows that increasing E_s also increases the natural frequencies significantly for Pasternak foundation. However, it is important to reveal the effects of shear layer on the natural frequencies of the model. Therefore, Fig. 2 which is plotted using the data in Tables 1-2, is presented to show the increment of first four natural frequencies by considering G_p in addition to k_w . According to Fig.2, the effects of Pasternak foundation become more observable for higher E_s values especially for higher modes. Although, the difference between the effects of lower and higher E_s values is not significant for the first mode. Fig.2 implies that effects of shear layer of Pasternak foundation are increased for higher modes up to third mode and second mode for higher and lower E_s values. The first four mode shapes are presented in Fig.3 for simply-supported Timoshenko beam resting on Pasternak foundation.

The proposed approach for calculation of time-history responses is verified using the results of FEM. For the verification purpose, the transverse displacement time-history of mid-span (y_{mid}) is calculated and presented in Figs.4a-b ignoring elastic foundation and considering Winkler foundation, respectively for $v = 50$ km/h. Figs.5a-b show the time-history response of y_{mid} taking $v = 100$ km/h for no foundation and Winkler foundation cases, respectively.

According to Figs.4-5, the accuracy of proposed approach on calculation of dynamic response of simply-supported Timoshenko beams resting on elastic foundation is very good. It is seen from Figs.4a-5a that increasing moving load velocity from 50 km/h to 100 km/h results in 34% higher peak of y_{mid} . However, the peak of y_{mid} is only 2.5% augmented by increasing the moving load velocity from 50 km/h to 100 km/h for the case of Winkler foundation according to Figs.4b-5b. Figs.4-5 show that elastic foundation dramatically limits the dynamic response when compared to without elastic foundation case for both v values considered. It should be noted that FEM results in Figs.4-5 are based on linear direct integration with Newmark-Beta method on SAP2000.

Figs.6a-b represent the effects of Winkler springs on the dynamic response of simply-supported Timoshenko beam on elastic foundation model for $v = 50$ km/h and $v = 100$ km/h, respectively. It is seen from Figs.6a-b that increase of E_s from 10 MPa to 45 MPa and 80 MPa reduces the peak of the y_{mid} by 72% and 82%, respectively. This behavior shows that the effects of spring stiffness values of elastic foundation on the dynamic response of beams subjected to moving loads would become limited when the strong foundation starts to behave like rigid.

Figs.7a-b-c are presented to show the effects of shear layer of elastic foundation on the dynamic response of simply-supported Timoshenko beam subjected to moving load considering $E_s = 10$ MPa, $E_s = 40$ MPa and $E_s = 80$ MPa, respectively.

Table 2. First four natural frequencies (rad/s) of simply-supported Timoshenko beam on Pasternak foundation

E_s (MPa)	1st mode	2nd mode	3rd mode	4th mode
10	123.286	178.050	310.189	511.895
45	255.095	299.650	409.810	593.932
80	339.039	384.549	489.562	665.939

Fig.2. Increment (%) of the first four natural frequencies by considering G_p in addition to k_w

Fig.3. First four mode shapes of Timoshenko beam on Pasternak foundation ($E_s = 45$ MPa)

Fig.4. Displacement time-history of y_{mid} comparatively for FEM and TMM: a) ignoring elastic foundation ($v = 50$ km/h), b) considering Winkler foundation ($v = 50$ km/h, $E_s = 10$ MPa)

Fig.5. Displacement time-history of y_{mid} comparatively for FEM and TMM: a) ignoring elastic foundation ($v = 100$ km/h), b) considering Winkler foundation ($v = 100$ km/h, $E_s = 10$ MPa)

Fig.6. Effects of k_w on the displacement time-history of y_{mid} ignoring G_p and considering a) $v = 50$ km/h, b) $v = 100$ km/h

Fig.7. Displacement time-history of y_{mid} considering Pasternak foundation and Winkler foundation comparatively a) $E_s = 10$ MPa, b) $E_s = 45$ MPa, c) $E_s = 80$ MPa ($v = 100$ km/h)

According to Figs.7a-c, the dynamic response of the beam for y_{mid} is lower for Pasternak foundation when compared to Winkler foundation considering E_s as 10 MPa, 45 MPa and 80 MPa, respectively. Fig.7a shows that the peak of y_{mid} is decreased by 5.3% considering Pasternak foundation instead of Winkler foundation for $E_s = 10$ MPa. According to Figs. 7b-c, taking E_s as 45 MPa and 80 MPa, there is reduction of peak of y_{mid} by 8.6% and 15.81%, respectively.

Fig.8a represents the effects of modal damping on the time-history responses of simply-supported Timoshenko beam on elastic foundation considering ξ as 0, 0.01, 0.03 and 0.05, respectively. According to Fig.8b which is the sub-figure showing the peak of Fig.8a, the increasing modal damping slightly decreases the peak of y_{mid}

however, this effect is not significant for the model considered in this study.

The vibrations induced by moving loads may become larger when the velocity reaches a specific value referred as critical/resonant velocity (v_{cr}) [22]. In the case of critical velocity, the free vibration part of the response after load leaves the beam can be important as they can exceed the original vibrations [23]. For the finite beams, the v_{cr} is defined as the velocity with the frequency of excitation be equal to a natural frequency of the beam that can excite the beam into resonance. The formulation for the lowest resonant velocity is presented in Eq. (27) where ω_1 represents the first natural frequency [24].

$$v_{cr} = \frac{L\omega_1}{\pi} \quad (27)$$

Fig.8. a) Effects of damping on the displacement time-history of y_{mid} considering $v = 100$ km/h ($E_s = 45$ MPa), b) sub-figure representing the zoom-in for the peak of Fig.8a.

Using the data in Tables 1-2 and Eq. (27), the lowest critical velocities for different elastic foundation scenarios as well as without elastic foundation case are presented in Table 3. According to Table 3, considering Winkler foundation increases the v_{cr} by 290% even for the lowest E_s considered in the numerical case study. Table 3 implies that considering shear layer of elastic foundation slightly increases v_{cr} . Table 3 shows that the v_{cr} values are extremely high for the model considered in this research. However, Fig. 9 is presented to show the displacement time-history of y_{mid} based on taking v as 282.52 km/h which is the v_{cr} for without elastic foundation case and can be considered as a reasonably high velocity.

According to Fig.9, the peak of y_{mid} is decreased by 75% and 85% when E_s is changed from 10 MPa to 45 MPa and 80 MPa, respectively. Fig.9 shows that the elastic foundation significantly limits the dynamic response of beams subjected to moving loads with high velocities as well as low and moderate velocities. However, it is seen from Fig.9 that significant free vibrations are observed for weak elastic foundation conditions considering $v = 282.52$ km/h.

It should be noted that the moving load velocities considered in the numerical case study are usual values for vehicles moving on the concrete pavements in real applications. However, these velocities are very low when compared to

critical velocities which are extremely high and cannot be reached in real applications.

6. Conclusions

The dynamic response analysis of Timoshenko beams resting on two-parameter elastic foundation subjected to moving load is performed using exact mode shapes obtained from transfer matrix formulations. The free vibrations and time-history responses are investigated for both Winkler and Pasternak foundation models. The natural frequencies and time-history responses of TMM are validated by using the results of FEM. The time-history responses are obtained by solving the modal equation that obtained from equivalent SDOF system which has generalized properties based on exact mode shapes. The effects of spring stiffness and shear layer of the elastic foundation on the dynamic response of the simply-supported Timoshenko beam model are revealed for various elastic foundation parameters. The proposed approach for the dynamic response analysis based on TMM can be extended to stepped beams, cracked beams and beams having rigid attachments and resting on elastic foundation as the size of the global transfer matrix of whole vibrating system is irrespective to number of members. The computation time for the root finding algorithm of TMM would not be significantly affected for more complicated structures.

Table 3. The lowest critical velocity values (km/h) for different elastic foundation scenarios

E_s (MPa)	10	45	80
Winkler	1100.42	2273.71	3021.36
Pasternak	1130.20	2338.54	3108.08
No foundation	282.52

Fig.9. Displacement time-history of y_{mid} considering Pasternak foundation ($v = 282.52$ km/h, $\xi=0$)

The following concluding remarks are based on the results of numerical case study:

- The natural frequencies are significantly increased by considering elastic foundation especially for lower modes.
- The beam on Pasternak foundation model provides higher natural frequencies when compared to beam on Winkler foundation model.
- The increment of the fundamental natural frequency by considering shear layer of foundation is not very sensitive to elastic modulus of soil.
- The peak of dynamic response is significantly lower for the case of Winkler foundation in comparison with no foundation case.
- Increasing elastic modulus of soil decreases the dynamic response of model by means of stiffness of Winkler springs and modulus of shear layer.
- The peak of dynamic response is slightly lower for Pasternak foundation in comparison with Winkler foundation.

- The effects of modal damping on the dynamic response of beam on Pasternak foundation are very limited.
- The critical velocities are extremely high for the beam on Winkler foundation model as well as beam on Pasternak foundation model.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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