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MEET REPRESENTATIONS IN UPPER CONTINUOUS
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1. INTRODUCTION

The study of representations of elements of the lattice of congruence relations of an algebra as intersections of other elements of this lattice is of interest for two reasons. In the first place, such a study offers an extension of the classical representation theory for the ideals of a noetherian ring. Secondly, there is a natural correspondence between the subdirect product representations of an algebra and the meet representations of the zero element of its lattice of congruence relations. The lattice of congruence relations of any algebra is compactly generated. (An element \( x \) of a lattice is compact if whenever \( x \leq \bigcup S \) there is a finite set \( F \subseteq S \) such that \( x \leq \bigcup F \). A complete lattice is said to be compactly generated if each element is the union of a set of compact elements.)

In order to study the subdirect product representations of an algebra it often suffices to investigate meet representations in compactly generated lattices. For example, the fact that any algebra has a subdirect product representation in terms of subdirectly irreducible factor algebras may be regarded as a consequence of the fact that each element of a compactly generated lattice has a meet representation in terms of completely meet irreducible elements. (An element \( x \) of a lattice is meet irreducible if \( x = s_1 \cap s_2 \) implies that either \( x = s_1 \) or \( x = s_2 \). It is completely meet irreducible if \( x = \bigcap S \) implies \( x \in S \).) A meet representation of an element, say \( x = \bigcap S \), is
irredundant if \( x < \bigwedge (S - s) \) for each \( s \in S \). Although any element of a compactly generated lattice has meet representations in terms of completely meet irreducible elements, it is not true that it must possess an irredundant one. For example, Dilworth and Crawley have characterized the compactly generated modular lattices in which each element has such a representation as those which are atomic (2). (A lattice is atomic if whenever \( x < y \) there exists \( x' \) such that \( x < x' \leq y \). Here, as throughout this paper, \( x < x' \) signifies that \( x \) is covered by \( x' \).) Our goal is to find compactly generated modular lattices in which each element has an irredundant representation in terms of meet irreducible elements (instead of completely meet irreducible elements.) Actually, for our purposes it will suffice to consider upper continuous modular lattices. (A complete lattice is upper continuous if for each element \( x \) and each chain of elements \( S \) the relation \( x \wedge \bigcup S = \bigcup_{s \in S} (x \wedge s) \) holds.) Any compactly generated lattice is upper continuous.

On one hand we will see that there is an analogy between the case dealing with meet irreducible elements and that involving completely meet irreducible elements. This may be seen by comparing Theorem 7.4 to Theorem 7.5, the latter being a restatement of the above mentioned result of Dilworth and Crawley. In a somewhat different vein we will show that a certain type of lattice homomorphism is useful in studying irredundant representations and the role played by covers in such representations. Adopting the terminology used in (3)
we say that a lattice homomorphism \( \phi : L \to L' \) is a complete join homomorphism if \( L \) and \( L' \) are complete and for each \( S \subseteq L \) we have \( \phi(\bigcup S) = \bigcup_{s \in S} \phi(s) \). Our principal results concerning such maps are the following. If \( L \) is an upper continuous modular lattice then there is an associated upper continuous modular lattice \( L^* \) which is the "largest" homomorphic image of \( L \) (under a complete join epimorphism) possessing no covers. (See Theorem 5.3) Each element of \( L \) has an irredundant representation in terms of meet irreducibles if and only if each element of \( L^* \) has such a representation. In particular, if \( L^* \) consists of one element then each element of \( L \) has an irredundant representation in terms of meet irreducible elements. The class of upper continuous modular lattices for which \( L^* \) consists of one element is large enough to include both the atomic ones, in which each element has an irredundant representation in terms of completely meet irreducible elements, and those satisfying the ascending chain condition, in which each element has an irredundant representation as the meet of a finite number of meet irreducible elements.
2. TORSION ELEMENTS, TORSION FREE ELEMENTS, AND COVERING CONDITIONS

We begin by generalizing two concepts from group theory.

Definition. An element $t$ of a lattice $L$ is a torsion element if for each $a < t$ there is a $b$ in $L$ such that $t \succ b \succ a$.

The element $0$ is always a torsion element. If $1$ is a torsion element, we will call $L$ a torsion lattice.

Definition. An element of $L$ is torsion free if no element of $L$ covers it.

Here $1$ is always torsion free.

Let $H$ be a subgroup of an abelian group $G$, and let $L(G)$ denote the lattice of subgroups of $G$. $H$ is a torsion element of $L(G)$ if and only if $H$ is a torsion group. $H$ is torsion free in $L(G)$ if and only if $G/H$ is a torsion free group. In fact the duals of the above definitions have counterparts in the theory of abelian groups. $H$ is dual torsion free in $L(G)$ if and only if $H$ is a divisible group. $H$ is a dual torsion element of $L(G)$ if and only if $G/H$ is a reduced group.

In general it is not the case that a lattice must possess maximal torsion elements or minimal torsion free elements. Examples are easily constructed by violating the following condition.

Definition. A lattice satisfies the lower covering condition if $a \succ a \cap b$ implies $a \cup b \succ b$.

The dual of this condition is termed the upper covering
condition. If $a$ and $b$ are elements of a modular lattice then the intervals $[a \land b, a]$ and $[b, a \lor b]$ are isomorphic. Thus a modular lattice satisfies both covering conditions.

Our first theorem is a direct generalization of a classical theorem of group theory.

Theorem 2.1 In a complete lattice which satisfies the lower covering condition there is an element which is both the largest torsion element and the smallest torsion free element.

Before proving this we will develop some lemmas. In the remainder of this section it is assumed that the lattice under discussion is complete and satisfies the lower covering condition.

Lemma 2.1 If $x$ is a torsion element of $L$, $z \geq x$, $z \geq y$, and $[y, z]$ has no atoms, then $y \geq x$.

Proof. Assume $x \land y < x$. Since $x$ is a torsion element there is a $c$ such that $x \geq c > x \land y$. Now $x \land y \leq c \land x \land y$. So that either $c \land y = c$ or $c \land y = x \land y$. If $c \land y = c$ we would have $c = c \land x \land y \land x < c$. Hence $c \land y = x \land y$ so that $c > c \land y$. By the lower covering condition, $c \lor y > y$. But $c \lor y \leq x \lor y \lor z$, contrary to the hypothesis that $[y, z]$ has no atoms. Thus $x \land y = x$ and $y \geq x$.

Lemma 2.1 has the following three consequences.

Lemma 2.2 If $x$ is a torsion element and $y$ is torsion free then $x \lor y$.

Proof: Let $z = 1$. Then $x, y$, and $z$ satisfy the hypo-
thesis of Lemma 2.1.

**Lemma 2.3.** If $T$ is a collection of torsion elements then $\bigcup T$ is a torsion element.

**Proof:** Suppose $y \leq \bigcup T$ and $[y, \bigcup T]$ has no atoms. If $t \in T$, $t \leq \bigcup T$ and Lemma 2.1 implies $y \geq t$. Then $y \geq \bigcup T$. Hence $y = \bigcup T$ and $\bigcup T$ is a torsion element.

**Lemma 2.4.** If $t$ is a torsion element of $L$ and $t' \succ t$ then $t'$ is also a torsion element.

**Proof:** Suppose $y \leq t'$ and $[y, t']$ has no atoms. By Lemma 2.1, $y \geq t$. But $y \succ t$ since $t' \succ t$. Hence $y = t'$ and $t'$ is a torsion element.

We next investigate the torsion free elements.

**Lemma 2.5.** If $F$ is a collection of torsion free elements then $\bigcap F$ is also torsion free.

**Proof.** Suppose $c \succ \bigcap F$. Let $f$ be any element of $F$. If $c \cap f = \bigcap F$ then $c > c \cap f$ and the lower covering condition implies $f \cup c > f$, contrary to the hypothesis. Hence $c \cap f = c$ and $c \leq f$. Thus $c \leq \bigcap F < c$, a contradiction.

We now prove Theorem 2.1. Let $T$ denote the collection of all torsion elements of $L$ and let $t = \bigcup T$. Lemma 2.3 implies $t$ is the largest torsion element of $L$. Similarly, if $F$ denotes the collection of all torsion free elements and $f = \bigcap F$ then, by the preceding lemma, $f$ is the smallest torsion free element of $L$. Lemma 2.2 implies $t \leq f$. If $t < f$, $t$ cannot be torsion free and hence there exists $t'$ such that $t' \succ t$. By Lemma 2.4, $t'$ is a torsion element, contrary to the maximality
of $t$. Hence $t=f$.

Note that the dual of Theorem 2.1 states that in a lattice which satisfies the upper covering condition there is an element which is both the largest dual torsion free (divisible) element and the smallest dual torsion (reduced) element.
3. ATOMIC LATTICES AND UPPER CONTINUITY

Suppose that $L$ is the lattice of subgroups of an abelian torsion group. If $x \in L$ then, since every subgroup of a torsion group is a torsion group, the sublattice $[0, x]$ is a torsion lattice and therefore $x$ is a torsion element of $L$. Hence each element of $L$ is a torsion element. A lattice has this property if, and only if, it is atomic. Any atomic lattice is a torsion lattice but the converse is not true. The first theorem of this section will give conditions which are sufficient to imply that a torsion lattice is atomic. Before stating these conditions it will be convenient to state a property of upper continuous lattices which we will use frequently.

Lemma 3.1. Suppose that in an upper continuous lattice there are elements $a$, $b$, and $c$ such that $b \wedge c = a$. Then there is an element $m \geq c$ which is maximal with respect to the property $b \wedge m = a$.

Proof. Let $P = \{ p \mid p \geq c, b \wedge p = a \}$. $P$ is not empty since it contains $c$. Suppose that $S \subseteq P$ and $S$ is a chain. Then $\cup S \geq c$ and, by upper continuity, $b \wedge \bigcup_{s \in S} (b \wedge s) = \bigcup_{s \in S} a = a$. The union of a chain of elements of $P$ is thus also an element of $P$. $P$ therefore contains maximal elements.

Theorem 3.1. An upper continuous torsion lattice which satisfies the upper covering condition is atomic.

Proof: Suppose that $x < y$. Since $y \wedge x = x$ the preceding
lemma implies the existence of an element \( z \) which is maximal with respect to the property \( y \cap z = x \). If \( z = 1 \) we have \( x = z \cap y = 1 \cap y = y \), contrary to \( x < y \). Therefore \( z < 1 \). Since \( 1 \) is a torsion element there is an element \( z' \) such that \( z' > z \). Now \( z < z \cup (y \cap z') \leq z' \). If \( z = z \cup (y \cap z') \) then \( y \cap z' \leq z \) and \( x = y \cap z \geq y \cap z' \geq y \cap z \) or \( y \cap z' = x \), contrary to the maximality of \( z \). Thus \( z \cup (y \cap z') = z' > z \). By the upper covering condition, \( y \cap z' > y \cap z' \cup z = y \cap z = x \). That is, \( y \geq y \cap z' > x \). Hence the lattice is atomic.

Suppose \( L \) is a complete lattice satisfying the lower covering condition. Then if \( x \) is any element of \( L \) the lattice \([x, 1]\) also satisfies the lower covering condition. Theorem 2.1 then implies the existence of an element \( t(x) \) which is both the largest torsion element and the smallest torsion free element of the lattice \([x, 1]\). Then \( t(x) \) must also be the smallest torsion free element of \( L \) above \( x \). Since \( t(x) \) is a torsion element of \([x, 1]\) the lattice \([x, t(x)]\) is a torsion lattice. Moreover, we have the following as an immediate consequence of the preceding theorem.

Lemma 3.2. If \( x \) is an element of an upper continuous lattice satisfying both covering conditions then the sublattice \([x, t(x)]\) is atomic.

In such lattices the torsion free elements have a regularity property.

Lemma 3.3. If \( x \) and \( y \) are elements of an upper continuous lattice satisfying both covering conditions then
Proof. Since $t(x \cap y)$ is the smallest torsion free element above $x \cap y$, Lemma 2.5 implies $t(x \cap y) \leq t(x) \cap t(y)$. We will complete the proof by showing that the opposite inequality also holds. Let $P = \{ p | x \leq p \leq t(x), p \cap y \leq t(x \cap y) \}$. $P$ is not empty since $x \in P$. Suppose that $S \subseteq P$ and $S$ is a chain. Then $x \leq \bigcup S \leq t(x)$. By upper continuity, $(\bigcup S) \cap y = \bigcup_{s \in S} (s \cap y) \leq t(x \cap y)$. Thus the union of a chain of elements of $P$ is again in $P$. Therefore $P$ contains maximal elements. Let $x'$ be such an element. Suppose $x' \prec t(x)$. Lemma 3.2 implies there is an element $c$ such that $x' \prec c \leq t(x)$. Now $c \cap y \leq t(x \cap y)$ would contradict the maximality of $x'$. In particular, since $x' \cap y \leq c \cap y$ and $x' \cap y \leq t(x \cap y)$ we have $x' \cap y \leq c \cap y$. Then $x' \prec x' \cup (c \cap y) \leq c$, and consequently $x' \prec x' \cup (c \cap y) = c$. By the upper covering condition, $c \cap y > x' \cap (c \cap y) = x' \cap y$. Now $x' \cap y \leq (c \cap y) \cap t(x \cap y) \leq c \cap y$ and since $c \cap y \not\leq t(x \cap y)$ we have $x' \cap y = (c \cap y) \cap t(x \cap y) < c \cap y$. By the lower covering condition, $(c \cap y) \cup t(x \cap y) > t(x \cap y)$. But this contradicts the fact that $t(x \cap y)$ is torsion free. Thus $x' = t(x)$ and $t(x) \cap y \leq t(x \cap y)$.

Similarly, there is an element $y'$ which is maximal with respect to the properties $y \leq y' \leq t(y)$ and $t(x) \cap y' \leq t(x \cap y)$. Again, it must be the case that $y' = t(y)$. Therefore we have $t(x) \cap t(y) \leq t(x \cap y)$.

Our next theorem is an example of how conditions on the torsion free elements have implications in a lattice.

Theorem 3.2. Let $L$ be an upper continuous lattice
which satisfies both covering conditions. If the torsion
free elements of L form a chain, L is modular.

Proof. A lattice is not modular if, and only if, it
contains elements a, b, and c such that a < b, a ∩ c = b ∩ c,
and a ∪ c = b ∪ c. We will first show that if L contains such
elements it also contains elements a', b', and c' such that
the following hold.

(i) a' < b'.

(ii) a' ∩ c' = b' ∩ c'.

(iii) a' ∪ c' = b' ∪ c'.

(iv) a' ∩ c' < a' and [a' ∩ c', a'] has no atoms.

(v) a' ∩ c' < c' and [a' ∩ c', c'] has no atoms.

Let P be the partially ordered set consisting of all those elements p such that a ∩ c ≤ p ∩ a ∪ c, p = (b ∪ p) ∩ (c ∪ p), and (a ∪ p) ∩ b = a. P is not empty since a ∩ c ∈ P. Suppose that S ⊂ P and S is a chain. Let s' = ∪ S. Clearly
a ∩ c ≤ s' ≤ a ∪ c. By upper continuity,

(b ∪ s') ∩ (c ∪ s') = (b ∪ S1 ∈ S1) ∩ (c ∪ S2 ∈ S2)

= [∪ S1 ∈ S (b ∪ S1)] ∩ [∪ S2 ∈ S (c ∪ S2)]

= ∪ S1 ∈ S [(b ∪ S1) ∩ ∪ S2 ∈ S (c ∪ S2)]

= ∪ S1, S2 ∈ S [(b ∪ S1) ∩ (c ∪ S2)].
Consider the term \((b \cup s_1) \cap (c \cup s_2)\). If \(s_1 \leq s_2\), \((b \cup s_1) \cap (c \cup s_2)\) \(\leq (b \cup s_2) \cap (c \cup s_2) = s_2 = s_1 \cup s_2\). Similarly, if \(s_2 \leq s_1\), \((b \cup s_1) \cap (c \cup s_2) \leq s_1 = s_1 \cup s_2\). Thus \(s' \leq (b \cup s') \cap (c \cup s')\)

\[
\leq \bigcup_{s_1, s_2 \in S} (s_1 \cup s_2) = \bigcup_{s \in S} s = s'\] and \(s' = (b \cup s') \cap (c \cup s')\). Also, \((a \cup s') \cap b = (a \cap \bigcup_{s \in S} s) \cap b = \bigcup_{s \in S} ([a \cup s] \cap b) = \bigcup_{s \in S} a = a\). Therefore the union of a chain of elements of \(P\) is again in \(P\) and \(P\) contains maximal elements. Let \(m\) be such an element and let \(a' = a \cup m\), \(b' = b \cup m\), and \(c' = c \cup m\).

Clearly \(a' \neq b'\). If \(a' = b'\), \(a \cup m = b \cup m\) and \(a = (a \cup m) \cap b = (b \cup m) \cap b = b\), contrary to \(a < b\). Therefore \(a' < b'\). Since \(m \leq (a \cup m) \cap (c \cup m) \leq (b \cup m) \cap (c \cup m) = m\) we have \(m = (a \cup m) \cap (c \cup m) = (b \cup m) \cap (c \cup m)\), or \(a' \cap c = b' \cap c = m\). Also, since \(m \leq a \cup c = b \cup c\), \((a \cup m) \cup (c \cup m) = a \cup c \cup m = a \cup c = b \cup c = b \cup c = b \cup m = (b \cup m) \cup (c \cup m)\), or \(a' \cup c = b' \cup c\). If \(a' = a' \cap c'\) then \(c' = a' \cap c' = b' \cap c'\) and \(b' = b' \cap c' = a' \cap c' = a\), contrary to \(a' < b'\). Hence \(a' \cap c' < a'\).

Similarly, if \(a' \cap c' = c'\) we have \(a' = a' \cup c' = b' \cup c'\) so that \(a' \geq b'\), again contradicting \(a' < b'\). Thus \(a' \cap c' < c'\). It only remains to show that \([a' \cap c', a']\) and \([a' \cap c', c']\) contain no atoms.

Suppose there is an \(m'\) in \(L\) such that \(a' \geq m' > a' \cap c'\). We will show that \(m' \in P\). Since \(m' > a' \cap c' = m\), this will contradict our choice of \(m\). Clearly \(a' \cap c' \leq a \cup c\). Also, since \(m \leq a \cup c = a \cup m\), we have \(a \cup m \leq a \cup m = a \cup c\). Then \((a \cup m') \cap b = (a \cup m) \cap b = a\). We now show that \((b \cup m') \cap (c \cup m') = m'\). Since \(a \cup m' = a \cup m\), \(b \cup m' = b \cup a \cup m' = b \cup a \cup m = b \cup m\). Consequently, \((b \cup m') \cap (c \cup m') = (b \cup m) \cap (c \cup m')\). Now \(m' \leq m' \cap (c \cup m) \leq m'\). If
13

m' ∩ (c ∪ m) = m' we would have m' ≤ c ∪ m and therefore
m' ≤ (c ∪ m) ∩ (b ∪ m) = m ∩ m', a contradiction. Thus
m = m' ∩ (c ∪ m) ≤ m'. By the lower covering condition,
c ∪ m < (c ∪ m) ∩ m' = c ∪ m'. Now c ∪ m ≤ (c ∪ m) ∩ [(b ∪ m) ∩ (c ∪ m')] ≤ c ∪ m'. If c ∪ m = (c ∪ m) ∩ [(b ∪ m) ∩ (c ∪ m')] then
(b ∪ m) ∩ (c ∪ m') ≤ c ∪ m. But this implies that m' ≤ m
= (b ∪ m) ∩ (c ∪ m') ≤ (b ∪ m) ∩ (c ∪ m') = (b ∪ m') ∩ (c ∪ m') ≤ m', a contradiction. Therefore c ∪ m < (c ∪ m) ∩ [(b ∪ m) ∩ (c ∪ m')]
= c ∪ m'. By the lower covering condition, (b ∪ m) ∩ (c ∪ m')
≤ (c ∪ m) ∩ [(b ∪ m) ∩ (c ∪ m')] = (b ∪ m) ∩ (c ∪ m) = m. That is,
(b ∪ m) ∩ (c ∪ m') = (b ∪ m) ∩ (c ∪ m') ≤ m. Since (b ∪ m') ∩ (c ∪ m')
≥ m' ≥ m we have m' = (b ∪ m') ∩ (c ∪ m'). Thus m' ∈ P and we
have the desired contradiction.

Similarly, if we assume the existence of an element m'
such that c' ≥ m' > a' ∩ c', we can show that m' ∈ P, contradicting
the maximality of m. Since m < m' ≤ c ∪ m, c ∪ m ≤ c ∪ m ≤ c ∪ m, or
c ∪ m' = c ∪ m. Also m < m' ≤ c ∪ m implies that m = (b ∪ m) ∩ m
≤ (b ∪ m) ∩ m' ≤ (b ∪ m) ∩ (c ∪ m) = m. That is, m' ∩ (b ∪ m) = m < m'. By
the lower covering condition, b ∪ m < m' ∩ (b ∪ m) = b ∪ m'. Now
b ∪ m ≤ (b ∪ m) ∩ [(b ∪ m') ∩ (c ∪ m)] ≤ b ∪ m'. If (b ∪ m') ∩ (c ∪ m)
≤ b ∪ m we would have m < m' ≤ (b ∪ m') ∩ (c ∪ m') = (b ∪ m') ∩ (c ∪ m)
≤ (b ∪ m) ∩ (c ∪ m) = m, a contradiction. Therefore b ∪ m < (b ∪ m)
[(b ∪ m') ∩ (c ∪ m)] = b ∪ m'. By the upper covering condition,
(b ∪ m') ∩ (c ∪ m) ≤ (b ∪ m)[(b ∪ m') ∩ (c ∪ m)] = (b ∪ m) ∩ (c ∪ m) = m.
Since c ∪ m' = c ∪ m, (b ∪ m') ∩ (c ∪ m') > m. Then (b ∪ m') ∩ (c ∪ m')
≥ m' > m implies m' = (b ∪ m') ∩ (c ∪ m'). We next show that
(a \cup m') \cap b = a. Since m' = m \cap (b \cup m) \geq m' \cap (a \cup m) \geq m we have m' \cap (a \cup m) = m < m'. By the lower covering condition,
(a \cup m') < m' \cup (a \cup m) = a \cup m'. Now a \cup m \leq (a \cup m) \cup [b \cap (a \cup m')] \leq a \cup m'. If (a \cup m) \cup [b \cap (a \cup m')] = a \cup m' then
b \cup (a \cup m) \cup [b \cap (a \cup m')] = b \cup (a \cup m') or b \cup m = b \cup m', contrary to b \cup m' \succ b \cup m. Therefore a \cup m = (a \cup m) \cup [b \cap (a \cup m')] < a \cup m'. Then b \cap (a \cup m') \leq a \cup m and b \cap (a \cup m') \leq b \cap (a \cup m).
Since b \cap (a \cup m') \geq b \cap (a \cup m) we have b \cap (a \cup m') = b \cap (a \cup m) = a.
Thus m' \in P and we again have the desired contradiction. We have therefore shown that a', b', and c' satisfy conditions (i) through (v).

Since the torsion free elements of L form a chain, either t(a') \leq t(c') or t(c') \leq t(a'). If t(a') \leq t(c'), Lemma 3.3 implies t(a' \cap c') = t(a') \cap t(c') = t(a'). We then have a' \cap c' < a' \leq t(a') = t(a' \cap c'). By Lemma 3.2 there is an element q such that a' \cap c' \leq q \leq a', contrary to condition (iv). Similarly, t(c') \leq t(a') contradicts condition (v). We therefore must conclude that the lattice does not contain elements a, b, and c such that a < b, an c = b \cap c, and a \cup c = b \cup c. That is, L is modular.

Corollary 3.1. An upper continuous torsion lattice which satisfies both covering conditions is atomic and modular.

Proof. Theorem 3.1 implies atomicity. Theorem 3.2 implies modularity since a torsion lattice contains only one torsion free element.
This corollary is a slight extension of a result due to Dilworth and Crawley (2).
4. COMPLETE JOIN HOMOMORPHISMS

Before proceeding to the representation theory it is necessary to establish several elementary facts concerning complete join homomorphisms. These homomorphisms will play a prominent role in the theory which is to follow and it will therefore be convenient to adopt certain notations concerning them. If $\theta$ is an equivalence relation in a lattice $L$ we let $\phi_\theta$ denote the natural mapping $L \to L/\theta$. For $x \in L$ let $v_\theta(x) = \bigcup\{ p \in L \mid p \equiv x \ (\text{mod } \theta) \} = \bigcup\{ p \in L \mid \phi_\theta(p) = \phi_\theta(x) \}$. We will use $C(L)$ to denote the collection of congruence relations $\theta$ for which the natural mapping $\phi_\theta : L \to L/\theta$ is a complete join epimorphism. The elements of $C(L)$ may be characterized in the following fashion.

Lemma 4.1. Suppose $L$ is a complete lattice, $S \subseteq L$, $1 \in S$, $S$ is closed under intersections, and letting $w(x) = \bigcap\{ s \in S \mid s \geq x \}$ the equation $w(x \wedge y) = w(x) \wedge w(y)$ holds for all $x, y \in L$. If a relation $\theta$ is defined in $L$ by saying $x \equiv y \ (\text{mod } \theta)$ if and only if $w(x) = w(y)$ then $\theta \in C(L)$, $w(x) = v_\theta(x)$, and $S = \{ x \in L \mid v_\theta(x) = x \}$. Conversely if $\theta \in C(L)$ then $S = \{ x \in L \mid v_\theta(x) = x \}$ is such a set and $v_\theta(x) = w(x)$ for each $x \in L$.

Supplement. For $\theta \in C(L)$ the following hold.

(i) $\phi_\theta(v_\theta(x)) = \phi_\theta(x)$.
(ii) $\phi_\theta(x) = \phi_\theta(y)$ if and only if $v_\theta(x) = v_\theta(y)$.
(iii) $v_\theta(v_\theta(x)) = v_\theta(x)$.
(iv) If $x \leq y$ then $v_\theta(x) \leq v_\theta(y)$.
\((v)\) \quad v_\theta(x \land y) = v_\theta(x) \land v_\theta(y).

\((vi)\) \quad \phi_\theta(x) < \phi_\theta(y) \text{ if and only if } v_\theta(x) < v_\theta(y).

\((vii)\) \quad If \(v_\theta(q) = q\) for each \(q \in Q\) then \(v_\theta(\bigcap Q) = \bigcap Q\) and

\[ \phi_\theta(\bigcap Q) = \bigcap_{q \in Q} \phi_\theta(q). \]

**Proof.** It is clear that \(\theta\) is an equivalence relation.

For \(x \in L\) let \(S_x = \{ s \in S \mid s \geq x \}\). Then \(x \leq w(x) = \bigcap S_x\). Note that if \(x \leq y\) then \(S_x \supseteq S_y, \bigcap S_x \subseteq \bigcap S_y\), and \(w(x) \leq w(y)\). Since \(S\) is closed under intersections, \(w(x) \in S_w(x)\) and \(w(w(x)) = w(x)\). Thus \(w(x) \equiv x \pmod{\theta}\) and \(v_\theta(x) \geq w(x)\). If \(y \equiv x \pmod{\theta}\) then \(y \leq w(y) = w(x)\). Consequently we also have \(v_\theta(x) \leq w(x)\) and it follows that \(v_\theta(x) = w(x)\). If \(x \in S\) then \(x = \bigcap S_x = w(x)\). Conversely, if \(x = w(x)\) then \(x = \bigcap S_x\) and \(x \in S\). Thus \(S = \{ x \mid w(x) = x \} = \{ x \mid v_\theta(x) = x \}\). If \(x \equiv x' \pmod{\theta}\) and \(y \equiv y' \pmod{\theta}\) then \(w(x \land y) = w(x) \land w(y) = w(x' \land y')\) and \(x \land y \equiv x' \land y' \pmod{\theta}\).

Thus \(\theta\) preserves finite intersections. We next verify that it preserves unions. If \(q \in Q\) then \(\bigcup Q \leq \bigcup_{q \in Q} w(q)\) and hence \(w(\bigcup Q) \leq w(\bigcup_{q \in Q} w(q))\). If \(q \in Q\) then \(q \leq \bigcup Q\) and \(w(q) \leq w(\bigcup Q)\). Thus \((\bigcup_{q \in Q} w(q)) \leq w(\bigcup Q)\) and \(w(\bigcup_{q \in Q} w(q)) \leq w(w(\bigcup Q)) = w(\bigcup Q)\). Consequently \(w(\bigcup Q) = w(\bigcup_{q \in Q} w(q))\). If \(x \equiv x' \pmod{\theta}\) and \(y \equiv y' \pmod{\theta}\) then \(w(x \lor y) = w(x) \lor w(y) = w(x') \lor w(y')\).

Thus \(\theta\) is a congruence relation and the natural map \(\phi_\theta : L \to L/\theta\) is a lattice epimorphism. Note that if \(x \in L\) then \(\phi_\theta(x) = \phi_\theta(w(x))\). Suppose \(A \subseteq L\). For \(a \in A\) we have \(a \leq \bigcup A\) and \(\phi_\theta(a) \leq \phi_\theta(\bigcup A)\). Thus \(\phi_\theta(\bigcup A)\) is an upper bound of the set \(\{ \phi_\theta(a) \}_{a \in A}\). Suppose that \(b\) is also an upper bound of this set and let \(b' \in L\) be such that \(\phi_\theta(b') = b\). For \(a \in A\) we have
Thus $\mathcal{U} A \triangleq w(b')$ and $\phi_\theta(\mathcal{U} A) \leq \phi_\theta(w(b')) = \phi_\theta(b') = b$. Hence $\phi_\theta(\mathcal{U} A)$ is the least upper bound of $\{\phi_\theta(a)\}_{a \in A}$. That is, $\phi_\theta(\mathcal{U} A) = \bigcup_{a \in A} \phi_\theta(a)$. Each subset of $L/\theta$, being the image of a subset of $L$ under $\phi_\theta$, has a least upper bound. Since $L/\theta$ is bounded below by $\phi_\theta(0)$ it follows that $L/\theta$ is complete. Thus $\phi_\theta: L \rightarrow L/\theta$ is a complete join epimorphism.

Before proving the converse portion of the lemma we will verify the properties listed in the supplement.

(i) Suppose $x \in L$ and $P = \{p \mid p \equiv x \ (\text{mod } \theta)\}$. Since $\phi_\theta$ preserves unions we have $\phi_\theta(v_\theta(x)) = \phi_\theta(U P) = \bigcup_{p \in P} \phi_\theta(p) = \mathcal{U} P \phi_\theta(x) = \phi_\theta(x)$.

(ii) If $\phi_\theta(x) = \phi_\theta(y)$ then $x \equiv y \ (\text{mod } \theta)$, $\{p \mid p \equiv x \ (\text{mod } \theta)\}$ = $\{p \mid p \equiv y \ (\text{mod } \theta)\}$, and therefore $v_\theta(x) = v_\theta(y)$. Conversely, if $v_\theta(x) = v_\theta(y)$ then $\phi_\theta(x) = \phi_\theta(v_\theta(x)) = \phi_\theta(v_\theta(y)) = \phi_\theta(y)$.

(iii) Since $\phi_\theta(v_\theta(x)) = \phi_\theta(x)$, (ii) implies $v_\theta(v_\theta(x)) = v_\theta(x)$.

(iv) If $x \leq y$ then $\phi_\theta(y) = \phi_\theta(x \cup y) = \phi_\theta(x) \cup \phi_\theta(y)$

$= \phi_\theta(v_\theta(x)) \cup \phi_\theta(v_\theta(y)) = \phi_\theta(v_\theta(x) \cup v_\theta(y))$ and therefore $v_\theta(x) \cup v_\theta(y) \leq v_\theta(y)$. It follows that $v_\theta(x) \leq v_\theta(y)$.

(v) By (iv), $v_\theta(x \cap y) \leq v_\theta(x)$ and $v_\theta(x \cap y) \leq v_\theta(y)$. Hence

$v_\theta(x \cap y) \leq v_\theta(x) \cap v_\theta(y)$. Since $\phi_\theta(v_\theta(x) \cap v_\theta(y)) = \phi_\theta(v_\theta(x)) \cap \phi_\theta(v_\theta(y)) = \phi_\theta(x) \cap \phi_\theta(y) = \phi_\theta(x \cap y)$ we also have $v_\theta(x) \cap v_\theta(y)$ \leq v_\theta(x \cap y)$. Thus $v_\theta(x \cap y) = v_\theta(x) \cap v_\theta(y)$.

(vi) If $v_\theta(x) \leq v_\theta(y)$ then $\phi_\theta(x) = \phi_\theta(v_\theta(x)) \leq \phi_\theta(v_\theta(y))$
Conversely, if $\phi_\theta(x) \leq \phi_\theta(y)$ then $\phi_\theta(x) = \phi_\theta(x) \cap \phi_\theta(y) = \phi_\theta(x \cap y)$, $v_\theta(x) = v_\theta(x \cap y) = v_\theta(x) \cap v_\theta(y)$, and $v_\theta(x) \leq v_\theta(y)$. Thus $\phi_\theta(x) \leq \phi_\theta(y)$ if and only if $v_\theta(x) \leq v_\theta(y)$. Then (ii) implies the desired result.

(vii) For $q \in Q$ we have $\bigcap Q \leq q$ and hence $v_\theta(\bigcap Q) \leq v_\theta(q) = q$. Therefore $v_\theta(\bigcap Q) \leq \bigcap Q$. Since the opposite inequality also holds we have $v_\theta(\bigcap Q) = \bigcap Q$. If $q \in Q$ then $q \leq \bigcap Q$ and $\phi_\theta(q) \geq \phi_\theta(\bigcap Q)$. Thus $\bigcap q \in Q \phi_\theta(q) \geq \phi_\theta(\bigcap Q)$. Let $p$ be such that $\phi_\theta(p) = \bigcap q \in Q \phi_\theta(q)$. Since $\phi_\theta(q) \geq \phi_\theta(p) \geq \phi_\theta(\bigcap Q)$ we have $v_\theta(q) \geq v_\theta(p) \geq v_\theta(\bigcap Q)$ and therefore $q \geq v_\theta(p) \geq \bigcap Q$. Then $\bigcap Q \geq v_\theta(p) \geq Q$, $\bigcap Q = v_\theta(p)$, and $\phi_\theta(\bigcap Q) = \phi_\theta(v_\theta(p)) = \phi_\theta(p) = \bigcap q \in Q \phi_\theta(q)$.

We now complete the proof of the lemma. Suppose $\theta \in C(L)$ and $S = \{s \mid v_\theta(s) = s\}$. For $x \in L$ let $S_x = \{s \in S \mid s \geq x\}$. Since $v_\theta(x) \in S_x$ we have $x \leq \bigcap S_x \leq v_\theta(x)$. Then $v_\theta(x) \leq v_\theta(\bigcap S_x) \leq v_\theta(v_\theta(x)) = v_\theta(x)$ and $v_\theta(x) = v_\theta(\bigcap S_x)$. But (vii) implies $v_\theta(\bigcap S_x) = \bigcap S_x$. Thus $v_\theta(x) = \bigcap \{s \in S \mid s \geq x\}$. Then (v) and (vii) imply $S$ has the desired properties.

The basic feature of complete join epimorphisms which makes them relevant to the study of meet representations is the following. If $L$ is a complete lattice and $\theta \in C(L)$ then the set $\{x \in L \mid v_\theta(x) = x\}$ determines a meet subsemilattice of $L$ which is isomorphic to $L/\theta$ when the latter is considered as a meet semilattice. It is therefore possible to obtain meet representations of some elements of $L$ from representations in $L/\theta$. The following two results illustrate this.
Lemma 4.2. If $\theta \in C(L)$ and $v_\theta(x) = x$ then $x$ is meet irreducible if and only if $\phi_\theta(x)$ is a meet irreducible element of $L/\theta$.

Proof. Suppose $x$ is meet irreducible and $\phi_\theta(x) = y_1 \cap y_2$. Let $x_1$ and $x_2$ be elements of $L$ such that $\phi_\theta(x_1) = y_1$ and $\phi_\theta(x_2) = y_2$. Then $\phi_\theta(x_1 \cap x_2) = \phi_\theta(x_1) \cap \phi_\theta(x_2) = y_1 \cap y_2 = \phi_\theta(x)$. Hence $v_\theta(x_1 \cap x_2) = v_\theta(x)$. Since $v_\theta(x_1 \cap x_2) = v_\theta(x_1) \cap v_\theta(x_2)$ and $v_\theta(x) = x$ we have $x = v_\theta(x_1) \cap v_\theta(x_2)$. The irreducibility of $x$ implies that either $v_\theta(x_1) = x$ or $v_\theta(x_2) = x$. If $v_\theta(x_1) = x$ then $\phi_\theta(x) = \phi_\theta(v_\theta(x_1)) = \phi_\theta(x_1) = y_1$. Thus $\phi_\theta(x)$ is meet irreducible.

Suppose $\phi_\theta(x)$ is meet irreducible and $x_1 \cap x_2 = x$. Then $\phi_\theta(x) = \phi_\theta(x_1 \cap x_2) = \phi_\theta(x_1) \cap \phi_\theta(x_2)$. Hence either $\phi_\theta(x_1) = \phi_\theta(x) = y_1 \cap y_2 = v_\theta(x)$ or $\phi_\theta(x_2) = \phi_\theta(x) = y_1 \cap y_2 = v_\theta(x)$. Since we also have $x_1 \lneq x$ it follows that $x_1 = x$. Thus $x$ is meet irreducible.

Lemma 4.3. Suppose $\theta \in C(L)$ and $v_\theta(x) = x$. Then $\phi_\theta(x)$ has an irredundant representation in terms of meet irreducible elements of $L/\theta$ if and only if $x$ has an irredundant representation in terms of meet irreducibles, $x = \bigcap R$, such that $v_\theta(r) = r$ for each $r \in R$.

Proof. Suppose $x = \bigcap R$ is a representation with the stated properties. Since $v_\theta(r) = r$ for each $r \in R$ we have $\phi_\theta(x) = \phi_\theta(\bigcap R) = \bigcap_{r \in R} \phi_\theta(r)$. The preceding lemma implies $\{\phi_\theta(r)\}_{r \in R}$ is a collection of meet irreducible elements of $L/\theta$. Suppose $r_0 \in R$ and $\bigcap_{r \in R} \phi_\theta(r) = \bigcap_{r \in R} r_0 \phi_\theta(r)$. Then
\[ \phi_\theta(\bigcap R) = \bigcap_{r \in R} \phi_\theta(r) = \bigcap_{r \in R - r_0} \phi_\theta(r) = \phi_\theta(\bigcap (R - r_0)) \]. Consequently, \( v_\theta(\bigcap R) = v_\theta(\bigcap (R - r_0)) \) and \( \bigcap R = \bigcap (R - r_0) \), contrary to the irredundancy of the representation of \( x \). Thus

\[ \bigcap_{r \in R} \phi_\theta(r) \neq \bigcap_{r \in R - r_0} \phi_\theta(r) \] for each \( r_0 \in R \) and we have an irredundant representation of \( \phi_\theta(x) \).

Conversely, suppose \( \phi_\theta(x) \) has an irredundant representation in terms of meet irreducible elements of \( L/\theta \), say \( \phi_\theta(x) = \bigcap_{a \in A} q_a \). For each \( a \in A \) let \( r_a = v_\theta(q'_a) \) where \( q'_a \) is any element of \( L \) such that \( \phi_\theta(q'_a) = q_a \). Since \( v_\theta(r_a) = r_a \) for each \( a \in A \) we have \( \phi_\theta(\bigcap_{a \in A} r_a) = \bigcap_{a \in A} \phi_\theta(r_a) = \bigcap_{a \in A} v_\theta(q'_a) = \bigcap_{a \in A} q_a = \phi_\theta(x) \). Consequently \( v_\theta(\bigcap_{a \in A} r_a) = v_\theta(x) \). Then \( \bigcap_{a \in A} r_a = v_\theta(\bigcap_{a \in A} r_a) = v_\theta(x) = x \).

The preceding lemma implies that each \( r_a \) is meet irreducible.

Suppose \( a_0 \in A \) and \( x = \bigcap_{a \in A - a_0} r_a \). Then \( \phi_\theta(x) = \phi_\theta(\bigcap_{a \in A - a_0} r_a) = \bigcap_{a \in A - a_0} \phi_\theta(r_a) = \bigcap_{a \in A - a_0} q_a \), contrary to the irredundancy of the representation of \( \phi_\theta(x) \). Hence \( x = \bigcap_{a \in A} r_a \) is a representation of \( x \) having the desired properties.

Another property of complete join epimorphisms which makes them valuable when one is dealing with upper continuous lattices is the following.

Lemma 4.4. If \( L \) is upper continuous and \( \theta \in C(L) \) then \( L/\theta \) is also upper continuous.

Proof. Suppose \( x \in L/\theta \), \( S \subseteq L/\theta \), and \( S \) is a chain.

Let \( x' \) be such that \( \phi_\theta(x') = x \) and for each \( s \in S \) let \( s' \) be such that \( \phi_\theta(s') = s \). Then \( \{ v_\theta(s') \} \subseteq S \) is a chain in \( L \). The upper continuity of \( L \) and the join preserving nature of \( \phi_\theta \) imply
Thus $L/\theta$ is upper continuous.

Since $C(L)$ is a subset of the lattice of all congruence relations of $L$, it is a partially ordered set. That is, $\theta_1 \leq \theta_2$ if $x \equiv y \pmod{\theta_1}$ implies $x \equiv y \pmod{\theta_2}$. Two results regarding this partial ordering which we will have need of later are given below.

**Lemma 4.5.** If $\theta_1, \theta_2 \in C(L)$, $\theta_1 \leq \theta_2$, and $v_{\theta_2}(x) = x$ then $v_{\theta_1}(x) = x$.

**Proof.** Since $\theta_1 \leq \theta_2$, $\{y \mid y \equiv x \pmod{\theta_1}\} \subseteq \{y \mid y \equiv x \pmod{\theta_2}\}$. Then $x \leq v_{\theta_1}(x) \leq v_{\theta_2}(x) = x$ and $v_{\theta_1}(x) = x$.

**Lemma 4.5.** Suppose that $\phi_1 : L \rightarrow L_1$ and $\phi_2 : L \rightarrow L_2$ are complete join epimorphisms and $\theta_1$ and $\theta_2$ are the congruence relations in $L$ determined by $\phi_1$ and $\phi_2$. Then the following are equivalent.

(i) $\theta_1 \leq \theta_2$. 

\[
x \cap U \phi(s') = \phi(x') \cap \bigcup_{s' \in \phi} \phi(s') \\
= \phi(x') \cap \bigcup_{s' \in \phi} \phi(v_{\phi}(s')) \\
= \phi(x') \cap \bigcup_{s' \in \phi} \phi(\phi(s')) \\
= \phi(x') \cap \bigcup_{s' \in \phi} \phi(v_{\phi}(x')) \\
= \bigcup_{s' \in \phi} \phi(x') \cap \bigcup_{s' \in \phi} \phi(v_{\phi}(s')) \\
= \bigcup_{s' \in \phi} \phi(x') \cap v_{\phi}(x') \\
= \bigcup_{s' \in \phi} \phi(x') \cap \phi(s') \\
= \bigcup_{s' \in \phi} \phi(x') \cap \phi(s').
\]
(ii) There is a complete join epimorphism \( \psi : L_1 \to L_2 \) such that \( \psi \phi_1 = \phi_2 \).

(iii) There is a function \( \psi : L_1 \to L_2 \) such that \( \psi \phi_1 = \phi_2 \).

Proof. (iii) implies (ii). Suppose \( S \subseteq L_1 \). For each \( s \in S \) let \( s' \in L \) be such that \( \phi_1 (s') = s \). Then

\[
\psi(\bigcup_{s \in S} \phi_1 (s')) = \psi \phi_1 (\bigcup_{s \in S} s') = \psi \phi_2 (\bigcup_{s \in S} s') = \bigcup_{s \in S} \phi_1 (s') = \bigcup_{s \in S} \psi (s) .
\]

In a similar fashion it can be shown that \( \psi \) preserves finite intersections. Thus \( \psi \) is a complete join homomorphism. If \( x \in L_2 \) there exists \( x' \in L \) such that \( \phi_2 (x') = x \). Then \( \psi \phi_1 (x') = x \). Thus \( \psi \) is an epimorphism.

(ii) implies (i). If \( x \equiv y \pmod{\theta_1} \) then \( \phi_2 (x) = \psi \phi_1 (x) = \psi \phi_1 (y) = \phi_2 (y) \) and \( x \equiv y \pmod{\theta_2} \). Thus \( \theta_1 \leq \theta_2 \).

(i) implies (iii). If \( x \in L_1 \) let \( \psi (x) = \phi_2 (x') \) where \( x' \) is any element of \( L \) such that \( \phi_1 (x') = x \). If \( x'' \) is such that \( \phi_1 (x'') = x \) then, since \( \theta_1 \leq \theta_2 \), \( \phi_2 (x'') = \phi_2 (x') \). Thus \( \psi \) is a function from \( L_1 \) to \( L_2 \) and by definition \( \psi \phi_1 (x') = \phi_2 (x') \) for each \( x' \in L \).

We will also have occasion to discuss suprema and infima of subsets of \( C(L) \).

Lemma 4.7. \( C(L) \) is a complete lattice.

Proof. We will first show that any subset of \( C(L) \) has a greatest lower bound. Suppose \( \{ \theta_\alpha \}_{\alpha \in A} \subseteq C(L) \). Let \( S \) be the collection of all those elements which can be expressed as the intersection of a subset of \( \{ x | v_\theta (x) = x \} \) for
some $\alpha \in A$. Then $l \in S$ and it is clear that $S$ is closed under intersections. Note that each element of $S$ may be expressed in the form $\bigcap_{\alpha \in A} x_{\alpha}$ where $v_{\theta_a}(x_{\alpha}) = x_{\alpha}$. For $x \in L$ let $S_x = \{s \in S \mid s \geq x\}$. Consider the element $w(x) = \bigcap S_x$. For each $\theta_a$ we have $v_{\theta_a}(x) \in S_x$ and therefore $w(x) = \bigcap S_x \leq v_{\theta_a}(x)$. Suppose $s \in S_x$ and let $s = \bigcap_{\alpha \in A} s_{\alpha}$ where $v_{\theta_a}(s_{\alpha}) = s_{\alpha}$. Since $x \leq s \leq s_{\alpha}$ we have $v_{\theta_a}(x) \leq v_{\theta_a}(s_{\alpha}) = s_{\alpha}$. Hence $\bigcap_{\alpha \in A} v_{\theta_a}(s_{\alpha}) = \bigcap S_x = w(x)$. Thus $w(x) = \bigcap_{\alpha \in A} v_{\theta_a}(x)$. If $x, y \in L$ then $w(x \cap y)$
\[= \bigcap_{\alpha \in A} v_{\theta_a}(x \cap y) = [\bigcap_{\alpha \in A} v_{\theta_a}(x)] \cap [\bigcap_{\alpha \in A} v_{\theta_a}(y)]
= w(x) \cap w(y).\]
Thus $S$ satisfies the hypothesis of Lemma 4.1.

Then there exists $\theta \in C(L)$ such that $S = \{x \in L \mid v_{\theta_a}(x) = x\}$ and $w(x) = v_{\theta_a}(x)$. Note that $v_{\theta_a}(x) = \bigcap_{\alpha \in A} v_{\theta_a}(x)$. We will show that $x \equiv y \pmod{\theta_a}$ if and only if $x \equiv y \pmod{\theta_a}$ for each $\theta_a$. It then follows that $\theta = \bigcap_{\alpha \in A} \theta_a$. If $x \in L$ then $x \leq v_{\theta_a}(x)$ for each $\theta_a$. Then $v_{\theta_a}(x) \leq v_{\theta_a}(v_{\theta_a}(x)) \leq v_{\theta_a}(x)$ for each $\theta_a$. Then if $x \equiv y \pmod{\theta_a}$ we have $x \equiv v_{\theta_a}(y) \equiv y \pmod{\theta_a}$ for each $\theta_a$. Conversely, if $x \equiv y \pmod{\theta_a}$ for each $\theta_a$ then $v_{\theta_a}(x) = \bigcap_{\alpha \in A} v_{\theta_a}(x) = \bigcap_{\alpha \in A} v_{\theta_a}(y) = v_{\theta_a}(y)$ and $x \equiv y \pmod{\theta}$. Thus $\theta = \bigcap_{\alpha \in A} \theta_a$. Since each subset of $C(L)$ has a greatest lower bound and $C(L)$ is bounded above by the unit congruence relation it follows that $C(L)$ is complete.

The following result describes the join of a subset of $C(L)$ in the case when $L$ is upper continuous.

Lemma 4.8. Suppose $L$ is an upper continuous lattice,
\[ \{ \theta_a \}_a \in A \subset C(L), \text{ and } \theta = \bigcup_a \theta_a. \text{ Then } \{ x \mid v_{\theta_a}(x) = x \} = \{ x \mid v_{\theta_a}(x) = x \text{ for each } \theta_a \}. \]

Proof. Let \( S = \{ x \mid v_{\theta_a}(x) = x \text{ for each } \theta_a \} \). Then \( 1 \in S \)
and it is clear that \( S \) is closed under intersections. For \( x \in L \) let \( S_x = \{ s \in S \mid s \supseteq x \} \) and let \( w(x) = \bigcap S_x \). Note that since \( S \) is closed under intersections we have \( w(x) \in S_x \). This implies that \( v_{\theta_a}(w(x)) = w(x) \) for each \( \theta_a \). It also implies \( w(w(x)) = w(x) \) for each \( x \in L \). In order to verify that \( S \)
determines an element of \( C(L) \) we must show that \( w(x \cap y) = w(x) \cap w(y) \) for all \( x, y \in L \). If \( a \leq b \) then \( S_a \supseteq S_b \), \( \bigcap S_a \)
\( \subseteq \bigcap S_b \), and \( w(a) \leq w(b) \). Since \( x \cap y \leq x \) and \( x \cap y \leq y \) we therefore have \( w(x \cap y) \leq w(x) \cap w(y) \). Let \( P = \{ p \mid x \leq p \leq w(x), p \cap y \)
\( \leq w(x \cap y) \} \). \( P \) is not empty since \( x \in P \). Suppose \( Q \subset P \)
and \( Q \) is a chain. Then upper continuity implies \( \bigcap Q = \bigcup_{q \in Q} \bigcap_{p \leq q} \bigcup_{q \in Q} w(x \cap y) = w(x \cap y) \). Since \( x \leq \bigcup Q \leq w(x) \), \( \bigcup Q \in P \). Thus \( P \) contains maximal elements. Let \( m \) be such an element. Since \( x \leq m \leq w(x) \) we have \( x \leq m \leq v_{\theta_a}(m) \leq v_{\theta_a}(w(x)) = w(x) \) for each \( \theta_a \). Also, \( v_{\theta_a}(m \cap y) \leq v_{\theta_a}(w(x \cap y)) = w(x \cap y) \).

Furthermore \( w(x \cap y) \geq v_{\theta_a}(m \cap y) = v_{\theta_a}(m) \cap v_{\theta_a}(y) \geq v_{\theta_a}(m \cap y) \). Thus \( v_{\theta_a}(m) \in P \) for each \( \theta_a \). By the maximality of \( m \)
therefore have \( v_{\theta_a}(m) = m \) for each \( \theta_a \). Since \( x \leq m \leq w(x) \) it follows that \( w(x) \leq w(m) \leq w(w(x)) = w(x) \) and \( w(x) = w(m) = m \). Thus \( w(x) \cap y \leq w(x \cap y) \). Similarly, we may take \( m' \) maximal with respect to the properties \( y \leq m' \leq w(y) \) and \( w(x) \cap m' \leq w(x \cap y) \). Again it must be the case that \( m' = w(y) \) and therefore \( w(x) \cap w(y) \leq w(x \cap y) \). Since the opposite inequality also
holds we have \( w(x) \cap w(y) = w(x \cap y) \). Then Lemma 4.1 implies the existence of \( \theta \in C(L) \) such that \( w(y) = v_\theta(y) \) and \( S = \{ x \mid v_\theta(x) = x \} \). For each \( x \in L \) and each \( \theta_a \) we have \( x \leq v_\theta(x) \leq w(x) \) and therefore \( v_\theta(x) = x \) (mod \( \theta \)). If \( x \equiv y \) (mod \( \theta_a \)), then \( x \equiv v_\theta(x) = v_\theta(y) \equiv y \) (mod \( \theta \)). Thus \( \theta \leq \theta_a \) for each \( \alpha \in A \).

Suppose \( \theta' \in C(L) \) is such that \( \theta' \geq \theta_a \) for each \( \alpha \in A \). Then Lemma 4.5 implies \( v_{\theta_a}(v_{\theta_a}(x)) = v_{\theta_a}(x) \) for each \( \alpha \in A \). Hence \( v_{\theta_a}(x) \in S_x \) and \( v_{\theta'_a}(x) \geq w(x) \geq x \). Consequently \( x \equiv w(x) \) (mod \( \theta' \)). If \( x \equiv y \) (mod \( \theta \)) then \( x \equiv w(x) = w(y) \equiv y \) (mod \( \theta' \)). Thus \( \theta' \geq \theta \).

We therefore have \( \theta = \bigcup_{\alpha \in A} \theta_a \).
5. TWO HOMOMORPHISMS

In some lattices the torsion free elements determine a congruence relation of the type discussed in the preceding section.

Theorem 5.1. If L is an upper continuous lattice satisfying both covering conditions then there exists \( \theta \in C(L) \) such that \( v_\theta(x) = t(x) \) for each \( x \in L \).

Proof. In Lemma 4.1 take \( S \) to be the collection of all torsion free elements of \( L \). By Lemma 2.5, \( S \) is closed under intersections. Then \( w(x) = \bigcap \{ s \in S \mid s \geq x \} = t(x) \).

By Lemma 3.3, \( t(x \land y) = t(x) \land t(y) \) for all \( x, y \in L \). Thus there exists \( \theta \in C(L) \) such that \( v_\theta(x) = w(x) = t(x) \) for each \( x \in L \).

Henceforth we will use \( L^t \) to denote the lattice \( L/\theta \), where \( \theta \) is the congruence relation given by the above theorem. One can easily characterize the upper continuous lattices satisfying both covering conditions for which \( L^t \) consists of one element. In such a lattice one has \( \phi_\theta(x) = \phi_\theta(l) \) for each \( x \in L \). Hence \( t(x) = t(l) = l \) for each \( x \in L \) and the only torsion free element is \( l \). Thus \( L \) is a torsion lattice and Corollary 3.1 implies that \( L \) is an atomic, modular, upper continuous lattice. It is readily seen that the converse also holds.

More generally, \( L^t \) may thought of as the lattice which is obtained by collapsing each of the covers in \( L \) upward.
We illustrate by constructing $L^t$ in a simple case. Let $L$ be the collection of sequences $(x_1, x_2, x_3, \ldots)$ where each $x_n$ is either 0 or 1. If these are ordered lexicographically then $L$ is a chain and therefore satisfies the hypothesis of the above theorem. The only covers which occur in $L$ have the form

$$(x_1, \ldots, x_n, 1, 0, 0, 0, \ldots) \geq (x_1, \ldots, x_n, 0, 1, 1, 1, \ldots).$$

If $\theta$ is the congruence relation of the preceding theorem than a $\theta$ class either contains only one element or consists of a pair of elements having the above form. Then $\theta$ is precisely the equivalence relation one would use to obtain the numbers of the closed real unit interval as limits of binary approximations. Thus $L^t$ is isomorphic to that interval.

Incidentally, this same example serves to show that compact generation, unlike upper continuity, need not be preserved by a complete join epimorphism. $L$ is compactly generated, the compact elements being those sequences which are zero from some point on. However the only compact element of $L^t$ is 0.

Theorem 5.2. Suppose $\psi : L_1 \to L_2$ is a complete join epimorphism, where $L_1$ and $L_2$ are upper continuous lattices satisfying both covering conditions. Then there is a complete join epimorphism $\psi' : L_1^t \to L_2^t$ such that the diagram
is commutative.

**Proof.** We first show that \( \phi_2 \psi(x) = \phi_2 \psi(t(x)) \) for each \( x \in L \). Let \( P = \{ p | x \leq p \leq t(x), \phi_2 \psi(p) = \phi_2 \psi(x) \} \). Suppose \( S \subseteq P \) and \( S \) is a chain. It is clear that \( x \leq \bigcup S \leq t(x) \). The map \( \phi_2 \psi \) is a complete join epimorphism since both \( \phi_2 \) and \( \psi \) are. Then \( \phi_2 \psi(\bigcup S) = \bigcup_{s \in S} \phi_2 \psi(s) = \bigcup_{s \in S} \phi_2 \psi(x) \).

\[ = \bigcup_{s \in S} \phi_2 \psi(x) = \phi_2 \psi(x) \text{ and } \bigcup S \in P. \]

Thus \( P \) contains maximal elements. Let \( m \) be such an element. If \( m < t(x) \) then, by Lemma 3.2, there exists \( m' \) such that \( m \sim m' \leq t(x) \). If \( \psi(m') = \psi(m) \) we have \( \phi_2 \psi(m') = \phi_2 \psi(m) = \phi_2 \psi(x) \), contrary to the maximality of \( m \). Thus \( \psi(m) < \psi(m') \). Suppose \( \psi(m) < y \leq \psi(m') \) and let \( y' \) be such that \( \psi(y') = y \). Then \( m \leq m \cup (m' \cap y') \leq m' \).

If \( m = m \cup (m' \cap y') \) we have \( m \geq m' \cap y' \) and \( \psi(m) \geq \psi(m') \cap \psi(y') \) = \( \psi(m') \cap y = y \), contrary to \( \psi(m) < y \). Therefore \( m \cup (m' \cap y') = m' \) and \( \psi(m') = \psi(m) \cup [\psi(m') \cap \psi(y')] = \psi(m) \cup (\psi(m') \cap y) \).

\[ = \psi(m) \cup y = y. \]

Thus \( \psi(m') > \psi(m) \). Hence \( \psi(m) \sim \psi(m') \leq t(\psi(m)) \).

Then \( \phi_2 \psi(m) \leq \phi_2 \psi(m') \leq \phi_2 [t(\psi(m))] \). But by the nature of \( \phi_2 \) we have \( \phi_2 \psi(m) = \phi_2 [t(\psi(m))] \). Therefore \( \phi_2 \psi(m') = \phi_2 \psi(m) = \phi_2 \psi(x) \), contrary to the maximality of \( m \). We are thus led to conclude that \( m = t(x) \) and \( \phi_2 \psi(x) = \phi_2 \psi(t(x)) \). Let \( \theta_1, \theta_2 \in \mathcal{C}(L) \) be the congruence relations corresponding to the
homomorphisms \( \phi_1 \) and \( \phi_2 \), respectively. If \( x \equiv y \) (mod \( e_1 \)) then \( t(x) = t(y) \), \( \phi_2^t(x) = \phi_2^t(t(x)) = \phi_2^t(t(y)) = \phi_2^t(y) \), and \( x \equiv y \) (mod \( e_2 \)). Thus \( e_1 \leq e_2 \). Lemma 4.6 then guarantees the existence of the desired homomorphism.

Corollary 5.1. Suppose \( \psi : L_1 \rightarrow L_2 \) is a complete join epimorphism, where \( L_1 \) is atomic, modular, and upper continuous. Then \( L_2 \) is also atomic, modular, and upper continuous.

Proof. \( L_2 \) is modular since any homomorphic image of a modular lattice is again modular. Lemma 4.4 implies that \( L_2 \) is upper continuous. Then by the preceding theorem there is a complete join epimorphism \( \psi' : L_1^t \rightarrow L_2^t \). Since \( L_1 \) is atomic, \( L_1^t \) consists of one element. Since \( \psi' \) is an epimorphism it follows that \( L_2^t \) consists of one element. Thus \( L_2 \) is atomic.

Suppose now that \( L \) is an upper continuous modular lattice and \( \theta \in C(L) \). Then \( L/\theta \) is again an upper continuous modular lattice. Hence the torsion free elements of \( L/\theta \) determine an element of \( C(L/\theta) \). The mapping \( L \rightarrow L/\theta \rightarrow (L/\theta)^t \) is a composite of two complete join epimorphisms and is therefore also a mapping of this type. Thus there is an element of \( C(L) \) corresponding to the mapping \( L \rightarrow (L/\theta)^t \). We will use \( \theta^t \) to denote the element of \( C(L) \) obtained from \( \theta \) in this manner. Thus \( L^t = L/\theta^t \). It is clear that \( \theta^t \geq \theta \) for each \( \theta \in C(L) \).

Lemma 5.1. Suppose \( L \) is an upper continuous modular lattice, \( \theta_1, \theta_2 \in C(L) \), and \( \theta_1 \leq \theta_2 \). Then \( \theta_1^t \leq \theta_2^t \).
Proof. Consider the following diagram.

Since $\theta_1 \leq \theta_2$, Lemma 4.6 implies the existence of the complete join epimorphism $\psi$ making the left half commutative. Then Theorem 5.2 guarantees the existence of the complete join epimorphism $\psi'$ such that the right half commutes. Again applying Lemma 4.6, the existence of the mapping $\psi'$ implies $\theta_1^t \leq \theta_2^t$.

Lemma 5.2. Suppose $L$ is an upper continuous modular lattice, $\theta \in C(L)$, and $v_\theta(x) = x$. Then $v_\theta(x) = x$ if and only if $\phi_\theta(x)$ is a torsion free element of $L/\theta$.

Proof. Let $\phi'$ denote the mapping $L/\theta \to (L/\theta)^t$.

Suppose $\phi_\theta(x)$ is torsion free. If $x \equiv y \pmod{\theta^t}$ then $\phi' \phi_\theta(x) = \phi' \phi_\theta(y)$ and therefore $t(\phi_\theta(x)) = t(\phi_\theta(y))$. Hence $\phi_\theta(y) \leq t(\phi_\theta(y)) = t(\phi_\theta(x)) = \phi_\theta(x)$. Then $v_\theta(y) \leq v_\theta(x)$ and we have $y \leq v_\theta(y) \leq v_\theta(x) = x$. Thus $v_\theta(x) = \bigcup \{y \mid y \equiv x \pmod{\theta^t}\}$.

Suppose $\phi_\theta(x)$ is not torsion free. Then $\phi_\theta(x) < t(\phi_\theta(x))$.

Let $y \in L$ be such that $\phi_\theta(y) = t(\phi_\theta(x))$. Since $\phi_\theta(y) > \phi_\theta(x)$ we have $v_\theta(y) \geq v_\theta(x) = x$. But $v_\theta(y) \neq x$ since $\phi_\theta(v_\theta(y)) = \phi_\theta(y) = t(\phi_\theta(x)) > \phi_\theta(x)$. Note that $v_\theta(y) \equiv x \pmod{\theta^t}$ since $\phi_\theta(v_\theta(y)) = \phi'(\phi_\theta(y)) = \phi'(t(\phi_\theta(x))) = \phi'(\phi_\theta(x))$. Thus $v_\theta(x) = \bigcup \{p \mid p \equiv x \pmod{\theta^t}\} \geq v_\theta(y) > x$.

Lemma 5.3. If $L$ is an upper continuous modular
lattice and \( \theta \in C(L) \) then the following are equivalent.

(i) \( \theta^t = \theta \).

(ii) Each element of \( L/\theta \) is torsion free.

(iii) \( L/\theta \) contains no covers.

(iv) Between any two comparable elements of \( \{ x \mid v_{\theta}(x) = x \} \) there is a third.

Proof. (ii) implies (i). Let \( x \) be any element of \( L \). Since \( v_{\theta}(v_{\theta}(x)) = v_{\theta}(x) \) and \( \phi_{\theta}(v_{\theta}(x)) \) is torsion free the preceding lemma implies \( v_{\theta^t}(v_{\theta}(x)) = v_{\theta}(x) \). Since \( x \equiv v_{\theta}(x) \pmod{\theta} \) and \( \theta \leq \theta^t \) we have \( x \equiv v_{\theta}(x) \pmod{\theta^t} \), and hence \( v_{\theta^t}(x) = v_{\theta^t}(v_{\theta}(x)) \). Thus \( v_{\theta}(x) = v_{\theta}(x) \). If \( x_1 \equiv x_2 \pmod{\theta^t} \) then \( v_{\theta}(x_1) = v_{\theta^t}(x_1) = v_{\theta^t}(x_2) = v_{\theta}(x_2) \) and \( x_1 \equiv x_2 \pmod{\theta} \). Therefore \( \theta = \theta^t \).

(i) implies (ii). Let \( x \) be any element of \( L/\theta \) and let \( x' \in L \) be such that \( \phi_{\theta}(x') = x \). Since \( v_{\theta^t}(v_{\theta}(x')) = v_{\theta}(v_{\theta}(x')) = v_{\theta}(x) \) the preceding lemma implies \( \phi_{\theta}(v_{\theta}(x')) \) is torsion free. But \( \phi_{\theta}(v_{\theta}(x')) = \phi_{\theta}(x') = x \). Thus each element of \( L/\theta \) is torsion free.

It is clear that (ii) is equivalent to (iii). In view of the correspondence between \( \{ x \mid v_{\theta}(x) = x \} \) and the elements of \( L/\theta \) it is also clear that (iii) is equivalent to (iv).

Lemma 5.4. Suppose \( L \) is an upper continuous modular lattice and \( \theta \in C(L) \). Then there is a unique \( \theta^* \in C(L) \) such that the following hold.

(i) \( \theta^* \geq \theta \)
(ii) \((\theta^*)^t=\theta^*\)

(iii) If \(\theta_1 \geq \theta\) and \(\theta_1^t=\theta\) then \(\theta_1 \geq \theta^*\).

Proof. Let \(S=\{\theta' \in C(L) | \theta' \geq \theta, (\theta')^t=\theta'\}\). \(S\) is not empty since it contains the unit congruence relation. Let \(\theta^*=\cap S\). Then \(\theta^* \geq \theta\). For each \(\theta' \in S\) we have \(\theta^* \leq \theta'\) and by Lemma 5.1 \((\theta^*)^t \leq (\theta')^t = \theta'\). Thus \(\cap S \geq (\theta^*)^t = \cap S\) and \((\theta^*)^t = \theta^*\). If \(\theta_1\) is as in (iii) then \(\theta_1 \in S\) and therefore \(\theta_1 \geq \cap S = \theta^*\). \(\theta^*\) is unique since (iii) forces any two elements satisfying (i) and (ii) to be equal.

The congruence relation \(\theta^*\) will be of particular interest to us and we will adopt special notation pertaining to it. Let \(L^*\) denote the lattice \(L/\theta^*\), let \(\phi^*\) denote the homomorphism \(\phi_{\theta^*}:L \rightarrow L/\theta^*=L^*\), and let \(v^*(x) = v_{\phi^*}(x)\) for \(x \in L\). The lattice \(L^*\) may be characterized in the following fashion.

Theorem 5.3. If \(L\) is an upper continuous modular lattice then:

(i) \(L^*\) contains no covers.

(ii) If \(\psi:L \rightarrow L'\) is a complete join epimorphism and \(L'\) contains no covers then there is a complete join epimorphism \(\psi':L^* \rightarrow L'\) such \(\psi = \psi' \phi^*\).

Proof. Since \((\theta^*)^t = \theta^*\), Lemma 5.3 implies \(L^*\) contains no covers. Let \(\theta\) denote the congruence relation in \(L\) determined by the mapping \(\psi:L \rightarrow L'\). Then \(\theta \in C(L)\) and \(L/\theta = L'\). Since \(L'\) contains no covers, Lemma 5.3 implies \(\theta = \theta^*\). It follows from Lemma 5.4 that \(\theta \geq \theta^*\). Then Lemma 4.6 implies...
the existence of the desired mapping \( \psi^*:L_0/\theta \rightarrow L_0/\theta \).

**Theorem 5.4.** If \( \psi:L_1+L_2 \) is a complete join epimorphism, where \( L_1 \) and \( L_2 \) are upper continuous modular lattices, then there is a complete join epimorphism \( \psi':L_1^*+L_2^* \) such that the diagram

\[
\begin{array}{ccc}
L_1 & \xrightarrow{\phi_1^*} & L_1^* \\
\downarrow \psi & & \downarrow \psi' \\
L_2 & \xrightarrow{\phi_2^*} & L_2^*
\end{array}
\]

is commutative.

**Proof.** By (i) of the preceding theorem \( L_2^* \) contains no covers. Then the complete join epimorphism \( \phi_2^* : L_1 + L_2^* \) satisfies the hypothesis of (ii) of that theorem. Thus there is a complete join epimorphism \( \psi':L_1^*+L_2^* \) such that \( \psi' \phi_1^* = \phi_2^* \psi \).

We conclude this section by showing that two important types of lattices have the property that \( L^* \) consists of one element.

If \( L \) is a modular lattice which satisfies the ascending chain condition then \( L^* \) consists of one element. Assume the contrary. Then \( v^*(0) < 1 \). Lemma 5.3 (iv) implies the existence of an infinite chain \( v^*(0) < x_1 < x_2 \ldots < 1 \) where \( v^*(x_n) = x_n \) for each \( n \). Since this is contrary to the ascending chain condition we arrive at the desired conclusion.

If \( L \) is an atomic, modular, upper continuous lattice
then \( L^* \) consists of one element. Since \( L \) is atomic \( L^t \) consists of one element and \( 0^t \) is the unit congruence relation. Since \( 0^t \leq (0^*)^t = 0^* \) it follows that \( 0^* = 0^t \) and \( L^* \) consists of one element.
6. MEET IRREDUCIBLE ELEMENTS

In this section we list some elementary properties of meet irreducible elements in order to gain some insight into the manner in which such elements can arise.

Lemma 6.1. An element \( x \) is completely meet irreducible if and only if there exists \( x' \) such that \( x \prec x' \preceq y \) for each \( y > x \).

Proof. Suppose \( x \) is completely meet irreducible. Let \( S = \{ y | y > x \} \) and let \( x' = \bigcap S \). By hypothesis \( x' > x \). If \( y > x \) then \( y \in S \) and \( y \preceq \bigcap S = x' \). Therefore \( x' > x \). Conversely, if \( x' \) is an element with the stated properties then \( \bigcap \{ y | y > x \} = \bigcap \{ y | y \preceq x' \} = x' > x \). Thus \( x \) is completely meet irreducible.

There is a simple criterion for determining which meet irreducible elements are completely meet irreducible.

Lemma 6.2. A meet irreducible element is completely meet irreducible if and only if it is not torsion free.

Proof. By Lemma 6.1, no completely meet irreducible element is torsion free. Conversely, suppose \( x \) is meet irreducible and \( x' > x \). If \( y > x \) then \( x \preceq x' \cap y \preceq x' \). By hypothesis \( x' \cap y \neq x \). Hence \( x' \cap y = x' \) and \( y \preceq x' \). Thus Lemma 6.1 implies \( x \) is completely meet irreducible.

In an upper continuous lattice the following may be used to obtain meet irreducible elements above a given element.

Lemma 6.3. Suppose \( x' > x \), \( x \) is meet irreducible in
terms of elements of \([x,x']\), and \(r\) is maximal with respect to the property \(r \cap x' = x\). Then \(r\) is meet irreducible. If \(x\) is completely meet irreducible in terms of elements of \([x,x']\) then \(r\) is completely meet irreducible.

Proof. Suppose \(y_1 > r, y_2 > r, \text{ and } y_1 \cap y_2 = r\). Then \(x = r \cap x' = (y_1 \cap y_2) \cap x' = (y_1 \cap x') \cap (y_2 \cap x')\). Here \(y_1 \cap x', y_2 \cap x' \in [x,x']\) and, by the maximality of \(r\), \(y_1 \cap x' > x\) and \(y_2 \cap x' > x\). But this contradicts the hypothesis regarding \(x\). Hence \(r\) is meet irreducible. The second assertion may be verified in a similar fashion.

As an immediate consequence of this we have:

Lemma 6.4. If \(x' \geq x\) and \(r\) is maximal with respect to the property \(r \cap x' = x\) then \(r\) is completely meet irreducible.

For modular lattices there is a converse to Lemma 6.3.

Lemma 6.5. Suppose that \(r\) is a meet irreducible element of a modular lattice, \(x < x'\), and \(x' \cap r = x\). Then \(r\) is maximal with respect to the property \(x' \cap r = x\) and \(x\) is meet irreducible in terms of elements of \([x,x']\). If \(r\) is completely meet irreducible then \(x\) is completely meet irreducible in terms of elements of \([x,x']\).

Proof. Suppose \(s > r\) and \(x' \cap s = x\). By modularity, \((r \cup x') \cap s = r \cup (x' \cap s) = r \cup x = r\). Since \(r\) is meet irreducible it follows that \(r \cup x' = r\). Then \(x' = x' \cap r = x\), a contradiction. Thus \(r\) is maximal with respect to the stated property.

Now suppose that \(x \leq y_1 \leq x'\), \(x \leq y_2 \leq x'\), and \(y_1 \cap y_2 = x\). Let
\( p_1 = y_1 \cap (y_2 \cup r) \) and \( p_2 = y_2 \cap (y_1 \cup r) \). Then \( p_1 \cap p_2 = y_1 \cap (y_2 \cap r) \cap y_2 \cap (y_1 \cap r) = y_1 \cap y_2 \cap x \). Also, since \( x \leq p_1 \cup p_2 \), we have \( (p_1 \cup p_2) \cap r = x \). By modularity, \( r \cup p_1 = r \cup y_1 \cap (r \cup y_2). \) Similarly, \( r \cup p_2 = (r \cup y_1) \cap (r \cup y_2) \). Thus \( r \cup p_1 = r \cup p_2 = r \cup y_1 \cup p_2 \). Consequently \( (p_1 \cup r) \cap (p_1 \cup p_2) = (p_1 \cup p_2) \cap (p_1 \cup p_2) = p_1 \cup p_2 \).

On the other hand, modularity implies \( (p_1 \cup r) \cap (p_1 \cup p_2) = p_1 \cup x = p_1 \). Hence \( p_1 = p_1 \cup p_2 \) and \( p_2 = p_1 \cap p_2 = x \). Then \( r = r \cup x = r \cup p_2 = r \cup y_2 \cap (r \cup y_1) = (r \cup y_2) \cap (r \cup y_1) \).

It follows from the irreducibility of \( r \) that either \( r \cup y_1 = r \) or \( r \cup y_2 = r \). If, say, \( r \cup y_1 = r \) then \( y_1 = r \cap y_1 = x \). Thus \( x \) is meet irreducible in terms of elements of \( [x, x'] \).

If, in addition, \( r \) is completely meet irreducible then, by Lemma 6.1, there exists \( r' \) such that \( r' \succ r \). Then \( r \leq r \cup (x' \cap r') \leq r' \). If \( r \cup (x' \cap r') = r \) then \( x' \cap r' = r \cap (x' \cap r') = x' \cap r = x \), contrary to the maximality of \( r \). Hence \( r \cup (x' \cap r') = r' \succ r \). The upper covering condition implies \( x' \cap r' \succ r \cap (x' \cap r') = x' \cap r = x \). Since \( x \prec x' \cap r' \prec x' \) and \( r \) is meet irreducible in terms of elements of \( [x, x'] \), by applying Lemma 6.2 in the sublattice \( [x, x'] \) we see that \( r \) is completely meet irreducible in terms of elements of this interval.

The following result will be needed later.

Lemma 6.6. If \( x \) is a torsion free element of a modular lattice and \( x = \cap R \) is an irredundant representation in terms of meet irreducible elements then each element of
R is torsion free.

Proof. Suppose \( r \in \mathcal{R} \) is not torsion free and \( r' \not\geq r \). By Lemma 6.2, \( r \) is completely meet irreducible. Then since \( \bigcap (R - r) \cap x \) and \( r \cap \bigcap (R - r) = x \) the preceding lemma implies that \( x \) is completely meet irreducible in terms of elements of \([x, \bigcap (R - r)]\). Applying Lemma 6.1 in the sublattice \([x, \bigcap (R - r)]\) we see there must exist \( x' \) such that \( x \not< x' \leq \bigcap (R - r) \), contrary to the hypothesis that \( x \) is torsion free. Hence each element of \( R \) is torsion free.

Theorem 6.1. Suppose that \( L \) is a modular lattice in which each element has an irredundant representation in terms of meet irreducible elements. Then for each \( x \not< l \) there exists \( x' \not> x \) such that \( x \) is meet irreducible in terms of elements of \([x, x']\). If, in addition, \( L \) is upper continuous then whenever \( x \not< y \) there exists \( x' \) such that \( x \not< x' \leq y \) and \( x \) is meet irreducible in terms of elements of \([x, x']\).

Proof. Suppose \( x \not< l \) and let \( x = \bigcap R \) be an irredundant representation in terms of meet irreducibles. Then \( R \) is not empty. Let \( r \) be any element of \( R \) and let \( x' = \bigcap (R - r) \). Since \( x' \not> x \) and \( x' \cap r = x \), Lemma 6.5 implies that \( x \) is meet irreducible in terms of elements of \([x, x']\).

Now suppose \( L \) is upper continuous and \( x \not< y \). By Lemma 3.1 there exists an element \( z \) which is maximal with respect to the property \( z \cap y = x \). If \( z = l \) then \( x = l \cap y = y \), contrary to \( x \not< y \). Hence \( z \not< l \) and by the first part of this theorem there exists \( z' \not> z \) such that \( z \) is meet irreducible in terms of
elements of \([z, z']\). The maximality of \(z\) implies \(z' \cap y > x\). Again applying Lemma 3.1, let \(r\) be maximal with respect to the property \(r \cap z' = z\). By Lemma 6.3, \(r\) is meet irreducible. Since \(r \cap (z' \cap y) = z \cap y = x\), Lemma 6.5 implies that \(x\) is meet irreducible in terms of elements of \([x, z' \cap y]\). Thus \(x' = z' \cap y\) is the desired element.

The above result concerning meet irreducible elements is analogous to the following one, basically due to Dilworth and Crawley, dealing with completely meet irreducible elements (2).

Theorem 6.2. Suppose that \(L\) is a modular lattice in which each element has an irredundant representation in terms of completely meet irreducible elements. Then \(L\) is a torsion lattice. If, in addition, \(L\) is upper continuous then \(L\) is atomic.

Proof. Suppose \(x\) is torsion free and \(x = \bigcap R\) is an irredundant representation in terms of completely meet irreducible elements. By Lemma 6.6 each element of \(R\) is torsion free. But Lemma 6.2 implies that each element of \(R\) is not torsion free. Hence \(R\) is empty and \(x = \bigcap R = 1\). Thus \(L\) is a torsion lattice. Theorem 3.1 then yields the second assertion.
7. IRREDUNDANT REPRESENTATIONS

In this section we will obtain various conditions for the existence of irredundant representations in upper continuous lattices. The technique used is an extension of one developed by Crawley (1). In attempting to obtain a representation of an element $x$ of a lattice $L$ we will consider quadruples of the form $(x,y,R,θ)$ where:

(i) $y \in L$, $R \subseteq L$, and $θ \in C(L)$.
(ii) Each $r \in R$ is meet irreducible.
(iii) $x = y \cap R$.
(iv) $y$ is maximal with respect to (iii).
(v) $x < y \cap (R-r)$ for each $r \in R$.
(vi) $v_θ(y) = y$.

Note that with each element $x$ there is associated a trivial quadruple $(x,x,\emptyset,0)$. Also note that a quadruple of the form $(x,1,R,1)$ corresponds to an irredundant representation of $x$ in terms of meet irreducible elements.

We may define a relation among these quadruples by saying $(x,y_1,R_1,θ_1) ≤ (x,y_2,R_2,θ_2)$ if and only if:

(vii) $y_1 ≤ y_2$, $R_1 \subseteq R_2$, $θ_1 ≤ θ_2$.
(viii) $(y_1,y_2,R_2-R_1,θ_2)$ is an admissible quadruple.
(ix) $v_{θ_1}(r) = r$ for each $r \in R_2-R_1$.

Thus if $(x,y_1,R_1,θ_1) ≤ (x,y_2,R_2,θ_2)$ then the representation $x = y_2 \cap \bigcap R_2$ is a refinement of the representation $x = y_1 \cap \bigcap R_1$. We first verify that this relation is a partial
ordering. It is clear that it is reflexive and antisymmetric. We will show that it is transitive. Assume 
\((x, y_1, R_1, \theta_1) \leq (x, y_2, R_2, \theta_2)\) and \((x, y_2, R_2, \theta_2) \leq (x, y_3, R_3, \theta_3)\).

Then \((y_1, y_2, R_2 - R_1, \theta_2)\) and \((y_2, y_3, R_3 - R_2, \theta_3)\) are admissible, 
\(v_{\theta_1}(r) = r\) for each \(r \in R_2 - R_1\), and \(v_{\theta_2}(r) = r\) for each \(r \in R_3 - R_2\).

We must verify that \((y_1, y_3, R_3 - R_1, \theta_3)\) is admissible and 
that \(v_{\theta_1}(r) = r\) for each \(r \in R_3 - R_1\). The latter assertion is clear in view of Lemma 4.5. Condition (iii) is satisfied since \(y_3 \cap (R_3 - R_1) = y_3 \cap (R_3 - R_2) \cap (R_2 - R_1) = y_2 \cap (R_2 - R_1) = y_1\).

Suppose \(y' \geq y_3\) and \(y' \cap (R_3 - R_1) = y_1\). Since \([y' \cap (R_3 - R_2)] \cap (R_2 - R_1) = y' \cap (R_3 - R_1) = y_1\) and \(y' \cap (R_3 - R_2) \geq y_3 \cap (R_3 - R_2) = y_2\), the maximality of \(y_2\) implies \(y' \cap (R_3 - R_2) = y_2\). Then the maximality of \(y_3\) implies \(y' = y_3\). Thus \(y_3\) is maximal with respect to the property \(y_3 \cap (R_3 - R_1) = y_1\) and \((y_1, y_3, R_3 - R_1, \theta_3)\) satisfies (iv). If \(r \in R_3 - R_2\) then \(y_2 < y_3 \cap (R_3 - R_2 - r)\) and by the maximality of \(y_2, y_3 \cap (R_3 - R_1 - r) = [y_3 \cap (R_3 - R_2 - r)] \cap (R_2 - R_1) > y_1\). Suppose \(r \in R_2 - R_1\) and \(y_3 \cap (R_3 - R_1 - r) = y_1\).

Then \(y_1 = y_3 \cap (R_3 - R_2) \cap (R_2 - R_1 - r) \cap (R_2 - R_1 - r)\), contrary to the admissibility of \((y_1, y_2, R_2 - R_1, \theta_2)\). Thus \(y_1 < y_3 \cap (R_3 - R_1 - r)\) for each \(r \in R_3 - R_1\) and \((y_1, y_3, R_3 - R_1, \theta_3)\) is admissible. Hence \((x, y_1, R_1, \theta_1) \leq (x, y_3, R_3, \theta_3)\) and the relation is transitive. Note that we have actually shown that if \((x, y, R, \theta)\) and \((y, z, S, \theta')\) are admissible then \((x, z, R \cup S, \theta')\) is admissible.

Lemma 7.1. Suppose \(L\) is an upper continuous lattice, \(x \in L\), and \(\theta_0 \in C(L)\). Then the partially ordered set
consisting of admissible quadruples of the form \((x, y, R, \theta)\), \(\theta \leq \theta_0\), contains maximal elements. If \((x, y, R, \theta)\) is a maximal quadruple then \(\phi_\theta(y)\) has the property that there exist no elements \(a, b \in L/\theta\) such that \(a\) is meet irreducible, \(b \geq \phi_\theta(y)\), and \(a \cap b = \phi_\theta(y)\). In particular \(\phi_\theta(y)\) is torsion free.

**Proof.** Let \(P\) be the collection of all chains of such triples which contain \((x, x, \emptyset, 0)\). Partially order \(P\) by set inclusion. \(P\) is not empty since it contains \(\{(x, x, \emptyset, 0)\}\). It is clear that the union of a chain of elements of \(P\) is again an element of \(P\). Thus \(P\) contains maximal elements. Suppose that \(M = \{(x, y_a, R_a, \theta_a)\}_{a \in A}\) is a maximal chain. We will show that \((x, \bigcup_{a \in A} y_a, \bigvee_{a \in A} R_a, \bigcup_{a \in A} \theta_a) \in M\). It then follows that this is a maximal quadruple.

Let \(R = \bigvee_{a \in A} R_a\) and \(\theta = \bigcup_{a \in A} \theta_a\). Since \(M\) is a chain the definition of the partial ordering implies that \(\{y_a\}_{a \in A}\) is also a chain. If \((x, y_a, R_a, \theta_a) \in M\) and \(r \in R - R_a\) then there exists \((x, y_\beta, R_\beta, \theta_\beta) \in M\) such that \((x, y_a, R_a, \theta_a) \leq (x, y_\beta, R_\beta, \theta_\beta)\) and \(r \in R_\beta - R_a\). Then \(y_a = y_\beta \cap (R_\beta - R_a) \leq x\). That is, if \(r \in R - R_a\) then \(r \geq y_a\). In particular, since \((x, x, \emptyset, 0)\) \(M\) we have \(r \geq x\) for each \(r \in R\). By upper continuity, \((\bigcup_{a \in A} y_a)\n R = \bigcup_{a \in A} (y_a \cap R) = \bigcup_{a \in A} [y_a \cap (\bigcap_{a \in A} (\bigcap_{a \in A} R_a))]\)

\[= \bigcup_{a \in A} [x \cap (\bigcap_{a \in A} (R_a))]\] = \(x\). Lemma 3.1 implies the existence of an element \(y \geq \bigcup_{a \in A} y_a\) which is maximal with respect to the property \(y \cap R = x\). We first show that
(x, y, R, θ) is an admissible quadruple. By our choice of y, R, and θ, conditions (i) through (iv) are satisfied. If 
(x, y_α, R_α, r_α) ∈ M then y ≥ y_α and r ≥ r_α for each r ∈ R - R_α. Thus 
y ∩ (R - R_α) ≥ y_α. Also y ∩ (R - R_α) ∩ R_α = y ∩ R = x. Since 
(x, y_α, R_α, θ_α) ≥ (x, x, θ, 0) the maximality of y_α implies 
yn ∩ (R - R_α) = y_α for each α ∈ A. If r ∈ R there exists 
(x, y_α, R_α, θ_α) ∈ M such that r ∈ R_α. Then 
y ∩ (R - R_α) ∩ (R_α - r) = y_α ∩ (R_α - r) ≥ x. Thus (x, y, R, θ) satisfies 
(v). If (x, y_α, R_α, θ_α) ∈ M and r ∈ R - R_α then there exists 
(x, y_β, R_β, θ_β) ∈ M such that (x, y_α, R_α, θ_α) ≤ (x, y_β, R_β, θ_β) and 
r ∈ R_β - R_α. By the definition of the partial ordering we have 
ν_α (r) = r. That is, if r ∈ R - R_α then ν_α (r) = r. Since 
v_α (y_α) = y_α and y = y ∩ (R - R_α) we have y_α = ν_α (y_α) = 
v_α (y ∩ (R - R_α)) = ν_α (y) ∩ ν_α (∩ (R - R_α)) = ν_α (y) ∩ ∩ (R - R_α).
Since ν_α (y) ≥ y and ν_α (y) ∩ R = ν_α (y) ∩ (R - R_α) ∩ R_α = y_α ∩ R_α = x our choice of y implies ν_α (y) = y. Since ν_α (y) = y for 
each θ_α, Lemma 4.8 implies ν_α (y) = y. Thus (x, y, R, θ) satisfies (vi).

We next show that (x, y, R, θ) ≥ (x, y_α, R_α, θ_α) for α ∈ A. Since 
we have already seen that ν_α (r) = r for each α it suffices to 
show that (y_α, y, R - R_α, θ) is an admissible quadruple. Con-
ditions (iii) and (vi) have already been verified. We next 
establish (iv). Suppose y' ≥ y and y' ∩ (R - R_α) = y_α. Then 
y' ∩ R = y' ∩ (R - R_α) ∩ R_α = y_α ∩ R_α = x and by our choice of y 
we have y' = y. It only remains to verify (v). Suppose 
r ∈ R - R_α and y ∩ (R - R_α - r) = y_α. Then y ∩ (R - r)
\[ y \cap R_\alpha \cap (R - R_\alpha - r) = y_\alpha \cap R_\alpha = x, \] contrary to the admissibility of \( (x, y, R, \theta) \). Thus \( y \cap (R - R_\alpha - r) > y_\alpha \) for each \( r \in R - R_\alpha \) and \( (x, y, R, \theta) > (x, y_\alpha, R_\alpha, \theta_\alpha) \).

It follows from the maximality of \( M \) that \( (x, y, R, \theta) \in M \).

It is then clear that this is a maximal quadruple such that \( \theta \leq \theta_0 \). The proof of the lemma will be complete when it has been shown that \( \phi_\theta(y) \) has the stated properties.

Suppose \( a \land b = \phi_\theta(y) \) where \( a \) is meet irreducible and \( b > \phi_\theta(y) \).

By Lemma 4.4, \( L/\theta \) is upper continuous. Lemma 3.1 then implies the existence of a \( b_1 \geq b \) which is maximal with respect to the property \( a \land b_1 = \phi_\theta(y) \). Let \( a' \) and \( b' \) be such that \( \phi_\theta(a') = a \) and \( \phi_\theta(b') = b_1 \). Let \( y' = v_\theta(b') \) and \( r' = v_\theta(a') \).

By Lemma 4.2, \( r' \) is meet irreducible. It is easily verified that \( (y, y', \{ r' \}, \theta) \) is an admissible quadruple.

Since \( v_\theta(r') = r' \) it follows that \( (x, y, R, \theta) \preceq (x, y', RV[r'], \theta) \).

Since \( \phi_\theta(y') = b_1 \succ \phi_\theta(y), v_\theta(y') = y', \) and \( v_\theta(y) = y \), we have \( y' \succ y \). Hence \( (x, y, R, \theta) < (x, y', RV[r'], \theta) \), contrary to the maximality of \( (x, y, R, \theta) \). Thus there cannot exist elements \( a, b \) with the stated properties. Suppose \( \phi_\theta(y) \) is not torsion free. Then there exists \( b \) such that \( b > \phi_\theta(y) \). Let \( a \) be maximal with respect to \( a \land b = \phi_\theta(y) \). Lemma 6.4 implies that \( a \) is completely meet irreducible. But this contradicts the property of \( \phi_\theta(y) \) which was just verified. Thus \( \phi_\theta(y) \) is torsion free.

Theorem 7.1. An upper continuous lattice \( L \) has the property that each element possesses an irredundant repre-
sentation in terms of meet irreducible elements if and only if for each \( y < 1 \) there exist elements \( a \) and \( b \) such that \( a \) is meet irreducible, \( b > y \), and \( ab = y \).

Proof. Suppose \( y \in L \), \( y < 1 \), and \( y \) has an irredundant representation in terms of meet irreducible elements, \( y = \bigcap R \). Then \( R \) is not empty and we may choose \( a \in R \) and \( b = \bigcap (R - a) \).

Suppose \( x \in L \) and take \( \theta = 0 \) in the preceding lemma. If \( (x, y, R, \theta) \) is a maximal quadruple then \( \theta = 0 \), \( x = y \bigcap \bigcap R \), \( x < y \bigcap \bigcap (R - r) \) for each \( r \in R \), and there exist no elements \( a \) and \( b \) such that \( a \) is meet irreducible, \( b > y \), and \( ab = y \).

If the only element of \( L \) which has the last of these properties is \( 1 \) we conclude that \( y = 1 \). Then \( x = \bigcap R \) is an irredundant representation.

One can use the above to obtain conditions which are sufficient for the existence of irredundant representations.

Theorem 7.2. Suppose that \( L \) is an upper continuous lattice and for each \( x < 1 \) there exists \( x' > x \) such that \( x \) is meet irreducible in terms of elements of \( [x, x'] \). Then each element of \( L \) has an irredundant representation in terms of meet irreducibles.

Proof. For \( x < 1 \) let \( x' \) be as in the statement of the theorem. By Lemma 3.1 there exists an element \( r \) which is maximal with respect to the property \( r \bigcap x' = x \). Lemma 6.3 implies \( r \) is meet irreducible. The preceding theorem there-
fore yields the desired conclusion.

The analogous result dealing with completely meet irreducible elements is the following.

**Theorem 7.3.** If $L$ is an upper continuous torsion lattice then each element of $L$ has an irredundant representation in terms of completely meet irreducible elements.

**Proof.** If $x' \geq x$ then $x$ is meet irreducible in terms of elements of $[x, x']$. The preceding theorem therefore implies that each element of $L$ has an irredundant representation in terms of meet irreducible elements. Suppose $y = \bigcap R$ is such a representation. Irredundancy implies $1 \notin R$. Hence no element of $R$ is torsion free. Lemma 6.2 then implies that each element of $R$ is completely meet irreducible.

Combining Theorem 7.2 with Theorem 6.1 one obtains the following.

**Theorem 7.4.** If $L$ is an upper continuous modular lattice then the following are equivalent.

(i) Each element of $L$ has an irredundant representation in terms of meet irreducible elements.

(ii) For each $x < 1$ there exists $x' > x$ such that $x$ is meet irreducible in terms of elements of $[x, x']$.

(iii) Whenever $x < y$ there exists $x'$ such that $x < x' < y$ and $x$ is meet irreducible in terms of elements of $[x, x']$.

Similarly, by combining Theorem 7.3 with Theorem 6.2 one has:

**Theorem 7.5.** If $L$ is an upper continuous modular
lattice then the following are equivalent.

(i) Each element of \( L \) has an irredundant representation in terms of completely meet irreducible elements.

(ii) For each \( x < l \) there exists \( x' \) such that \( x \prec x' \).

(iii) Whenever \( x < y \) there exists \( x' \) such that \( x \prec x' \prec y \).

We now state our principal result.

Theorem 7.6. If \( L \) is an upper continuous modular lattice then each element of \( L \) has an irredundant representation in terms of meet irreducible elements if and only if each element of \( L^* \) has such a representation.

Proof. Suppose each element of \( L^* \) has such a representation and let \( x \) be any element of \( L \). In Lemma 7.1 take \( \theta_0 = \theta^* \) to obtain a maximal quadruple \( (x, y, R, \theta) \), \( \theta \subseteq \theta^* \). Since \( v_\theta(y) = y \) and \( \phi_\theta(y) \) is torsion free it follows from Lemma 5.2 that \( v_{\theta^*}(y) = y \). Since \( \theta \subseteq \theta^* \) we have \( \theta \subseteq \theta^* \subseteq \theta^* \). Thus \( (x, y, R, \theta^t) \) is admissible and the maximality of \( (x, y, R, \theta) \) implies \( \theta = \theta^t \). It follows that \( \theta = \theta^* \) and \( v^*(y) = y \). By hypothesis \( \phi^*(y) \) has an irredundant representation in terms of meet irreducible elements of \( L^* \). Then Lemma 4.3 implies that \( y \) has an irredundant representation in terms of meet irreducibles, \( y = \bigcap S \), where \( v^*(x) = x \) for each \( s \in S \). Then \( (y, l, S, 0^*) \) is admissible and \( (x, l, R \setminus S, 0^*) \subseteq (x, y, R, 0^*) \). Hence \( x = \bigcap (R \setminus S) \) is a representation of the desired type.

Now suppose each element of \( L \) has such a representation and let \( x \) be any element of \( L^* \). Let \( x' \in L \) be such that
\( \phi'(x') = x \) and let \( y = \psi(x') \). Suppose \( y = \bigwedge R \) is an irredundant representation in terms of meet irreducibles. We will show that \( \psi(r) = r \) for each \( r \in R \). Since \( \phi'(y) = \phi'(x') = x \) it then follows from Lemma 4.3 that \( x = \bigwedge_{r \in R} \phi'(r) \) is a representation having the desired properties. Let \( S = \{ \theta' \in C(L) \mid \theta' \leq 0^* \}, \ 
 \psi_{\theta'}(r) = r \) for each \( r \in R \} \) and let \( \theta = \bigcup S \). Lemma 4.8 implies \( \theta \in S \). By Lemma 4.3, \( \phi_{\theta}(y) = \bigwedge_{r \in R} \phi_{\theta}(r) \) is an irredundant representation of \( \phi_{\theta}(y) \) in terms of meet irreducible elements of \( L/\theta \). Since \( \theta \leq 0^* \) we have \( \theta \leq 0^t \leq 0^* \). Since \( \psi(y) = y \) it follows from Lemma 4.5 that \( \psi_{\theta}(y) = \psi_{\theta t}(y) = \psi(y) = y \). Then Lemma 5.2 implies \( \phi_{\theta}(y) \) is a torsion free element of \( L/\theta \).

By Lemma 6.6, \( \{ \phi_{\theta}(r) \}_{r \in R} \) is a collection of torsion free elements of \( L/\theta \). Since \( \psi_{\theta}(r) = r \) for each \( r \in R \) it follows from Lemma 5.2 that \( \psi_{\theta t}(r) = r \) for each \( r \in R \). Then \( \theta t \in S \) and \( \theta = \bigcap S \geq \theta t \). Hence \( \theta = \theta t = 0^* \) and \( \psi(r) = r \) for each \( r \in R \).

More generally, one can show that if \( L \) is an upper continuous modular lattice, \( \theta \in C(L) \), and \( \theta \leq \theta' \leq \theta^* \), then each element of \( L/\theta \) has an irredundant representation in terms of meet irreducible elements if and only if each element of \( L/\theta' \) has such a representation.

As an immediate consequence of Theorem 7.6 we have the following.

**Theorem 7.7.** If \( L \) is an upper continuous modular lattice and \( L^* \) consists of one element then each element of \( L \) has an irredundant representation in terms of meet irreducible elements.
Note that this implies the existence of representations in modular lattices which satisfy the ascending chain condition and in atomic, upper continuous, modular lattices.

We conclude this section with an example of a very well behaved lattice which contains elements having no irredundant representation in terms of meet irreducible elements. Let \( L \) be the collection of all monotone non-decreasing functions of the closed real unit interval into itself. If suprema and infema are defined pointwise then \( L \) becomes a complete lattice. One can then easily verify the following remarks.

(i) \( L \) inherits very strong regularity properties from the unit interval, including modularity and upper continuity.

(ii) The meet irreducible elements of \( L \) are those functions which have the form:
\[
\begin{align*}
  f(x) &= y_0 \text{ if } x < x_0, \\
  f(x_0) &= \text{either } y_0 \text{ or } 1, \\
  f(x) &= 1 \text{ if } x > x_0,
\end{align*}
\]
where \( 0 \leq x_0 \leq 1 \) and \( 0 \leq y_0 \leq 1 \).

(iii) Among others, the function \( f(x) = x \) does not possess an irredundant representation in terms of meet irreducible elements.
8. APPLICATIONS

We will now give some results which sometimes allow one to infer the existence of irredundant representations in an upper continuous modular lattice from properties of a set of generators of the lattice. Such results have natural applications in compactly generated modular lattices. In particular, these results apply to module theory and we will give examples in this field.

Theorem 8.1. Suppose \( L \) is an upper continuous modular lattice, \( G \subseteq L \), \( \bigcup G = 1 \), and each of the lattices \( L_g = [0, g], g \in G \), has the property that each element possesses an irredundant representation in terms of meet irreducible elements of \( L_g \). Then each element of \( L \) has an irredundant representation in terms of meet irreducible elements.

Proof. We will show that \( L \) satisfies (ii) of Theorem 7.4. Suppose \( x < 1 \). Then there exists \( g \in G \) such that \( g \not\leq x \). It follows that \( x \land g < g \). Since each element of \( L_g \) has an irredundant representation in terms of meet irreducible elements of \( L_g \), Theorem 7.4 (ii) implies the existence of \( y \) such that \( x \land g < x \land y \) and \( x \land g \) is meet irreducible in terms of elements of \( [x \land g, y] \). It is easily seen that \( x \land y = x \land g \). Modularity therefore implies that the intervals \( [x \land g, y] \) and \( [x, x \land y] \) are isomorphic. Hence \( x < x \land y \) and \( x \) is meet irreducible in terms of elements of \( [x, x \land y] \).

The converse to the above is somewhat trivial. For
suppose \( L \) is an upper continuous modular lattice and \( g \in L \). If each element of \( L \) has an irredundant representation in terms of meet irreducible elements then, by Theorem 7.4 (iii), whenever \( x < g \) there exists \( x' \) such that \( x < x' \leq g \) and \( x \) is meet irreducible in terms of elements of \([x, x']\). Then the lattice \( L_g = [0, g] \) satisfies condition (ii) of Theorem 7.4. Therefore each element of \( L_g \) has an irredundant representation in terms of meet irreducible elements of \( L_g \).

As an immediate consequence of Theorem 8.1 we have the following.

**Corollary 8.1.** Suppose \( L \) is a compactly generated modular lattice and each of the lattices \( L_k = [0, k] \), \( k \) compact, has the property that each element possesses an irredundant representation in terms of meet irreducible elements of \( L_k \). Then each element of \( L \) has an irredundant representation in terms of meet irreducible elements.

**Corollary 8.2.** Suppose \( L \) is a compactly generated modular lattice in which every element which lies below a compact element is compact. Then each element of \( L \) has an irredundant representation in terms of meet irreducible elements.

**Proof.** If \( k \) is a compact element such that each element below \( k \) is also compact then the ascending chain condition holds in \([0, k]\). Thus Corollary 8.1 yields the
desired conclusion.

An example of a lattice satisfying the hypothesis of the preceding corollary would be the lattice of submodules of a left module over a left noetherian ring.

Results analogous to those given above also allow one to conclude that a lattice \( L \) has the property that \( L^* \) consists of one element.

Theorem 8.2. Suppose \( L \) is an upper continuous modular lattice, \( G \subseteq L, \bigcup G = 1 \), and each of the lattices \( L_g = [0,g], g \in G, \) has the property that \( L_g^* \) consists of one element. Then \( L^* \) consists of one element.

The proof requires the following lemma.

Lemma 8.1. An upper continuous modular lattice \( L \) has the property that \( L^* \) consists of one element if and only if the only subset \( S \) of \( L \) which has the properties listed below is the set \( \{1\} \).

(i) \( 1 \in S \).

(ii) \( S \) is closed under intersections.

(iii) Letting \( w(x) = \bigcap \{ s \in S | s \geq x \} \) the equation \( w(xny) = w(x) \cap w(y) \) holds for all \( x, y \in L \).

(iv) Between any two comparable elements of \( S \) there is a third.

Proof. Lemma 4.1 and Lemma 5.3 imply that \( \{v^*(x)\}_{x \in L} \) is a set having these properties. If \( \{1\} \) is the only such set then \( v^*(x) = 1 \) for each \( x \in L \). Consequently \( \phi^*(x) = \phi^*(v^*(x)) = \phi^*(1) \) for each \( x \in L \) and \( L^* \) consists of one
Conversely, suppose \( L^* \) consists of one element and \( S \) has the listed properties. By Lemma 4.1 there exists \( \theta \in C(L) \) such that \( S = \{ v_{\theta}(x) \} \). Lemma 5.3 implies that \( L/\theta \) contains no covers. It follows from Theorem 5.3 that there is a complete join epimorphism \( \psi : L^* \to L/\theta \). Since \( L^* \) consists of one element it must therefore be the case that \( L/\theta \) contains only one element. Then for each \( x \in L \) we have \( \phi_{\theta}(x) = \phi_{\theta}(1) \) and therefore \( v_{\theta}(x) = v_{\theta}(1) = 1 \). Thus \( S = \{ 1 \} \).

Proof of Theorem 8.2. For \( g \in G \) let \( S = \{ gnv^*(x) \} \). It can be verified that \( S \) is a subset of \( L_g \) which satisfies the conditions listed in the preceding lemma. Since \( L^*_g \) consists of one element it follows that \( S = \{ g \} \). Hence \( g \leq v^*(x) \) for each \( g \in G \) and each \( x \in L \). Then \( v^*(x) \leq U G = 1 \) and \( v^*(x) = 1 \) for each \( x \in L \). Consequently \( v^*(x) = v^*(v^*(x)) = \psi^*(1) \) for each \( x \in L \) and \( L^* \) consists of one element.

Corollary 8.3. Suppose \( L \) is a compactly generated modular lattice and each of the lattices \( L_k = [0,k] \), \( k \) compact, has the property that \( L_k^* \) consists of one element. Then \( L^* \) consists of one element.

Corollary 8.2 may be sharpened somewhat.

Corollary 8.4. Suppose \( L \) is a compactly generated modular lattice in which every element which lies below a compact element is compact. Then \( L^* \) consists of one element.

Proof. Since each of the lattices \( L_k = [0,k], k \)
compact, satisfies the ascending chain condition the preceding corollary yields the desired conclusion.

We will conclude by giving two examples of applications in module theory. By a ring \( R \) we will mean a commutative ring with identity. Suppose \( M \) is an \( R \) module, \( \{ P_a \}_{a \in A} \) is a collection of submodules of \( M \), and \( \bigcap_{a \in A} P_a = 0 \). Then \( M \) has a canonical subdirect product representation, \( M \cong \prod_{a \in A} M/P_a \). If \( P_a \) is a meet irreducible submodule then the rank of \( M/P_a \) is one. If the meet representation of \( 0 \) is irredundant then each of the factors \( M/P_a \) in the subdirect product representation has a non-zero intersection with \( M \). Thus results regarding the existence of representations in the lattice of submodules of \( M \) imply the existence of various types of subdirect product representations of \( M \).

The lattice of submodules of a module \( M \) is a compactly generated modular lattice.

**Theorem 8.3.** The following are equivalent.

(i) Each ideal of the ring \( R \) has an irredundant representation in terms of meet irreducible ideals.

(ii) If \( M \) is any \( R \) module then every submodule of \( M \) has an irredundant representation in terms of meet irreducible submodules.

**Proof.** Let \( L(R) \) denote the lattice of ideals of \( R \) and if \( M \) is an \( R \) module let \( L(M) \) denote the lattice of submodules of \( M \). The first condition is a trivial conse-
quence of the second since $R$ is itself an $R$ module and the submodules of $R$ are its ideals. Suppose $R$ satisfies (i) and $M$ is any $R$ module. For $m \in M$ let $(m)$ denote the submodule of $M$ generated by $m$. Then $I=\{r \in R | rm=0\}$ is an ideal of $R$ and there is a module isomorphism between $(m)$ and $R/I$. Hence the intervals $\[0,(m)\] \subseteq L(M)$ and $[I,R] \subseteq L(R)$ are isomorphic. Since each element of $[I,R]$ has an irredundant representation in terms of meet irreducible elements it follows that each element of the lattice $L_{(m)}=[0,(m)]$ has such a representation. The desired conclusion then follows from Theorem 8.1 by taking $G=\{(m)\}_{m \in M}$.

Theorem 8.4 The following are equivalent.

(i) The lattice $L(R)$ of ideals of the ring $R$ has the property that $L(R)^*$ consists of one element.

(ii) If $M$ is any $R$ module then the lattice $L(M)$ of submodules has the property that $L(M)^*$ consists of one element.

Proof. Again it is the case that (i) is a trivial consequence of (ii). Suppose (i) holds and $M$ is an $R$ module. If $m \in M$ there is an ideal $I$ of $R$ such that the interval $[I,R]$ of $L(R)$ is isomorphic to the interval $[0,(m)]$ of $L(M)$. The only subset of $L(R)$ satisfying the conditions listed in Lemma 8.1 is the set $\{R\}$. In particular, the only such subset of $[I,R]$ is $\{R\}$. Hence the only subset of the lattice $L_{(m)}=[0,(m)]$ satisfying those condi-
tions is \{(m)\}. Therefore \(L^*(m)\) consists of one element for each \(m \in M\) and the desired conclusion is obtained from Theorem 8.2 by taking \(G=\{(m)\}_{m \in M}\).

Theorem 8.2 implies that a ring which is a direct sum of noetherian rings satisfies condition (i) of the above theorem.
9. BIBLIOGRAPHY


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