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Some aspects of combinability of information

by

Kenneth Earl Merritt

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of DOCTOR OF PHILOSOPHY

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I. INTRODUCTION

One of the principal objectives of this dissertation is to study the problems of combining information from several different sources under a wide variety of population conditions. Developments made in linear estimation, variances, and estimates of variances are directed toward numerical implementation on a high-speed digital computer so that numerical, as well as mathematical, comparisons can be made between various proposed estimators. This research illustrates the importance and effectiveness of utilizing a high-speed computer for studying estimation techniques under various simulated model conditions. This approach is not needed or desired for problems that can be solved directly by general and complete mathematical formulation. For our problem of combining information, such formulation is unwieldy and difficult at best, as is adequately demonstrated in succeeding chapters of this dissertation.

A brief review of literature covering pertinent discussions of various aspects of the problem of combining information and applications to incomplete block designs is presented in Chapter II. Also, a brief discussion of minimal sufficiency in estimation of incomplete block parameters as well as some general estimability considerations are presented.

The research of succeeding chapters is essentially directed toward the central theme of formulating and estimating
unconditional variance for estimators of the parameter $\beta$ from the general linear model $Y = X\beta + \epsilon$, where $Y$ is an $(n \times 1)$ vector of observations, $X$ is an $(n \times p)$ matrix of known coefficients, $\beta$ is a $(p \times 1)$ vector of unknown parameters, and $\epsilon$ is an $(n \times 1)$ vector of residuals such that $E\epsilon\epsilon' = V$. Developments in estimation of variances are made in Chapter III under the above model assumptions with variance-covariance matrix $V$ being positive definite, symmetric, and nonsingular. Under the assumption of normality, a maximum likelihood estimate is obtained for $V$, denoted by $S$, and Wishart distributional properties are applied in obtaining the variance of a combined estimator of $\beta$ formed by utilizing the estimated elements of $V$.

Direct formulation of the $c^{th}$ moment of a random weight is presented in Chapter IV. These random weights are used in combining information under a diagonal variance-covariance population structure. Special attention is directed toward the case involving equal sampling from two independent and uncorrelated sources. In Chapter V applications of combining information using random weights are extended to a wide class of incomplete block designs. Special attention is given to the implementation of general algorithms for solving problems involving both single and multiple incomplete block experimental designs. Methods are discussed for obtaining and
utilizing design association parameters in estimation and analysis of variance.

A Monte Carlo study was conducted and results tabulated in Chapter VI to compare relative gains and/or losses for a variety of estimating procedures and population structures discussed in previous chapters. A great deal of interest is focused on the relative increase in variance of simple least squares estimators over either best linear unbiased or asymptotically efficient proposed estimators in heterogeneous populations. Particular interest is directed toward relative estimating techniques of variances in examining adequacy of estimation and model assumptions.

Finally in Chapter VII, results and conclusions of this research are summarized.
II. REVIEW OF LITERATURE

A. Preliminary Developments of Combinability

One of the important questions in statistics is: What is the best way to combine information from several sources, possibly having unequal variances? As was mentioned in the introduction, much of this dissertation is concerned with the problem of weighted estimates with special application to combining inter- and intra-block information in incomplete block designs. Developments are made under different model assumptions to give as wide a range of applicability as possible.

Perhaps the first effort to gain additional information by combining estimates for incomplete block designs was by Yates (1936, 1940). His procedure, developed for the combining of inter- and intra-block estimates for the balanced incomplete block design, has been widely applied to other incomplete block designs as well as to several independent estimates of a given estimable parameter under study.

Cochran (1937) found an approximation to the variance of a combined weighted mean for the case of a large number of independent estimates of the parameter $\mu$ having equal sample size $n$. The estimator has the form

$\bar{\mu} = \frac{1}{k} \sum_{i=1}^{k} \hat{\mu}_i,$

(2.1)
where $\hat{w}_i$ is the estimated weight and $\hat{\mu}_i$ is an unbiased estimate of $\mu$ from the $i^{th}$ source, $i = 1, 2, \ldots, k$. The approximation of the variance of $\bar{\mu}$ is

\[(2.2) \quad \text{Var}(\bar{\mu}) = \frac{1}{\hat{w}} [1 + \frac{2}{n-4}], \]

which can be estimated unbiasedly by

\[(2.3) \quad \text{Var}(\bar{\mu}) = \frac{1}{\hat{w}} [1 + \frac{4}{n-4}], \]

where

$$\hat{w} = \sum_{i=1}^{k} \frac{1}{\text{Var}(\hat{\mu}_i)}$$

and

$$\hat{\hat{w}} = \sum_{i=1}^{k} \frac{1}{\text{Var}(\hat{\mu}_i)}.$$ 

$\text{Var}(\hat{\mu}_i)$ is an unbiased estimate of the variance of $\hat{\mu}_i$, $i = 1, 2, \ldots, k$.

Meier (1953) considered the case of unequal sample sizes from $k$ independent sources. The estimator proposed was of the form (2.1). $\bar{\mu}$ was found to be an asymptotically efficient estimator of $\mu$ when $n_i$ was large for all $i$, $i = 1, 2, \ldots, k$. The method of steepest descent was used. An expression for
the first order asymptotic variance of \( \bar{\mu} \) was found to be

\[
\text{Var}(\bar{\mu}) = \frac{1}{w} \left( 1 + 2 \sum_{i=1}^{k} \frac{1}{n_i} w_i (1 - w_i) + o \left( \frac{1}{n_i} \right) \right),
\]

where \( w_i = \frac{1}{\sigma_i^2} \) with an approximate unbiased estimator of the form

\[
\text{Var}^*(\bar{\mu}) = \frac{1}{w} \left( 1 + 4 \sum_{i=1}^{k} \frac{1}{n_i} \hat{w}_i (1 - \hat{w}_i) \right),
\]

where \( w \) and \( \hat{w} \) are defined in (2.3).

Graybill and Deal (1959) exhibited the minimum degrees of freedom for the case of two independent sources of unequal sample size that would be necessary to yield a combined estimator that would be uniformly better than either of the individual estimators. The combined estimator is of the form

\[
\bar{\mu} = \frac{x_{\hat{w}_1} + x_{\hat{w}_2}}{\hat{w}_1 + \hat{w}_2},
\]

where \( \hat{w}_i = n_i/s_{i}^2 \), and \( s_i^2 \) is an unbiased estimate of the variance of an individual observation from the \( i \)th source, \( i = 1, 2 \).

Under conditions of mutual independence of \( \overline{x}_1 \), \( \overline{x}_2 \), \( s_1^2 \), \( s_2^2 \) and normality, a necessary and sufficient condition that the unbiased estimator \( \bar{\mu} \), defined by (2.6), be uniformly
better than either $\bar{x}_1$ or $\bar{x}_2$ alone, is that the degrees of freedom of $s^2_1$ and $s^2_2$ be both greater than or equal to nine. The variance of $\mu$ was not found explicitly in a general form but bounds on the variance of $\mu$ were obtained for specific degrees of freedom of $s^2_1$ and $s^2_2$ by evaluating the F-distribution at particular points in the expectation

\begin{equation}
\text{Var}(\mu) = E_{s^2_1 s^2_2} \{ \text{Var}(\mu)/s^2_1, s^2_2 \}
\end{equation}

\begin{equation}
= E \left[ \left( \frac{\hat{w}_1^2 \sigma^2}{n_1} + \frac{\hat{w}_2^2 \sigma^2}{n_2} \right) / \left( \left( \hat{w}_1 + \hat{w}_2 \right)^2 \right) \right].
\end{equation}

By evaluating the expected value in (2.7), it is possible to find the point or set of points of degrees of freedom for which

\begin{equation}
\text{Var}(\mu) \leq \min \left( \frac{\sigma^2}{n_1}, \frac{\sigma^2}{n_2} \right)
\end{equation}

will be satisfied uniformly for $k = \sigma^2_1/\sigma^2_2 > 0$. This method is not readily extendible to the combinability of several independent estimates.

It is possible, as shown by Mehta and Gurland (1969), to reduce the number of degrees of freedom required by Graybill and Deal (1959) to obtain a uniformly better combined estimate. If $k = \sigma^2_2/\sigma^2_1$ is known to be greater than or equal to one, then
the degrees of freedom of three or more for both $s_1^2$ and $s_2^2$ in (2.6) will be sufficient to insure that the combined estimator (2.6) is uniformly better than either individual estimator $\bar{x}_1$ or $\bar{x}_2$ would be alone. The method they used to find the variance of the combined estimators proceeded along the lines of Graybill and Deal (1959), which involved evaluating the F-distribution at particular values of degrees of freedom.

The work by Sprott (1956) and extensions made by Martin and Zyskind (1966) on combining uncorrelated estimators by simple weighting is pertinent to the overall background of combining information. The weights of the individual estimators are assumed known; consequently, the problem of finding a general formula for the unconditional variance is greatly simplified at the expense of having to assume additional knowledge. The method of attack proceeds along the line of obtaining the form of estimable contrasts that are best combined by simple weighting for given matrices $X_1$ and $X_2$ under the two mutually uncorrelated linear models

(2.9) \[ Y_i = X_i \beta + \epsilon_i, \ i = 1, 2, \]

where $Y_i$ is ($n \times 1$) vector of observations,

$X_i$ is ($n \times p$) known matrix,

$\beta$ is ($p \times 1$) vector of unknown parameters, and

$\epsilon_i$ is ($n \times 1$) vector of residuals with assumptions

$E\epsilon_i = 0$, $E\epsilon_i \epsilon_i^\prime = V_i$, $E\epsilon_i \epsilon_j^\prime = \phi$, $i \neq j$, $i,j = 1, 2$. 
If the rank of \([R(X_1) \cap R(X_2)] = r\), where \(R(X_i)\) denotes the row space of design matrix \(X_i\), then it can be shown that there are exactly \(r\) independent \(\lambda\) vector or different possible contrasts, for which the best linear unbiased (b.l.u.) estimate of the estimable parametric function \(\lambda'\beta\) is obtainable by simple weighting of the form

\[
\lambda'\hat{\beta} = w\hat{\lambda}'\beta + (1-w)\tilde{\lambda}'\beta,
\]

where \(\hat{\lambda}'\beta\) is the b.l.u. estimate of \(\lambda'\beta\) from the data set \(Y_1 = X_1\beta + \epsilon_1\) and \(\tilde{\lambda}'\beta\) is the b.l.u. estimate of \(\lambda'\beta\) from the second set of data \(Y_2 = X_2\beta + \epsilon_2\).

For cases of incomplete block design, the intra-block and inter-block set of equations may be regarded as originating from two uncorrelated sources of information on the treatment parameter vector \(\tau\). The estimable contrast \(\lambda'\tau\) is best estimated from one source alone or by simple weighting of b.l.u. estimators from the respective sources if and only if \(\lambda\) is an eigenvector of the design matrix product \(X'ZZ'X = (\gamma_{ij})\) from the linear model

\[
Y = \mu 1 + X\tau + Z\beta + \epsilon
\]

with \(EY = \mu 1 + X\tau\), \(Var(Y) = ZZ'\sigma^2 + \sigma^2 I_n\). \(\gamma_{ij}\) is the number of times treatments \(i\) and \(j\) occur together in a block. Martin and Zyskind (1966) found conditions under which combinability of inter- and intra-block information by simple weighting is
not best. The estimable parametric function $\lambda^T$, under model (2.11), is best estimated from intra-block information alone if and only if

(2.12) \hspace{1cm} (X'ZZ'X)\lambda = 0,

and from inter-block information alone if and only if

(2.13) \hspace{1cm} (X'ZZ'X)\lambda = rk\lambda ,

where $r$ is the number of treatment replications and $k$ is the plot size of the incomplete block design.

When fixed weights are used to obtain a combined estimator of the form

(2.14) \hspace{1cm} \overline{\mu} = w\hat{\mu} + (1-w)\tilde{\mu}

of the estimable parameter $\mu$, the resulting estimator has been shown not to be uniformly better than the individual estimator alone that has smallest variance, since for any fixed $w$, $0 \leq w \leq 1$,

(2.15) \hspace{1cm} \text{Var}(\overline{\mu}) > \min(\text{Var}(\hat{\mu}), \text{Var}(\tilde{\mu}))

for at least some value of $k = \text{Var}(\hat{\mu})/\text{Var}(\tilde{\mu})$.

An extension of combining information to include multivariate structure was made by Williams (1967). The designs considered were restricted to equal replication. The observation vector $Y(m \times 1) = \{Y_1', Y_2', ..., Y_{r+1}'\}'$ is assumed
to be normally distributed with mean \((J_{r+1}^l \otimes X)\beta\) and dispersion matrix \((I_{r+1} \otimes V)\) where \(m = n(r+1)\). The matrix \(X\) is an \((n \times p)\) known matrix and \(V\) is an \((n \times n)\) unknown dispersion matrix. The notation \(\otimes\) denotes the Kronecker matrix product. Under the above conditions a weighted regression estimator for the estimable parameter \(\beta\) and its variance were found. An unbiased estimate \(S\) of \(V\) is obtained from the data and substituted for \(V\) in the set of generalized normal equations to obtain the weighted regression estimate \(\hat{\beta}_w\).

Under the full rank linear model

\[(2.16) \quad Y = X\beta + \epsilon,\]

\(Y\) \((n \times 1)\) vector of observations, \(X\) \((n \times p)\) matrix of known coefficients, \(\beta\) \((p \times 1)\) unknown vector of parameters, and \(\epsilon\) \((n \times 1)\) is such that \(E\epsilon = 0\) and \(E\epsilon\epsilon' = V\), where \(V\) is \((n \times n)\) non-singular positive definite known matrix, the b.l.u. estimator of the estimable parametric function \(\lambda'\beta\) was established by Aitken (1934) to be \(\lambda'\hat{\beta}\) where \(\hat{\beta}\) is any solution of the generalized normal equation

\[(2.17) \quad X'V^{-1}X\beta = X'V^{-1}Y.\]

The variance of \(\hat{\beta}_w\) was expressed by Williams (1967) in terms of the variance of the weighted regression estimator \(\hat{\beta}_w\) when \(V\) is known, to determine the additional variance of the estimator when weights are not known. Williams (1967) considered two
population structures for the variance-covariance matrix $V$. In the first case $V$ is assumed to be positive definite. In the second case $V$ is assumed to be diagonal. In both cases $V$ is assumed to be nonsingular.

An extension to the case of unequal sample size was made by Bement and Williams (1969). Following along the lines of Williams (1967), Bement and Williams formulated a weighted regression estimator $\hat{\beta}_w$ and obtained a general variance approximation for $\text{Var}(\hat{\beta}_w)$ for the case where the dispersion matrix $V$ is known to be diagonal.

Rao (1967) considered some problems in the estimation of and inference about unknown parameters in a linear model under various population structures of the dispersion matrix $V$. He showed that for some structures of $V$, an estimate of the unknown parameter $\beta(p \times 1)$ from the linear model (2.16), obtained by substituting an estimate $S$ for $V$ in the generalized least squares equation (2.17), does not always yield the "best" estimator. Certain improvements, however, can be made, depending on the known or inferred structure of $V$ and model assumptions. A test criterion for examining the adequacy of the nonrandom part of model (2.16) was obtained. The direction taken by Rao (1967) for increasing the efficiency of linear estimators followed along the general line of adjusting the least squares estimator for covariates. A selection of
suitable concomitant variables or their functions was made on
the basis of the given structure for $V$.

In cases where weights are not known exactly for use in
combining information but distributional properties are known
or can be assumed, then estimators can be improved by utiliz­
ing a prior knowledge in a Bayesian approach. Tiao and Zellner
(1964) considered the Bayesian estimation of multivariate
regression parameters in the model of the form

$$Y_i = X_i \beta + \epsilon_i, \quad i = 1, 2, \ldots, r,$$

where $Y_i(n \times 1)$ is the vector of observations,
$X_i(n \times p)$ is a known matrix of rank $p$,
$\beta(p \times 1)$ is a vector of unknown parameters, and
$\epsilon_i(n \times 1)$ is the vector of residual with

$$E \epsilon_i = 0, \quad E \epsilon_i \epsilon_i' = V, \quad \text{and} \quad E \epsilon_i \epsilon_j' = 0, \quad i \neq j, \quad i, j = 1, 2, \ldots, r.$$

The prior joint distribution of $\beta$ and $V$ that Tiao and Zellner
(1964) selected is of the form

$$p(\beta, V) = p(\beta)p(V),$$

$$= c \cdot \left| V \right|^{-\frac{r+1}{2}},$$

where $c$ is uninformative, a prior constant for $\beta$.

Rojas (1958) investigated the effects of combining infor­
mation from several designed experiments in which he assumed
the within error distribution of the $i^{th}$ design to be $N(0, \sigma^2_i)$, where $\sigma^2_i$ is independently distributed as gamma ($\alpha, \lambda$), $i = 1, 2, \ldots, p$. Weighted and semi-weighted means, having the form

$$\bar{\mu} = \frac{k \sum w_i \hat{\mu}_i}{w},$$

where

$$w_i = \frac{1}{\sigma^2_i}, \quad \sigma^2_i \sim \text{gamma} (\alpha, \lambda),$$

$$w = \frac{k}{\sum w_i},$$

were compared to unweighted means to find the gains in relative efficiency. The gain realized is somewhat marginal unless the gamma parameters ($\alpha, \lambda$) are known. If they are not known, they must be estimated. To obtain a combined estimator by weighting, the estimation process is then shifted from estimating the weights directly from the data to estimating the distribution of the weights. Once the distribution is determined, it can be used in the form of a prior information to obtain a Bayes solution for the combined estimator. The method of moments is used to obtain estimates for the parameter ($\alpha, \lambda$) of the gamma distribution.
The combination of information was studied by Nelder (1968) in what he termed generally balanced designs, with application of implementation on a high-speed electronic computer. In this class of designs there is often more than one estimate of a given parameter available, as exemplified by both inter- and intra-block information on treatment effects in an incomplete block design. Stratum variances, used as weights, are estimated by using an iterative procedure which usually converges in a finite number of steps.

The class of experimental situations that Nelder considered had orthogonal block structure and additive linear treatment structure. An experimental design is said to have orthogonal block structure if the identity decomposition of an arbitrary response has a matrix representation that satisfies the conditions of the well-known Cochran theorem. Nelder further defined linear treatment structures as those which can be specified by idempotent matrices.

Mexas (1970) discussed works by Nelder and Wilkinson with particular interest focused on their proposed algorithms of analysis of variance as applied to balanced complete design structures with computer implementation. Mexas was somewhat critical of Wilkinson's recursive algorithm applied to balanced complete structures and many nonorthogonal balanced
structures due to its lack of information on degrees of freedom for cases not having maximum rank and its order dependence of factor presentations. The algorithm seemed applicable to two factor arrangements only.

B. Combining Inter- and Intra-block Information

The problem of combining inter- and intra-block information is an important case of the general problem of combining several unbiased estimates of a given estimable parametric function. This problem has received wide and varied attention from many authors. The objective is to find the "best" estimate or to improve on the estimator or technique of estimation of the parameter under study. The problem of combining information from designed experiments, such as incomplete blocks, differs from many stratified sampling situations in that stratum sample size is often used in forming weights rather than estimated variances. This type of weights can be fixed or random, depending on how sample sizes are allocated.

Much of the effort in the investigation of recovery of inter-block information has been directed toward special cases of incomplete blocks, such as the balanced incomplete block (bib) and various lattice designs. Some of the techniques used for combining information in special cases can be applied toward solving problems for the more general class of incomplete block designs. Recovery of inter-block information was
examined by Roy and Shah (1962). They proceeded along the lines developed by Yates which consist of applying the method of weighted least squares to intra-block contrasts and inter-block contrasts of observations for the purpose of estimating treatments, weighting being inversely proportional to the variance of these contrasts. A maximum likelihood estimate corrected for bias of the ratio \( \rho \) of inter- and intra-block variances was used as a weight for obtaining unbiased estimates of treatment effects. An expression for the additional variance of treatment contrasts caused by using an estimate of \( \rho \) was also exhibited.

Martinez (1965) studied the problem of combining information from several designs, each of which contained a "common" and a "regular" set of treatments. Information for the set of common or similar treatments was combined across all designs by unweighted least squares, while the set of regular or dissimilar treatment effects was analyzed on an individual design basis.

A method for recovery of inter-block information that would result in a uniformly better combined estimator of treatment effects in maximal rank incomplete block designs was discussed by Shah (1964). He used a weight of \( \hat{\rho}^{-1} \) where \( \hat{\rho} = \hat{\sigma}^2 / \hat{\sigma}_0^2 \). \( \hat{\sigma}^2 \) and \( \hat{\sigma}_0^2 \) are unbiased estimators of inter-block variance \( \sigma_1^2 \) and intra-block variance \( \sigma_0^2 \), respectively. The condition \( \rho \geq 1 \) was imposed, thus
introducing some bias in the weights; but inter-block information will not get weighted more than the intra-block information. This would clearly not be the case if unbiased estimates were used since unbiased estimates of inter-block variance can be less than the unbiased estimate of intra-block variance.

The problem of variance component estimation, needless to say, plays an extremely important role in combining information in almost all cases.

The combined unbiased estimators of inter- and intra-block information proposed by Graybill and Deal (1959) was improved upon by Seshadri (1963a). He proposed, for the special case of balanced incomplete block design, a slightly different set of inter- and intra-block weights composed of a special linear combination of the inter- and intra-block variances and obtained the variance of the combined unbiased estimator:

\[(2.21) \quad \bar{T} = \hat{\tau} + \hat{\tau} \hat{w},\]

where \(\hat{w}\) is an unbiased estimator of

\[(2.22) \quad w = \frac{(r-\lambda)\sigma^2}{k(r\sigma^2 + \lambda \nu \sigma^2_\beta)}\]

and with distributional properties:

\[(2.23) \quad \text{Block effects and errors are random,}\]
(2.24) \[ \hat{\tau}_i \sim N(\tau_i, k\sigma^2/\lambda v), \text{ intra-block estimate,} \]

(2.25) \[ \tilde{\tau}_i \sim N(\tau_i, k(\sigma^2 + k\sigma^2_B)/(r-\lambda)), \text{ inter-block estimate,} \]

(2.26) \[ (\hat{\tau}_i - \tilde{\tau}_i) \sim N(0, k^2 (\sigma^2 + \lambda v \sigma^2_B)/\lambda v (r-\lambda)). \]

The variance of \( \tau_i \) was obtained by finding unconditional expectations over the estimated weights, assumed to be the ratio of two chi-square variables. This unconditional variance is

(2.27) \[ \text{Var}(\tau_i) = \frac{k}{\lambda v} \sigma^2 - \frac{2(v-3)(r-\lambda)(\sigma^2)^2}{\lambda v(v-1)(\sigma^2 + \lambda v \sigma^2_B)} + \frac{(v-3)(f+2)(r-\lambda)(\sigma^2)^2}{\lambda v(v-1)f (\sigma^2 + \lambda v \sigma^2_B)}, \]

where \( v, r, \) and \( \lambda \) are the number of treatments, number of replicates, and number of times any two different treatments appear together in the same block, respectively.

C. Minimal Sufficient Statistics for Incomplete Block Designs

In any discussion of estimation, various properties of the estimators, such as sufficiency, consistency, efficiency, completeness, invariance, as well as unbiasedness and minimum variance, must be considered. An estimator may possess some, none, or all of these properties. A combined estimator of statistics with a given set of the above properties may or may
not possess the same properties as the statistics from which it was found.

If a set of minimal sufficient statistics can be found for a given distribution, then by the Rao-Blackwell theorem, an estimator which is a minimum variance unbiased (m.v.u.) estimator must be an explicit function of this set. If this minimal set is also complete (Lehmann and Scheffé (1950)), then an unbiased estimator based on this complete set is the unique m.v.u. estimator. A minimal sufficient statistic (which always exists) for a family of probability densities is desirable because when such a statistic is found, then we have essentially all the information contained in the sample about the indexing parameter condensed as far as possible. Unfortunately, in cases where there are two or more unbiased estimates, each of which are functions of a minimal sufficient statistic for a given estimable parametric function, the Rao-Blackwell theorem does not tell us which has the smallest variance.

Graybill and Weeks (1962) exhibit a set of minimal sufficient statistics for the balanced incomplete block design under the assumption that the blocks, treatments, and errors are uncorrelated random variables; i.e., an Eisenhart Model II (Eisenhart, 1947). These results were extended slightly by Graybill and Weeks (1962) to obtain a set of minimal sufficient statistics for a more general class of two-way
experimental designs, including a subset of balanced and partially balanced incomplete block designs, under the model

\[ Y = \mu 1 + X\tau + Z\beta + \epsilon, \]

where \( l(n \times 1), X(n \times v), \) and \( Z(n \times b) \) are known coefficient matrices and \( \tau, \beta, \epsilon \) are uncorrelated random variables with distributions

1) \( \tau \sim N(0, \sigma^2_{\tau}I_v) \),

2) \( \beta \sim N(0, \sigma^2_{\beta}I_b) \),

3) \( \epsilon \sim N(0, \sigma^2I_n) \).

\( Y(n \times 1) \) is a vector of observations with \( EY = \mu 1, \ Var(Y) = XX'\sigma^2_{\tau} + ZZ'\sigma^2_{\beta} + \sigma^2I_n = V. \) This case differs somewhat from some of the cases we consider for combining information. Our interest is toward the estimation of the variance components \( \sigma^2_{\tau} \) and \( \sigma^2 \) from incomplete block designs, of which functions can be used as weights in models where treatment effects are fixed.

The investigation of minimal sufficient statistics was extended by Graybill and Hultquist (1965) to include the two-way classification mixed model design with maximal rank having unequal subclass frequency. The model considered is of the form (2.28), but with treatments fixed. The observation vector \( Y(n \times 1) \) has the following expectation and variance:
In order to obtain the set of minimal sufficient statistics, one must first pick a set of statistics, then show that the picked set is both sufficient and minimal. The procedure followed by Graybill and Weeks (1962) under model (2.28) was to find an orthogonal matrix $P$, which would diagonalize the design matrix $NN'$ where $N = (1, X, Z)$, thus providing a basic set of vectors in determining the form of the matrix product $P'V_1P$ and $P'(Y - E(Y))$. These two products are needed in order to define a set of sufficient statistics for the given family of probability densities,

$$f(Y; \theta) = K \exp\left\{ -\frac{1}{2} (Y - E(Y))' P P^{-1} P' (Y - E(Y)) \right\},$$

where

$$K = \left| V_1 \right|^{-\frac{n}{2}} (2\pi)^{-\frac{nP}{2}},$$

$$\theta = \{ \mu, \sigma^2, \sigma^2, \sigma^2 \},$$

$n$ = total number of observations,

$p$ = number of parameters.

A set of estimators $\hat{\theta}$, exhibited by Graybill and Weeks (1962), was shown to be sufficient for the parameter $\theta$ by the well-known factorization theorem for the family of densities (2.30).
To show the set of sufficient statistics \( \hat{\theta} \) is indeed a minimal set, a theorem by Lehmann and Scheffé (1950) is applied. This method proving that a function of the type

\[
\lambda(y, y_0) = \frac{L_f(y; \hat{\theta})}{L_f(y_0; \hat{\theta})},
\]

which is independent of parameters, implies \( \hat{\theta} = \hat{\theta}_0 \), where \( \hat{\theta} \) is a set of sufficient statistics for the family of densities \( f(y; \theta) \) and is generally considered as a proposed minimal set. \( \hat{\theta}_0 \) is obtained from \( f(y_0; \theta) \) in the same way that \( \hat{\theta} \) was obtained from \( f(y; \theta) \). \( L \) is an operator on a density function which reduces the dimension of the space of the sufficient statistics. The operation used by Graybill and Weeks (1962) was to apply an orthogonal transformation \( P \).

It was established by Graybill and Hultquist (1965) that the dimension of a minimal sufficient set, under model assumptions (2.29), is a function of the ranks of certain submatrices of design matrix \( X'Z \). The procedure for establishing sufficiency for the mixed model with mean and variance (2.29) is to find an orthogonal transformation that will diagonalize the quadratic form of sum of squares and cross products found in the exponent of the density (2.30) into a sum of products independent of unknown variance components.

In discussing the special class of balanced incomplete block designs with treatment effects fixed, Rao (1947) stated
there are "best" estimators although the equations leading to them are quite complicated. He apparently was implying the use of the method of maximum likelihood to obtain estimators, but it is interesting to note that the only claim that can validly be made with regard to maximum likelihood estimators is that they are based on a minimal sufficient set of statistics.

Graybill and Weeks (1959) showed that the combined estimator, proposed by Yates (1940), of treatment effects, which Rao (1947) endorsed as a good approximate method, is based on a minimal sufficient set. It is reasonably apparent that in the problem of combining information, minimal sufficient statistics play an extremely important role; but this does not fully answer questions pertaining to utilizing inter-block information with regard to increased precision of estimators. The importance lies more in the fact that linear and quadratic estimation must be considered together rather than a separate problem of estimation in the general incomplete block designs under the mixed or Eisenhart Model III, in which treatments are fixed.

Seely (1969) considered both biased and unbiased quadratic estimators of variance components under the conditions of a finite parameter space. We do not limit the spaces considered in our problem to that extent, but some of Seely's results provide background to some of the general problems of
combining information, which we consider, under certain known variance-covariance structures. The form of the parameters Seely (1969) considered in the space $\Omega$ are of the form

$$(2.32) \quad \beta^* = \{(X\beta, \sum_{i=1}^{m} \alpha_i V_i) : (\beta, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m))' \in \Omega\},$$

where $X$ is known $(n \times p)$, each $V_i$ is a known $(n \times n)$ symmetric matrix. The parameter space $\Omega$ is assumed to be given explicitly or described implicitly through the random vector $\varepsilon$ of the linear model $y = X\beta + \varepsilon$, such that $E\varepsilon = 0$, and $E\varepsilon\varepsilon' = \sum_{i=1}^{m} \alpha_i V_i$. The set of quadratic forms of the vector $Y$ can be expressed as

$$(2.33) \quad \{Y'AY : A = A'\},$$

where the set $\{A : A = A'\}$ is a vector space.

An approach along the lines of that of Graybill and Hultquist (1961), with some generalizations by Basson (1965), is pursued by Seely (1969) to find the best estimator of variance components under various conditions and model assumptions. The notion of inner products, denoted by $<, >$, and $(,)$ is used to describe estimators over designated finite dimensional vector spaces.

The question of consistency of the equations leading to estimation and existence of unbiased estimation with regard
to variance components was pursued by Harville (1967) and by Seely (1969). The main emphasis is placed on finding conditions for unbiasedness and on obtaining unbiased estimators of linear functions of variance components. Empirical results presented by Leone and Nelson (1966) provided evidence of a nontrivial frequency of occurrence of negative estimates of variance components in cases of unbiased estimates. The results are of special importance in the problem of combining information in incomplete blocks, where the unbiased estimate of the block weight may be less than intra-block variance.
III. COMBINABILITY OF REGRESSION ESTIMATORS

A. General Estimability and Least Squares Considerations

One may surmise from Chapter II that there are many directions from which one may approach the problem of combining information, depending on the type and properties of estimators and what is known about the given population structures under study. In this chapter and in the succeeding chapter, we formulate and examine relative variances of several possible estimators for given parametric functions. Particular interest is focused on utilizing high-speed computers to aid researchers in finding solutions to problems involving combining information.

We direct attention first to some general aspects of estimability of the parametric function $\lambda'\tau$ under the general linear model

$$(3.1) \quad Y = X\tau + \varepsilon,$$

where $Y$ is an $(n \times 1)$ vector of observations,

$X$ is an $(n \times p)$ known matrix,

$\tau$ is a $(p \times 1)$ vector of unknown parameters, and

$\varepsilon$ is an $(n \times 1)$ vector of residuals,

such that $E\varepsilon = 0$ and $E\varepsilon\varepsilon' = V\sigma^2$. The $(n \times n)$ matrix $V$ is a known symmetric, positive definite matrix.
The parametric function $\lambda' \tau$ is said to be estimable if and only if there exists a vector $a(n \times 1)$ such that $E(a'Y) = \lambda' \tau$. $a'Y$ is also the best linear unbiased (b.l.u.) estimator, under model (3.1) with $V = I$, if and only if $a \in C(X)$ where $C(X)$ denotes the column space of the matrix $X$. If $a \in C(X)$, then there exists a vector $\rho(p \times 1)$ such that $a = X\rho$, so that

$$E(a'Y) = a'X\tau = \rho'X'X\tau = \lambda' \tau,$$

where $\lambda = X'X\rho$ belongs to $C(X')$ since $C(X')$ is identical with the column space of $X'X$. The b.l.u. estimator of $\lambda' \tau$ can be expressed as $\rho'X'Y$ where $\rho$ is any solution of the conjugate normal equations

$$X'X\rho = \lambda. \tag{3.2}$$

When the variance-covariance matrix (3.1) is the identity matrix $I$, it is known that the b.l.u. estimator of $\lambda' \tau$ is identical to the simple least squares estimator $\lambda' \hat{\tau}$, where $\hat{\tau}$ is any solution to the normal equations $X'X\hat{\tau} = X'Y$. For $X$ of full rank, the estimator of $\lambda' \tau$ may be expressed as

$$\lambda' \hat{\tau} = \lambda'(X'X)^{-1}X'Y. \tag{3.3}$$
For the case of a more complex error structure under model (3.1) where \( V \) is a known \((n \times n)\) nonsingular matrix, the b.l.u. estimator of \( \lambda' \tau \) is

\[
(3.4) \quad \lambda' \hat{\tau} = \gamma' X' V^{-1} y,
\]

where \( \gamma \) is any vector solution of the generalized conjugate normal equations

\[
(3.5) \quad X' V^{-1} X \gamma = \lambda.
\]

There are several conditions on the structure combination of the given matrix \( X \) and the variance-covariance matrix \( \sigma^2 \) that when satisfied the simple least squares estimator of every estimable parametric function \( \lambda' \tau \) will be the best linear unbiased estimator. The following conditions, each of which is both necessary and sufficient for the equality of these two estimators, are equivalent.

(1) The \( C(X) \) is an invariant subspace of the matrix \( V \);
    i.e., if the vector \( a \in C(X) \), then the vector \( Va \in C(X) \).

(2) A matrix \( R \) exists such that \( VX = XR \).

(3) A basis for \( C(X) \) can be found by a subset of \( r \) eigenvectors to the matrix \( V \), where \( r = \text{rank} (X) \).

(4) A full-rank reparametrization exists so that the expected value of \( Y \) can be expressed as \( EY = X\tau = W\beta \)
    where every column of \( W(n \times r) \) is an eigenvector of the matrix \( V \).
(5) There exists an orthogonal matrix \( O = \begin{pmatrix} O_1 & O_2 \end{pmatrix} \) that will diagonalize the matrix \( V \) so that

\[
V = ODO' = \begin{pmatrix} O_1 & O_2 \end{pmatrix} D \begin{pmatrix} O_1 & O_2 \end{pmatrix}',
\]

where \( O_1 \) is any orthonormal basis of \( C(X) \), \( O_2 \) is an orthonormal basis of \( C^\perp(X) \), and \( D \) is a diagonal matrix with positive elements. \( C^\perp(X) \) denotes the orthogonal complement of \( C(X) \).

(6) If \( P \) is an orthogonal matrix projection operator on \( C(X) \), then the matrices \( P \) and \( V \) commute; i.e., \( VP = PV \) and \( VP = (VP)' \).

[Statements of the above conditions (1-6) were given by Zyskind (1962) and by Zyskind, et al. (1964). They were extended further by Zyskind (1967, 1969) to include more general models with arbitrary nonnegative (possibly singular) covariance matrix \( \sigma^2 V \). Aspects of these questions have also been studied by Rao (1967) and Kruskal (1968).]

Further generalizations for finding best linear unbiased estimators in the presence of various known variance-covariance structures are possible, and there exists some literature on the subject. In cases where \( V \) is singular, one procedure for finding a best linear unbiased estimator follows along the line of substituting a special member of the class of conditional inverses of \( V \) in place of \( V^{-1} \) (Zyskind and
Martin (1969), then of obtaining solutions of the resulting generalized normal equations. In many cases, the simple least squares estimator is not b.l.u. However, if the variance-covariance matrix is known, can be inferred to within a scalar multiple, or can be estimated, it is possible to improve upon the simple least squares estimator.

B. Weighted Least Squares Estimation

Under a Positive Definite Dispersion Matrix

If $V$ is not known exactly, there does not in general exist a best linear unbiased estimator for the estimable parametric function $\lambda' \tau$ under the linear model (3.1). We seek the estimator or set of estimators that utilizes available or obtainable information to improve upon possible alternative candidates. This, of course, means the use of estimated variance-covariance structure, sample size, and preliminary testing in proposing possible estimators.

Suppose a physical situation can be properly described by the linear model

$$Y_i = X_i \tau + \varepsilon_i, \ i = 1, 2, \ldots, r,$$

where $Y_i$ is an $(n \times 1)$ vector of observations,

$X_i$ is an $(n \times p)$ known design matrix with full rank $p$,

$\tau$ is a $(p \times 1)$ vector of unknown parameters,

$\varepsilon_i$ is an $(n \times 1)$ vector of residuals,
with $Ee_i = 0$, $Ee_i e'_i = V(n \times n)$, $Ee_i e'_j = 0$, $i \neq j$, $i, j = 1, 2, \ldots, r$.

We consider the case when the dispersion matrix $V$ is nonsingular positive definite and has finite but unknown elements. For this case, the complete set of observations can be expressed by the vector

$$Y' = (Y'_1, Y'_2, \ldots, Y'_r),$$

which is assumed to have a normal distribution, expressed as

$$Y \sim \text{NI}(X\tau, V),$$

where

$$X' = (X'_1, X'_2, \ldots, X'_r)$$

and

$$V = I_r \otimes V.$$

For the case where $V$ and $X_i$'s have full rank, we consider the following estimators for the $(p \times 1)$ vector $\tau$.

(3.7) Simple least squares estimator:

$$\hat{\tau} = (X'X)^{-1}X'Y$$

$$= \left( \sum_{i=1}^{r} X'_i X_i \right)^{-1} \sum_{i=1}^{r} X'_i Y_i$$

$$= (X'X)^{-1}X'Y, \text{ if } X_i = X, i = 1, 2, \ldots, r.$$
(3.8) Weighted least squares estimator with $V$ known:

$$
\hat{\tau}_W = (X'V^{-1}X)^{-1}X'V^{-1}Y
$$

$$
= \left( \sum_{i=1}^{r} X_i'V^{-1}X_i \right)^{-1} \left( \sum_{i=1}^{r} X_i'V^{-1}Y_i \right)
$$

$$
= (X'V^{-1}X)^{-1}X'V^{-1}Y, \text{ if } X_i = X, \quad i = 1, 2, \ldots, r.
$$

(3.9) Weighted least squares estimator with $V$ unknown but with an unbiased estimate $\hat{S}$ of $V$ available and independent of $Y$:

$$
\hat{\tau}^\wedge_W = (X'S^{-1}X)^{-1}X'S^{-1}Y
$$

$$
= \left( \sum_{i=1}^{r} X_i'S^{-1}X_i \right)^{-1} \left( \sum_{i=1}^{r} X_i'S^{-1}Y_i \right)
$$

$$
= (X'S^{-1}X)^{-1}X'S^{-1}Y \quad \text{if } X_i = X, \quad i = 1, 2, \ldots, r.
$$

For each of the above estimators, $\overline{Y}$ is the $(n \times 1)$ vector of means expressible as

(3.10) $$
\overline{Y} = \frac{1}{r} \sum_{\alpha=1}^{r} Y_\alpha.
$$

The estimator $S$ used in expression (3.9) is obtained under the condition $X_i = X$, $i = 1, 2, \ldots, r$ by the method of maximum likelihood corrected for bias and can be expressed as
The distribution of \((r-l)S = A\) is Wishart, denoted by \(W(V, r-l)\), where \(V\) is the dispersion matrix and \((r-l)\) is the number of degrees of freedom for \(A\). With \(m = r - 1\) and \(r \geq n\), the density is

\[
(3.12) \quad w(A; V, m) = K^{-1} |A|^{\frac{m-n-1}{2}} e^{\frac{1}{2} tr AV^{-1}} \quad \text{for } A \text{ positive definite}
\]

\[
= 0, \quad \text{otherwise},
\]

where \(K = 2^\frac{mn}{2} \pi^{n(n-1)/4} \left|V\right|^{\frac{m}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{m+l-i}{2}\right)\).

The distributional properties of \(A\) will be used later in obtaining the unconditional variance of estimator (3.9). One of the problems pursued in this chapter is the clarification and extension of works by Williams (1967) on combining information and the comparisons of proposed estimators. Conclusions will be drawn and proposals made with regard to the estimation of the parameter vector \(\beta\). One method commonly used to compare estimators is to look at the estimated relative efficiency, which is a ratio of estimators of common functions of estimated variances. In Section III A, some conditions under which estimator (3.7) would be the same
as (3.8) were presented. One of the principal reasons of great importance for establishing the necessary and sufficient conditions under which the simple least squares estimator (3.7) and the weighted least squares estimator (3.8) are the same is the ease of which estimator (3.7) can be computed beyond estimator (3.8). If $V$ is not known, (3.8) cannot be computed; consequently, it is then even more important to establish that the easily computable simple least squares estimator is the best linear unbiased estimator. Under a heterogeneous variance-covariance population structure, the simple least squares estimator can be verified not to be in general the best linear unbiased estimator. If the actual population dispersion matrix is unknown, an unbiased estimate can sometimes be used to improve on the least squares estimator.

The dispersion matrix of the simple least squares estimator (3.7) under model (3.6) is

$$(3.13) \quad \text{Var}(\hat{\tau}) = E(\hat{\tau} - E\hat{\tau})(\hat{\tau} - E\hat{\tau})'$$

$$= (X'X)^{-1}X'VX(X'X)^{-1}/r.$$ 

The $E\hat{\tau}$ is easily obtained to be

$$E\hat{\tau} = E[(X'X)^{-1}X'Y]$$

$$= \tau,$$
which shows \( \hat{\tau} \) to be an unbiased estimator of parameter vector \( \tau \).

Proceeding under the same model assumptions as above, estimator (3.8) is also easily shown to be unbiased

\[
E_{\tau_w} = E[(X'V^{-1}X)^{-1}X'V^{-1}Y]
\]

\[
= (X'V^{-1}X)^{-1}X'V^{-1}E(Y) = \tau,
\]

and has dispersion matrix

\[
(3.14) \quad \text{Var}(\hat{\tau}_w) = E(\hat{\tau}_w - \tau)(\hat{\tau}_w - \tau)',
\]

\[
= (X'V^{-1}X)^{-1}X'V^{-1}VV^{-1}X(X'V^{-1}X)^{-1}/r
\]

\[
= (X'V^{-1}X)^{-1}/r.
\]

The generalized variance (Wilks (1932)) can be obtained for the dispersion matrices (3.13) and (3.14) to form scalars which can be used to calculate the relative efficiency based on these variances. This relative efficiency,

\[
(3.15) \quad \text{R.E.}(\hat{\tau}, \hat{\tau}_w) = \frac{|(X'V^{-1}X)^{-1}|}{|(X'X)^{-1}X'VX(X'X)^{-1}|},
\]

can be verified to be unity for each of the conditions listed in Section III A, thus indicating that estimators (3.7) and (3.8) are the same under those particular conditions. The general comparison of these two estimators is not the principal
objective of this part of research, but rather to make comparisons which lead to possible improvements of the combined estimators of the type (3.9) over (3.7). Also, we seek formulation for calculating loss in precision that is incurred when an estimate of the dispersion matrix is used from the estimator (3.8) in which the dispersion matrix is known.

By applying the method of conditional expectation, the estimator (3.9) can be shown, under model (3.6), to be unbiased for the vector $\tau$. We have

\begin{equation}
E(\hat{\tau}_w^\wedge) = E_\w^\w E_{\tau_w}^\w^\w(\hat{\tau}_w^\w^\w/\w)
\end{equation}

\begin{equation}
= E_\w^\w(x's^{-1}x)-1x's^{-1}x\tau
\end{equation}

\begin{equation}
= \tau,
\end{equation}

since the expected value of a constant does not depend on the particular estimated weight that is used. The weight $S$ is statistically independent of $Y$ and can be calculated from either the same data or from a different set of data similar to that used to calculate $Y$. The unconditional variance of estimator (3.9) can be expressed in terms of conditional variances and expectations as follows:

\begin{equation}
\text{Var}(\hat{\tau}_w^\w^\w) = E_\w^\w [\text{Var}_{\tau_w}^\w^\w (\hat{\tau}_w^\w^\w/\w)] + \text{Var}_\w^\w [E_{\tau_w}^\w^\w (\hat{\tau}_w^\w^\w/\w)].
\end{equation}
The expected value of $\hat{\tau}_w^\wedge$ for a given weight $\hat{w}$ is shown in (3.16) to be constant, and the variance of a constant is zero. Consequently, the unconditional variance of $\hat{\tau}_w^\wedge$ is simply the expected value over the random weights, which in the present case are the random elements of the estimated dispersion matrix $S$. The difficulty in evaluating (3.17) is in obtaining closed forms for this expectation. It is necessary to obtain this unconditional variance in order to form a basis for comparisons with other proposed estimators. We have

\begin{equation}
(3.18) \quad r\text{Var}(\hat{\tau}_w^\wedge) = rE_w^\wedge[\text{Var}_{\tau_w^\wedge}(\hat{\tau}_w^\wedge/\hat{w})] \\
= rE_w^\wedge[E_{\tau_w^\wedge}(\hat{\tau}_w^\wedge - \tau)(\hat{\tau}_w^\wedge - \tau)'/\hat{w}] \\
= E_w^\wedge[(X'S^{-1}X)^{-1}X'S^{-1}VS^{-1}VX(X'S^{-1}X)^{-1}].
\end{equation}

To evaluate this expectation, we proceed to a certain extent along the direction proposed by Williams (1967) in making a transformation that will decompose the entire product into two orthogonal parts. The first part will be the contribution to the unconditional variance due to estimating the parameter vector $\tau$, while the second will be the additional variance caused by using the estimated dispersion matrix $S$. The details of the procedure followed by Williams (1967) are somewhat sketchy and relatively obscure. There is much need for clarification and extension, which we attempt to do. The
studies he presented are incomplete and to some extent inconclusive with regard to establishing properties for a combined estimator of the form $\hat{\tau}_W$.

Since the dispersion matrix $V$ is nonsingular positive definite, there exists a nonsingular matrix $P(n \times n)$ such that $PVP' = I$. Let this $P$ be a matrix operator to transform $S$ into $S_P = PSP'$. If $S$ is distributed according to Wishart, $W(V,r-1)$, then $S_P$ can be shown (Anderson (1958)) to be distributed as $W(I_n,r-1)$, where $I_n = PVP'$, which is the identity matrix. With $Z_\alpha = Y_\alpha - \bar{Y}$, $(r-1)S$ can be expressed as the sum $\sum_{\alpha=1}^{r} Z_\alpha Z'_\alpha$. Consequently,

$$S_P = PSP' = \sum_{\alpha=1}^{r} (PZ_\alpha)(PZ'_\alpha)' ,$$

where

$$PZ_\alpha \sim N(0,(1-\frac{1}{r})I_n).$$

Now, $S = P^{-1}S_P P'^{-1}$ implies for $S$ nonsingular that $S^{-1} = P'S_P^{-1}P$. Substituting for $S^{-1}$ in (3.18), we can write

$$E_w[(X'S_P^{-1}PX)^{-1}X'P'S_P^{-1}PVP'S_P^{-1}PX(X'S_P^{-1}PX)^{-1}] .$$
The next step is to find a transformation on the matrix product \( PX(n \times p) \) that will decompose (3.19) into two orthogonal parts.

**Theorem 3.1**: If \( (PX)(n \times p), p \leq n \) is such that rank \( (PX) = p \), and \( (PX)'(PX) \) is symmetric positive definite, then there exists a decomposition \( (PX)' = ADB \), where \( A(p \times p) \) and \( B(p \times n) \) are such that \( A'A = AA' = BB' = I_p \) and \( D(p \times p) \) is diagonal.

By Theorem 3.1 (Roy (1957)), there exist matrices \( A(p \times p) \) and row orthonormal matrix \( B(p \times n) \) such that

\[
(3.20) \quad (PX)' = ADB,
\]

where \( D(p \times p) \) is a diagonal nonsingular matrix. The matrix \( B \), expressed in terms of matrices \( P \), \( A \), and \( X \) as

\[
(3.21) \quad B = D^{-1}A'(PX)',
\]

is such that \( BB' = I_p \) and \( A'(PX)'(PX)A = D^2 \). The diagonal elements of the diagonal matrix \( D^2 \) are the eigenvalues of the matrix product \( (PX)'(PX) \).

To the row orthonormal matrix \( B \), add \( (n-p) \) orthonormal rows, denoted by the \( (n-p) \times n \) matrix \( F \), to provide a basis for an orthogonal decomposition of expression (3.19). We have

\[
(3.22) \quad [-F_B]'
\]

\[
[-F_B] = F'F + B'B = I_n.
\]
Multiplying on the left and right of (3.22) by $S_P^{-1}$, we obtain

\[(3.23) \quad S_P^{-1} S_P^{-1} = S_P^{-1} (F'F + B'B) S_P^{-1}.\]

Substituting for $(PX)$ and $S_P^{-1} S_P^{-1}$ in (3.19) and recalling that $PVP' = I_n$, we obtain

\[(3.24) \quad E^w[H_i^{-1} ADBS_P^{-1} (F'F + B'B) S_P^{-1} B'DA'H_i^{-1}],\]

where

\[H_i^{-1} = (ADBSP^{-1}B'DA')^{-1} = AD^{-1}(BS_P^{-1}B')^{-1}D^{-1}A'.\]

Since $A$ is an orthonormal matrix such that $A' A = I_p$, expression (3.24) can be written

\[(3.25) \quad AD^{-1}E^w[H_2^{-1}BS_P^{-1}F'FSP^{-1}B'H_2^{-1} + H_2^{-1}BS_P^{-1}B'BS_P^{-1}B'H_2^{-1}]D^{-1}A'],\]

where

\[H_2^{-1} = (BS_P^{-1}B')^{-1}.\]

Expression (3.25) can be expressed as the sum of expected values
\[ (3.26) \quad AD^{-1}[E_{wH_3}]D^{-1}A' + AD^{-1}[E_{wH_4}]D^{-1}A', \]

where

\[ H_3 = H_2^{-1}BSP^{-1}F'FSP^{-1}B'H_2^{-1} \]

and

\[ H_4 = H_2^{-1}H_2H_2^{-1} = I_p. \]

The expected value of \( H_4 \) is simply the identity matrix \( I_p \), which leaves the second term of (3.26) expressed as the matrix product

\[ (3.27) \quad AD^{-1}D^{-1}A' = AD^{-2}A'. \]

From (3.20), form the matrix product

\[ (3.28) \quad (PX)'(PX) = ADBB'DA' 
\]
\[ = ADDA' \]
\[ = AD^2A'. \]

Since the matrix product (3.28) is nonsingular, the inverse exists and is written

\[ (3.29) \quad [(PX)'(PX)]^{-1} = AD^{-2}A'. \]
or

\[ AD^{-2}A' = (X'P'PX)^{-1}. \]

Recalling that \( PVP' = I \), we have \( V = P^{-1}P' \), which implies \( V^{-1} = P'P \). Substituting \( V^{-1} \) for \( P'P \), expression (3.29) can be written

\[(3.30) \quad V_2 = AD^{-2}A' = (X'V^{-1}X)^{-1} \cdot \]

The part of expression (3.26) \( AD^{-1}[E^\omega H]D^{-1}A' \), denoted by \( V_3 \), is the contribution due to using the estimated dispersion matrix \( S \) rather than the matrix \( V \). For the present case \( V_3 \) can be expressed in terms of the population dispersion matrix \( V \) so that a direct comparison of the combined estimator (3.8) can be made with (3.9). As an initial approach to simplify \( V_3 \), we need the following matrix theorems (Anderson (1958)):

**Theorem 3.2:** If a matrix \( A \) is positive definite, there exists a triangular matrix \( T \) such that \( TAT' = I \).

The proof of Theorem 3.2 essentially makes repetitive use of Theorem 3.3, stated as follows.

**Theorem 3.3:** Let the symmetric positive definite matrix \( A \) be partitioned as

\[(3.31) \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} \]
so that $A_{11}$ is square, and let

$$\text{(3.32)} \quad B = \begin{pmatrix} I - A_{12} A_{22}^{-1} \\ 0 \end{pmatrix},$$

then

$$\text{(3.33)} \quad BAB' = \begin{pmatrix} A_{11} - A_{12} A_{22}^{-1} A_{21} \\ 12 22 21 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ A_{22} \end{pmatrix}.$$

By letting $A_{22}$ in (3.32) be the element $a_{nn}$ of the $(n \times n)$ matrix $A$, the resulting matrix, denoted by $B_n$, is used to obtain the product

$$\text{(3.34)} \quad B_n AB_n' = \begin{pmatrix} A_{11} - A_{12} A_{22}^{-1} A_{21} \\ 12 22 21 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ a_{nn} \end{pmatrix} = \begin{pmatrix} A_{11} n \quad 0 \\ 0 \quad a_{nn} \end{pmatrix}.$$

Proceed recursively to define $B_{n-1}$, $B_{n-2}$, ..., $B_2$ as above, using submatrices $A_{11 \cdot n}$, $A_{11 \cdot n-1, n}$, ..., $A_{11 \cdot 3, 4}$, ..., $n$ to pick the bottom diagonal positioned element. Then

$$\text{(3.35)} \quad B_{n-1} A_{11 \cdot n} B_{n-1}' = \begin{pmatrix} A_{11 \cdot n-1, n} \quad 0 \\ 0 \quad a_{n-1, n-1 \cdot n} \end{pmatrix}$$

...$$\text{(3.36)} \quad B_j A_{11 \cdot j+1, \ldots, n} B_j' = \begin{pmatrix} A_{11 \cdot j, \ldots, n} \quad 0 \\ 0 \quad a_{jj \cdot j+1, \ldots, n} \end{pmatrix}.$$
The dimensions of these products decrease by steps of 1 from \( n \) to 2. Let \( C_i \) be the \((n \times n)\) matrix partitioned as

\[
C_i = \begin{pmatrix} B_i & 0 \\ 0 & I_{n-i} \end{pmatrix},
\]

where \( B_i \) has dimension \( i + 1 \) and \( I \) is an \((n - i)\) identity matrix, \( i = 2, 3, \ldots, n \), with \( C_n = B_n \). Now forming the matrix product and substituting quantities (3.34) to (3.37), we have

\[
CAC' = C_2 C_3 \ldots C_{n-1} C_n A C_1 C_2 \ldots C_n C_2,
\]

which can be shown to be the diagonal matrix \( \text{Diag}(a_{11}, \ldots, a_{n}, \ldots, a_{nn}) \). This procedure indicates clearly how one may obtain the triangular matrix \( T \) numerically so that for \( A(n \times n) \) symmetric, positive definite and nonsingular, we have

\[
TAT' = I_n. \quad \text{Let } T = (CAC)^{-\frac{1}{2}}, \text{ where } C = (C_2 C_3 \ldots C_n) \text{ is triangular since it is the product of triangular matrices; then the condition}
\]

\[
TAT' = (CAC')^{-\frac{1}{2}} CAC(CAC')^{-\frac{1}{2}} = (CAC')(CAC')^{-1} = I_n
\]
is satisfied. Since $T$ is nonsingular, $A$ is expressible as the product $T^{-1}(T')^{-1}$. $A$ is also expressible as the product $LL'$, where $L$ is an $(n \times n)$ triangular matrix. The conditions of $A$ for which such an $L$ exists may be stated as follows.

**Theorem 3.4:** If $A(n \times n)$ is symmetric and positive definite, then there exists a nonsingular triangular matrix $L(n \times n)$ such that $LL' = A$ and, furthermore, $L$ is unique except for a post factor.

Now we combine the results of Theorems 3.3 and 3.4, proved in Roy (1957), to establish a relationship between the two triangular matrices $T$ and $L$.

**Lemma 3.1:** If $A$ is an $(n \times n)$ symmetric, positive definite matrix, then there exist triangular matrices $T$ and $L$ and a diagonal matrix $D$ such that $T = DL$ where $T$ and $L$ have the same configuration and are such that $TAT' = I_n$ and $LL' = A$.

**Proof:** Since $T$ and $L$ are nonsingular, $L^{-1}$ and $T^{-1}$ exist so that

$$A = T^{-1}T' = LL'.$$

Multiply on the left by $T$ and on the right by $L^{-1}$ to get $T'^{-1}L'^{-1} = TL$. The inverse of a triangular matrix has the same configuration as the matrix itself. Also, the product of two triangular matrices of the same configuration has a similar
configuration while a product involving opposite configurations is not triangular. Suppose $T$ and $L$ have the same triangular configuration. Then the product $TL$ has the configuration of $L$ while $T'{}^1L'{}^1$ has the configuration of $L'$, which in general is opposite of that of $TL$. The only way $T'{}^1L'{}^1$ can equal $TL$ is for them both to be equal to matrix $D$ so that

$$D = T'{}^1L'{}^1 = TL.$$  

Consequently, $T = DL^{-1}$ or $L = DT^{-1}$. Obviously, this relationship can hold only if $T$ and $L$ have similar configurations or are diagonal, and the proof is completed.

We next apply the preceding lemma in evaluating the part of the unconditional variance of estimator (3.9), denoted by $V_3$. Now

$$V_3 = AD^{-1}[E_3^WH_3D^{-1}]A',$$  

where

$$H_3 = H_2^{-1}BS_P^{-1}F'FS_P^{-1}B'H_2^{-1}$$  

and

$$H_2 = BS_P^{-1}B'.$$
To simplify $H_3$, which is symmetric, positive definite, express the $(n \times n)$ matrix product $(F',B')'S_p(F',B')$ in terms of its triangular factorization

$$(3.41) \quad (F_B)S_p(F',B') = LL',$$

where $L$ is such that

$$(3.42) \quad LL' = \begin{pmatrix} L_{11} & 0 \\ \frac{L_{21}}{L_{22}} & \frac{L_{22}}{L_{22}} \end{pmatrix} \begin{pmatrix} L'_1 & \frac{L'_2}{L_{22}} \\ 0 & \frac{L_{22}}{L_{22}} \end{pmatrix}.$$

The matrix $L_{11}$ is $(n-p) \times (n-p)$, $L_{21}$ is $p \times (n-p)$, and $L_{22}$ is $(p \times p)$.

Since the $(n \times n)$ matrix $(F',B')$ is orthonormal and $S_p$ is nonsingular, the inverse of $LL'$ can be written

$$(3.43) \quad (F_B)S_p^{-1}(F',B') = (LL')^{-1},$$

$$\begin{pmatrix} F_{SP}^{-1}F' \\ BS_{SP}^{-1}F' \end{pmatrix} \begin{pmatrix} F_{SP}^{-1}B' \\ BS_{SP}^{-1}B' \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix} \begin{pmatrix} -L_{12} & L_{22} \\ -L_{22} & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & L_{22} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{pmatrix} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & L_{22} \end{pmatrix} \begin{pmatrix} -L_{12} & L_{22} \\ -L_{22} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & L_{22} \end{pmatrix}. $$

Consequently,

$$(3.45) \quad BS_{SP}^{-1}B' = L_{22}^{-1}.$$
and

\[(3.46) \quad B^{S^{-1}}_{F} = -L^{1}_{22}{L^{-1}_{22}}L^{1}_{21}L^{-1}_{21}.
\]

Substituting (3.45) and (3.46) into \(H_{3}\) \((p \times p)\), we have

\[(3.47) \quad H_{3} = (L^{1}_{22}{L^{-1}_{22}})^{-1}(-L^{1}_{22}{L^{-1}_{22}}L^{1}_{21}L^{-1}_{21})(-L^{1}_{21}L^{1}_{22}) (L^{1}_{22}{L^{-1}_{22}})^{-1}
\]

\[= (L^{1}_{22}L^{1}_{22}L^{1}_{21}L^{-1}_{21}) (L^{1}_{22}L^{1}_{22}L^{1}_{21}L^{-1}_{21})^{-1}
\]

\[= L^{1}_{21}L^{-1}_{11}L^{1}_{21}L^{-1}_{11}
\]

\[= (L^{1}_{21}L^{-1}_{11})(L^{1}_{21}L^{-1}_{11})
\]

\[= \left(\sum_{k=1}^{n-p} \sum_{j=k}^{n-p} \bar{\varepsilon}_{ij} \bar{\varepsilon}_{jk}, \sum_{j'=k}^{n-p} \bar{\varepsilon}_{mj, \bar{\varepsilon}_{j', k}}\right)_{i,m}
\]

\[= \left(\sum_{k=1}^{n-p} \sum_{j=1}^{n-p} \sum_{j'=1}^{k} \varepsilon_{ij, \bar{\varepsilon}_{jk}, \bar{\varepsilon}_{j', k}}\right)_{i,m},
\]

where

\[L_{21}^{1} = [\varepsilon_{ij}], \quad i = 1, 2, \ldots, p, \quad j = 1, 2, \ldots, n-p,
\]

\[L^{-1}_{11} = [\bar{\varepsilon}_{ij}], \quad i, j = 1, 2, \ldots, n-p.
\]

The expected value of the \((i,m)^{th}\) element of \(H_{3}\) can be expressed as
which can be evaluated by finding the distribution of the individual elements of the triangular matrix $L$, which we now proceed to do. The $(n \times n)$ symmetric matrix $LL'$, defined by (3.41), is distributed as a central Wishart with an expected value of $I_n$ and $r - 1$ degrees of freedom provided the $(n \times n)$ estimated dispersion matrix $S$ is distributed as Wishart with expected value $V$ and $r - 1$ degrees of freedom since

$$LL' = (r - 1)^{-1}OS_0'$$

$$= (r - 1)^{-1}OPSP'0'$$

$$= (r - 1)^{-1}QSQ',$$

where $Q = OP$ is an $(n \times n)$ orthonormal matrix such that $QQ' = I_n$ and $P(n \times n)$ is a nonsingular matrix, such that $PVP' = I_n$. The distribution of $LL'$ given $V$, obtained from (3.12) by applying transformation (3.49), can be written

$$f(\frac{r-n-2}{2} \frac{1}{2} u) = f(\frac{r-n-2}{2} \frac{1}{2} u)$$

where

$$K = \left(\frac{n(r-1)}{2}\right)^{\frac{n(n-1)}{4}} \prod_{i=1}^{n} \Gamma\left(\frac{r-i}{2}\right)$$
and

\[ u = \text{tr}Q'LL'Q'V^{-1} \]
\[ = \text{tr}LL'(QVQ')^{-1} \]
\[ = \text{tr}LL'. \]

The Jacobian \( J(S : LL') \) of the orthogonal transformation \( Q : S+LL' \) is unity.

Next, we seek the distribution of the individual elements of the \((n \times n)\) triangular matrix \( L = [l_{ij}] \) where \( LL' = (r-1)QSQ' \) has distribution (3.50). The Jacobian of the transformation \( M : (LL')_L^r \rightarrow L \) (Roy (1957)) is

\[
(3.51) \quad J(LL':L) = 2 \prod_{i=1}^{n} \frac{\xi_{ii}^{n-i}}{(r-1)^{2'}} \prod_{i=1}^{n} \xi_{ii}^{n-i} = K \prod_{i=1}^{n} \xi_{ii}^{n-i},
\]

where \( K = 2^{\frac{n(n-1)}{4}} \frac{\prod_{i=1}^{n} \xi_{ii}^{n-i}}{(r-1)^{2'}} \). The mapping of the elements of \((LL')_L^r\) into the lower triangular matrix \( L \) can be shown to be \( \frac{r(n+1)}{2} \) to 1 where \((LL')_L\) is the lower triangular part of the symmetric matrix \( LL' \). By applying the Jacobian (3.51) to density (3.50), the joint density of the individual elements of the \((n \times n)\) triangular matrix \( L \) given \( V \) can be expressed
\[(3.52)\]

\[ f_1(L/V) = K|L|^{2\left(\frac{r-n-2}{2}\right)} e^{-\frac{1}{2} u J(\text{LL}' : L)} \]

\[ = K \prod_{i=1}^{n} \lambda_{i i}^{r-n-2} e^{\frac{1}{2} \text{tr} \text{LL}' \text{LL}'} \prod_{i=1}^{n} \lambda_{i i}^{-1} \]

\[ = K \prod_{i=1}^{n} \lambda_{i i}^{r-i-2} e^{\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i} \lambda_{i j}^2} \]

\[ = K \prod_{i=1}^{n} \left( \lambda_{i i}^{r-i} e^{\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i} \lambda_{i j}^2} \right) \prod_{j=1}^{n} e^{\frac{1}{2} \lambda_{j j}^2} \]

\[ = \left( \prod_{i=1}^{n} g(\lambda_{i i}^2) \right) \left( \prod_{j=1}^{n} h(\lambda_{j j}) \right) , \]

where

\[ g(\lambda_{i i}^2) = \frac{1}{2^\frac{r-i-1}{2}} \left( \lambda_{i i}^2 \right)^{\frac{r-i-1}{2}} e^{-\frac{\lambda_{i i}^2}{2}} , \quad i = 1, 2, \ldots, n , \]

\[ h(\lambda_{i j}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \lambda_{i j}^2} , \quad i > j = 1, 2, \ldots, n , \]
with \( K_2 \) expressible as the product

\[
K_2 = KK_1
\]

\[
= \frac{n(r-1)}{2} \frac{(r-1)}{n(r-1)} \frac{n(n-1)}{2} \frac{2}{\pi^n} \frac{\Gamma\left(\frac{r-i}{2}\right)}{(r-1)^2}
\]

\[
= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{n} \left(\frac{1}{\sqrt{2\pi}}\right).
\]

Since the joint probability density of the set of individual elements of \( L \) (3.52) can be factored into the product of chi-square densities for the square of diagonal elements and normal densities for off-diagonal elements, then the complete set of elements of \( L \) are mutually independently distributed. Each diagonal element of \( L \) is independently distributed as the square root of a chi-square random variable with \( r - i \) degrees of freedom, and each off-diagonal element is normal independently distributed with mean zero and variance one.

Applying the above results, expression (3.48) can now be evaluated. We have

\[
\hat{E}_w(x_{ik}'x_{mk}'x_{jk}'x_{jk}') = \hat{E}_w(x_{ik}'x_{mk}')\hat{E}_w(x_{jk}'x_{jk}')
\]

\[
= \delta_{ik} \hat{E}_w(x_{jk}'x_{jk}')
\]
where

\[ \delta_{ik}^{mk'} = 1 \quad \text{if } (i,k) = (m,k') \]

\[ = 0, \text{otherwise,} \]

so that

\[ (3.53) \quad E_w(H_3)_{ii} = [E_w^{n-p} \left( \sum_{i=1}^{n-p} \sum_{k=1}^{p} (\overline{v}_{ik})^2 \right)], \quad i = 1, 2, \ldots, p \]

\[ = [E_w^{tr}(L^{-1}L^T_{11})], \quad i = 1, 2, \ldots, p. \]

To evaluate (3.53) we make use of the following. Let A be an \((n \times n)\) triangular matrix consisting of positive diagonal elements that are distributed as chi-square with \(f\) degrees of freedom and off-diagonal elements that are functions of random normal and chi-square variables; then the expected value of \(trA^{-1}\) is

\[ E(trA^{-1}) = \frac{n\Gamma\left(\frac{f}{2}-1\right)}{2\Gamma\left(\frac{f}{2}\right)} = \frac{n}{f-2}. \]

Proof: We assume that \(A^{-1}\) exists almost everywhere for the random matrix A such that \(AA^{-1} = A^{-1}A = I_n\). Since the elements are independently distributed, we need consider only the distributional properties of the diagonal elements where
the $i^{th}$ diagonal element of $A^{-1}$ is $a_{ii}^{-1}$, and $a_{ii} > 0$ is the $i^{th}$ diagonal element of $A$. We have $a_{ii} \sim \chi^2(f)$. Consequently, $Ea_{ii}^{-1}$ can be expressed as

$$g_i = Ea_{ii}^{-1} = \int_0^\infty a_{ii}^{-1} dF(a_{ii})$$

$$= \int_0^\infty \frac{a_{ii}}{\Gamma(\frac{f}{2})} e^{-\frac{a_{ii}}{2}} da_{ii}$$

$$= \frac{\Gamma(\frac{f}{2}-1)}{2\Gamma(\frac{f}{2})} = \frac{1}{f-2},$$

and since the $a_{ii}$'s are identically distributed, the result follows directly.

Finally, we consider the exact form of the inverse $B_{11} = \{b_{ij}\}$ of the $(n-p) \times (n-p)$ triangular matrix $L_{11}$. It can easily be shown that the elements of the $k^{th}$ row of $B$ have the form

$$b_{k,k} = \frac{1}{\ell_{kk}},$$

$$b_{k,k-1} = -\sum_{j=k-1}^{k-1} \frac{\ell_{kj}}{\ell_{kk}} b_{j,k-1},$$

$$b_{k,k-2} = -\sum_{j=k-2}^{k-1} \frac{\ell_{kj}}{\ell_{kk}} b_{j,k-2}. $$
\[ b_{k,i} = -\sum_{j=i}^{k-1} \frac{\ell_{ki}}{k_{kk}} b_{ji}, \]

\[ b_{k,1} = -\sum_{j=1}^{k-1} \frac{\ell_{ki}}{k_{kk}} b_{j1}, \text{ and } b_{kj} = 0, j > k. \]

Let \( a_{kk} \) be the \( k \textsuperscript{th} \) diagonal element of the matrix product 

\[ L^\textsuperscript{-1} L^\textsuperscript{-1}. \]

Then for \( k = 2, \ldots, n-p, \)

\[ \begin{equation}
(3.55) \quad a_{kk} = \sum_{m=1}^{k} b_{km}^2
\end{equation} \]

\[ = \frac{1}{\ell_{kk}^2} + \sum_{m=1}^{k-1} \left( \sum_{j=m}^{k-1} \frac{\ell_{kj} b_{jm}}{\ell_{kk}^2} \right)^2. \]

The first moment of \( a_{kk} \) can be partially obtained by applying previous results to the expected value of the inverse of a chi-square random variable. Now we have

\[ \begin{equation}
(3.56) \quad E(a_{kk}) = E\left(\frac{1}{\ell_{kk}^2}\right) + \sum_{m=1}^{k-1} \sum_{j=m}^{k-1} E\left(\frac{\ell_{kj} b_{jm}}{\ell_{kk}^2}\right), \quad k = 2, \ldots, n-p,
\end{equation} \]

since the \( \ell_{ij} \)'s are independently distributed \( \text{N}(0,1) \) with

\[ E\ell_{ij} \ell_{ij}' = E\ell_{i}'\ell_{i}, \quad \ell_{ij} = 0, \] and the \( \ell_{ij}^2 \)'s are independently distributed as chi-square with one degree of freedom. Also, the ratios of the form \( (\ell_{ij}^2/1)/(\ell_{ii}^2/f_i) \) are independently distributed as a Snedecor \( F \) with \( (1,f_i) \) degrees
of freedom, denoted by $F(1, f_i)$, where $f_i = r - i$. Let

$$F'_{ij} = \frac{\ell_{ij}^2}{\ell_{ii}^2} = \frac{1}{f_i} \frac{\ell_{ij}^2}{f_i} = \frac{1}{f_i} F_{ij},$$

where $F_{ij} \sim F(1, f_i)$. Then the expected value of $F'_{ij}$ is

$$(3.57) \quad g_2 = E F'_{ij} = \frac{1}{f_i} E F_{ij} = \frac{1}{f_i} \left( \frac{f_i}{1} \right) \frac{\Gamma(1+\frac{1}{2})}{\Gamma\left(\frac{1}{2}\right)} \left( \frac{\frac{f_i}{2}-1}{\frac{f_i}{2}} \right) = \frac{1}{f_i-2}.$$

This value can be used to find a value for $E(U_{jkm}^2)$ where $U_{jkm}^2 = \ell_{kj}^2 b_{jm}^2 / \ell_{kk}^2$ can be represented as a sum of products of independently distributed $F$ random variables and inverse chi-square random variables $G_i$ of the form $\Sigma G_i F_{ij}$. The expected value of $U_{jkm}^2$ can be expressed as

$$EU_{jkm}^2 = E \sum_{i=2}^{q} G_i F_{ij}$$

$$= \sum_{i=2}^{q} \alpha_i \left( \frac{f_i}{2} - 1 \right) \frac{\Gamma\left(\frac{f_i}{2}\right)}{\Gamma\left(\frac{f_i}{2}-1\right)},$$

where $\alpha$ is a function of $k$.

The trace of $A = (L_{11}^{-1} L_{11}^T)$ can now be expressed as $\sum_{k=1}^{n-p} a_{kk}$ with an expected value of
\[(3.58)\]
\[
E_{\text{trA}} = \sum_{k=1}^{n-p} a_{kk}
\]
\[
= \mathbb{E} \left[ \frac{1}{\ell_{11}} + \sum_{k=2}^{n-p} \left( \frac{1}{\ell_{kk}} + \sum_{m=1}^{k-1} \sum_{j=m}^{k-1} \left( \frac{\ell_{kj}\ell_{jm}}{\ell_{kk}} \right)^2 \right) \right]
\]
\[
= \sum_{k=1}^{n-p} \mathbb{E} \left( \frac{1}{\ell_{kk}^2} \right) + \sum_{k=2}^{n-p} \sum_{m=1}^{k-1} \sum_{j=m}^{k-1} \mathbb{E} \left( \frac{\ell_{kj}\ell_{jm}}{\ell_{kk}^2} \right)
\]
\[
= \sum_{k=1}^{n-p} \mathbb{E} \left( \frac{1}{\ell_{kk}^2} \right) + \sum_{k=2}^{n-p} \sum_{m=1}^{k-1} \sum_{j=m}^{k-1} \mathbb{E} \left( \frac{\ell_{kj}\ell_{jm}}{\ell_{kk}^2} \right)
\]
\[
+ \sum_{k=3}^{n-p} \mathbb{E} \left( \frac{\ell_{kj}\ell_{km}}{\ell_{kk}^2} \right) + \sum_{k=4}^{n-p} \mathbb{E} \left[ \frac{\ell_{kj}\ell_{km}}{\ell_{kk}^2} \right]
\]
\[
+ \frac{\ell_{kj}\ell_{km}}{\ell_{kk}^2} \left( \frac{\ell_{k-1,k-3}}{\ell_{k-1,k-3}^2} \right)
\]
\[
+ \frac{\ell_{k-1,k-2}\ell_{k-2,k-3}}{\ell_{k-1,k-2}^2} \right) + R_5,
\]
where

$$R = \sum_{k=5}^{n-p-k} \sum_{m=1}^{k} \sum_{j=k-m}^{k-1} E \left( \frac{\chi^2_{k-j} \chi^2_{j-m}}{\chi^2_{k-m}} \right)$$

provided $n - p \geq 5$; otherwise, $R = 0$. Expression (3.58) can be transformed into a sum of products of the form (3.57) and evaluated very easily by simply observing the combinatorial patterns involved in expressing the inverse elements in terms of the $\ell_{ij}$'s. For example, consider the combinations of the first column of the matrix $B_{11} \ast B_{11}$, where $\ast$ denotes the Hadamard product defined as $A \ast B = [a_{ij}b_{ij}]$, where $A(n \times p) = [a_{ij}]$ and $B(p \times m) = [b_{ij}]$. The diagonal element $b^2_{11}$ has the form $X^{-1}_{11} \sim \chi^2(f_1)$. The first off-diagonal element $b^2_{21}$ is expressible as $X_{21}(X_{11})$ where $X_{21} \sim F(1, f_2)$ independently of $X_{11}$; the second off-diagonal element $b^2_{31}$ is expressible as the function of $b^2_{11}$ and $b^2_{21}$, denoted as $X_{31}(X_{11} + X_{21}(X_{11}))$, where $X_{31} \sim F(1, f_3)$ independent of $X_{11}$ and $X_{21}$; similarly, the third off-diagonal element $b^2_{41}$ is expressible as $X_{41}(X_{11} + X_{21}(X_{11}) + X_{31}(X_{11} + X_{21}(X_{11})))$, where $X_{41} \sim F(1, f_4)$ independent of $X_{11}$, $X_{21}$, and $X_{31}$. Continuing in this manner, each term of $B_{11} \ast B_{11}$ is expressible as a sum of products of inverse chi-square and $F$ random variables. In evaluating expected values of this sum of products, it will be convenient to rearrange the order of addition of terms in (3.58). First, find the
expected value of column totals of $B_{\text{11}} \times B_{\text{11}}$. These totals are expressible as $c_k = \sum_{i=k}^{n-p} b_{i,k}^2$, $k = 1, 2, \ldots, n-p$. For $k = n-p$ we have

\begin{equation}
E(c_{n-p}) = E(b_{n-p,n-p}^2) = E(F_{0,n-p}) = \frac{1}{r - (n-p) - 2}
\end{equation}

by applying result (3.57) to the one nonzero element in the $(n-p)^{th}$ column total of $B_{\text{11}} \times B_{\text{11}}$. The expected value of the $(n-p-1)^{th}$ column total $c_{n-p-1}$ is

\begin{equation}
Ec_{n-p-1} = Eb_{n-p-1,n-p-1}^2 + Eb_{n-p,n-p-1}^2 = EF_{0,n-p-1} + E(F_{n-p,n-p-1} F_{0,n-p-1}) = \frac{1}{r - (n-p-1) - 2} + \frac{1}{r - (n-p) - 2} = \frac{1}{r - (n-p) - 2}.
\end{equation}

These results suggest the possibility that the $c_i$'s have a constant expected value of $1/(r - (n-p) - 2)$. For the $(n-p-m)^{th}$ column total $c_{n-p-m}$, the expected value is expressible as
where $k = n - p - m$. The reordering of addition of terms (3.61) greatly simplifies the collection of subsets of terms, as will be seen.

Expanding each term of (3.61) utilizing (3.54), we have the following expected values. For the $k^{th}$ term we have

\begin{equation}
Eb_{k,k} = E(F_{k,0})
= \left(\frac{1}{r - (k+0) - 2}\right).
\end{equation}

For the $(k+1)^{th}$ term we have

\begin{equation}
Eb_{k+1,k} = EF_{k+1,k}F_{k,0}
= \left(\frac{1}{r - (k+1) - 2}\right)\left(\frac{1}{r - (k+0) - 2}\right).
\end{equation}

For the $(k+2)^{th}$ term we have

\begin{equation}
Eb_{k+2,k} = EF_{k+2,k}(1 + F_{k+1,k})F_{k,0}
= \left(\frac{1}{r - (k+2) - 2}\right)(1 + \frac{1}{r - (k+1) - 2})\left(\frac{1}{r - (k+0) - 2}\right)
= \left(\frac{1}{r - (k+2) - 2}\right)\left(\frac{1}{r - (k+1) - 2}\right).
\end{equation}
For term $b^2_{k+3,k}$ we have

\[(3.65) \quad E b^2_{k+3,k} = E F_{k+3,k} \left( (1 + F_{k+1,k}) + F_{k+2,k+1} (1 + F_{k+1,k}) \right) F_{k,0} \]

\[= \frac{1}{r - (k+3) - 2} \left( \left(1 + \frac{1}{r - (k+1) - 2} \right) + \frac{1}{r - (k+2) - 2} \left(1 + \frac{1}{r - (k+1) - 2} \right) \right) \]

\[= \frac{1}{r - (k+0) - 2} \left( \left(1 + \frac{1}{r - (k+1) - 2} \right) + \frac{1}{r - (k+2) - 2} \left(1 + \frac{1}{r - (k+1) - 2} \right) \right) \]

\[= \frac{1}{r - (k+3) - 2} \frac{1}{r - (k+1) - 2} \frac{1}{r - (k+2) - 2} \left(1 + \frac{1}{r - (k+1) - 2} \right) \]

\[= \frac{1}{r - (k+3) - 2} \frac{1}{r - (k+2) - 2} . \]

Continuing in the above manner, the expected value of $b^2_{k+j,k}$ is

\[(3.66) \quad E b^2_{k+j,k} = \frac{1}{r - (k+j) - 2} \frac{1}{r - (k+j - 1) - 2} \]

for $j \leq n - p - k$ since

\[E b^2_{k+j,k} = E F_{k+j,k} \left\{ \left( (1 + F_{k+1,k}) + F_{k+2,k+1} (1 + F_{k+1,k}) \right) \right. \]

\[+ F_{k+3,k} (1 + F_{k+1,k}) + F_{k+2,k} (1 + F_{k+1,k}) \]

\[+ \ldots + F_{k+j-1,k} (1 + F_{k+1,k}) + F_{k+2,k} (1 + F_{k+1,k}) \]

\[+ \ldots + \frac{k+j-2}{\Pi_{i=1}^{I} F_{k+i,k}} \right\} F_{k,0} , \]
\[
\frac{1}{r-(k+j)-2}\left[\frac{1}{r-(k+0)-2} + \frac{1}{r-(k+1)-2}\left(\frac{1}{r-(k+0)-2}\right)\right] + \ldots + \frac{1}{r-(k+j-2)-2}\left(\frac{1}{r-(k+j-1)-2}\right)\].
\]

The result follows by collecting succeeding pairs of terms in the above expression. The \(a\)th step, for example, yields

\[
(3.67) \quad \frac{1}{r-(k+a)-2} + \frac{1}{r-(k+a+1)-2} \frac{1}{r-(k+a)-2} = \frac{1}{r-(k+a+1)-2}.
\]

Applying result (3.67) along with results (3.62) to (3.66), we can now evaluate (3.61) to get

\[
(3.68) \quad E_{c_{n-p-m}} = \sum_{i=k}^{n-p} b_i^2 = \frac{1}{r-(n-p)-2} + \frac{n-p}{r-(k+0)-2} \frac{1}{r-(k+1)-2} \frac{1}{r-(k+i-1)-2}
\]

\[
= \frac{1}{r-(n-p)-2}.
\]

An exact value for (3.58) can be obtained by summing over the \((n-p)\) \(c_i\)'s to get

\[
(3.69) \quad E_{trA} = \sum_{i=1}^{n-p} c_i = (n-p) \frac{1}{r-(n-p)-2} = a(r, n, p),
\]
which is a result stated but not shown by Williams (1967).
An incorrect statement regarding the distribution of the
diagonal elements of a matrix of the type \((L_{11} L'_{11})^{-1}\) was made
by Williams (1967). Upon examining expression (3.58), it is
clear that the diagonal elements of \((L_{11} L'_{11})^{-1} = L'_{11}L_{11}^{-1}\) are
not inverse of a chi-square variable but are a sum of products
of chi-square and F random variables, which may be treated as
a sum of products of inverse chi-square due to mutual indepen­
dence and the fact that the expected value of all random
variables appearing in the numerator is unity.

The unconditional variance of estimator (3.9) can now be
written in terms of a scalar multiple of the unconditional
variance of estimator (3.8) as

\[
(3.70) \quad \text{Var}(\hat{\tau}_w) = r^{-1}(1+\alpha(r,n,p))(X'V^{-1}X)^{-1},
\]

where \(\alpha(r,n,p)\) is of the form (3.69).

We note that \(\alpha(r,n,p) > 0\) for all finite \(r, n,\) and \(p\)
satisfying the restrictions \(r > (n-p) - 2\) and \(n > p\), so \(\hat{\tau}_w\) is not
a "best" unbiased function for \(\tau\). The question is how does \(\hat{\tau}_w\)
compare with \(\hat{\tau}_w\) over the range of combinations of \(r, n,\) and \(p\).
For fixed \(n\) and \(p\), we have

\[
\lim_{r \to \infty} \alpha(r,n,p) = 0,
\]
which shows that $\hat{\tau}_w$ is asymptotically efficient. On the other hand, $\alpha(r,n,p)$ approaches a maximum value when $r+(n-p)-2$ at the same time as $n-p+\infty$, thus causing $\alpha(r,n,p)\to\infty$. This indicates that weighting is very poor in cases where one wishes to estimate a few parameters from data collected from a large number of correlated and widely variable normal populations. On the other hand, weighting can be profitable for cases where $r-(n-p)-2$ is large relative to $n-p$. Results of empirical studies comparing several possible estimators of $\tau$ are presented in Chapter VI.

The estimate

\begin{equation}
(3.71) \quad \text{Var}(\hat{\tau}_w) = r^{-1}(1+\alpha(r,n,p))(X'S^{-1}X)^{-1}
\end{equation}

can be shown to be unbiased for (3.70) by applying (3.20), (3.21), (3.28), and (3.29), along with the fact that the $(n\times n)$ nonsingular matrix $P$ is such that $E(PSP') = I_n$. The expected value of $(X'S^{-1}X)^{-1}$ is expressed as

\begin{equation}
(3.72) \quad E(X'S^{-1}X)^{-1} = E[X'P'P^{-1}S^{-1}P^{-1}PX]^{-1}
\end{equation}

\begin{equation}
= E[(PX)'(PSP')^{-1}(PX)]^{-1}.
\end{equation}

Upon substituting ADB for $(PX)'$ in (3.72) and applying result (3.30), we have
(3.73) \[ E(X'S^{-1}X)^{-1} = AD^{-1}BE(PSP')B'D^{-1}A' \]

\[ = AD^{-2}A' \]

\[ = (X'V^{-1}X)^{-1}, \]

where A, D, and B are matrices from (3.20), thus indicating that (3.71) is an unbiased estimate for (3.70).

For cases of nonfull rank models where the (n x p) matrix X has rank q (q < p), it is known that the best linear estimator of any estimable set, \( \lambda'\tau \) say, is uniquely given by

(3.74) \[ \lambda'\hat{\tau}_w = \lambda'(X'V^{-1}X)^*X'V^{-1}Y, \]

where \( (X'V^{-1}X)^* \) is any conditional inverse of \( (X'V^{-1}X) \), i.e., is such that \( (X'V^{-1}X)(X'V^{-1}X)^*(X'V^{-1}X) = (X'V^{-1}X) \). The dispersion matrix for (3.74) is

(3.75) \[ \text{Var}(\lambda'\hat{\tau}) = \lambda'(X'V^{-1}X)^*\lambda/r. \]

The operator

(3.76) \[ X(X'V^{-1}X)^*X'V^{-1} \]

is a unique projection on \( C(X) \) and is independent of the conditional inverse \( (X'V^{-1}X)^* \), which implies that (3.76) is unique. If Z is any matrix such that \( C(Z) = C(X) \), then

(3.77) \[ Z(Z'V^{-1}Z)Z' = X(X'V^{-1}X)^*X'. \]
So, if we pick $Z$ to be full rank, then the vector of observations $Y$ can be expressed in terms of the full rank reparametrized model

\[(3.78) \quad Y = X\tau + \epsilon = Z\delta + \epsilon.\]

The estimable function $\lambda'\tau = a'X\tau$ then corresponds to $v'\delta = a'Z\delta$ and the estimator $\lambda'\hat{\tau}$ is equal to $v'\hat{\delta}$ and thus the variance of $\lambda'\hat{\tau}$ is equal to the variance of $v'\hat{\delta}$. Now, since $Z$ has full rank we have

\[(3.79) \quad (Z'V^{-1}Z)^\# = (Z'V^{-1}Z)^{-1},\]

so (3.77) is expressible in terms of a unique inverse as

\[(3.80) \quad X(X'V^{-1}X)^X = Z(Z'V^{-1}Z)^{-1}Z'.\]

Expressions (3.79) and (3.80) also hold when $S^{-1} = (\text{Est} V)^{-1}$ is substituted for $V^{-1}$ so that an estimate for the variance of $\lambda'\hat{\tau}$ is expressible as

\[(3.81) \quad \text{Var}(\lambda'\tau) = \text{Var}(v'\delta)\]

\[= (1 + \alpha(r,n,p))v' \text{Var}(\delta)v\]

\[= (1 + \alpha(r,n,p))v'(Z'S^{-1}Z)^{-1}v\]

\[= (1 + \alpha(r,n,p))a'Z(Z'S^{-1}Z)^{-1}Z'a\]

\[= (1 + \alpha(r,n,p))a'X(X'S^{-1}X)^Xa\]

\[= (1 + \alpha(r,n,p))\lambda'(X'S^{-1}X)^\#\lambda.\]
IV. COMBINING INFORMATION UNDER POPULATION CONDITIONS OF A DIAGONAL DISPERSION MATRIX

A. Introduction

When the individual observations denoted by the vector $Y_j (m \times 1)$ are uncorrelated, then this information should be taken into account in estimating the dispersion matrix of $Y_j$. For this case we assume the data can be described by the linear model (3.6) with a dispersion matrix $V = \text{Diagonal} (\sigma_{11}, \sigma_{22}, \ldots, \sigma_{mm}), \sigma_{ii} > 0$ for each $i, i = 1, 2, \ldots, m$.

As in III B, we seek a good estimator for the $(p \times 1)$ vector $\tau$ of the linear model $Y_j = X_j \tau + e_j, j = 1, 2, \ldots, r$, when the diagonal dispersion matrix is unknown. The best linear unbiased estimator is of the form (3.8) for the case where $V$ is known. If one chooses to ignore possible gains derived from estimated weights, the simple least squares estimator of the form (3.7) is used.

The weighted estimator used in Chapter III is of the form (3.9), which is

$$\hat{\tau}_w = (X'S^{-1}X)^{-1}X'S^{-1}Y.$$  

The sample dispersion matrix $S$ may then be estimated by the method of maximum likelihood corrected for bias, given by (3.11), but since the observations within each replicate are uncorrelated, this method would be somewhat inefficient.
because, in effect, it estimates the known off diagonal zeros. An alternative is to estimate only the diagonal elements of the dispersion matrix. However, the complexion of the problem would then be changed from an essentially multivariate to an essentially univariate one and would deny the valid use of the Wishart distribution properties in obtaining the variance-covariance matrix of the combined estimator.

B. Random Sampling with Equal Replication of Two Sources

For this experimental situation we have the linear model

\[(4.2) \quad Y_j = X_j \tau + \varepsilon_j, \quad j = 1, 2, \ldots, r, \]

where \(Y_j (m \times l) = (y_{ij}) \) is a vector of observations, with \(i \) as a running subscript,

\[X_{(m \times p)} = X_j, \quad l \neq j, \text{is an (m \times p) known design matrix,} \]

with \(m > p \) and \( \text{rank} X_j = p \),

\(\tau \) is a \((p \times 1)\) unknown parameter vector, and

\(\varepsilon_j \) is an \((m \times 1)\) vector of residuals such that

\[E\varepsilon_j = 0, \quad E\varepsilon_j\varepsilon_j' = 0, \quad E\varepsilon_j\varepsilon_j' = \text{Diag}(\sigma_{11}, \ldots, \sigma_{mm}), \]

\(l, j = 1, 2, \ldots, r, \quad l \neq j.\)

The estimator we wish to examine is of the form \((4.1)\)

where only diagonal elements are estimated. These elements of within sample variance are calculated as
(4.3) \[ s_{ii} = \frac{1}{r-1} \sum_{j=1}^{r} (y_{ij} - \bar{y}_i)^2, i = 1, 2, \ldots, m, \]

which are independently distributed as \( \sigma_{ii}^2 (r-1) \) and are unbiased estimates of \( \sigma_{ii}^2 \).

The unconditional variance to our modification of (4.1) is expressible as

(4.4) \[ r \text{Var}(\hat{\tau}_w) = rE_w[\text{Var}(\hat{\tau}_w/\hat{w})] \]
\[ = E_w[(X'S_D^{-1}X)^{-1}X'S_D^{-1}vS_D^{-1}X(X'S_D^{-1}X)^{-1}], \]

where

\[ S_D = \text{Diag}(s_{11}, s_{22}, \ldots, s_{mm}). \]

For the special design matrix \( X_j = J_m^1 \), the parameter vector \( \tau \) reduces to an unknown scalar, say \( \mu \). In this case the unconditional variance of \( \hat{\mu}_w \) is expressible as

(4.5) \[ r \text{Var}(\hat{\mu}_w) = E_w[(\sum_{i=1}^{m} \frac{1}{s_{ii}})^{-1} (\sum_{i=1}^{m} \frac{1}{s_{ii}} \sigma_{ii}) (\sum_{i=1}^{m} \frac{1}{s_{ii}})^{-1}] \]
\[ = E_w \sum_{i=1}^{m} \frac{1}{s_{ii}^2} \sigma_{ii} \]
\[ \sum_{i=1}^{m} \hat{\alpha}_i^2 \sigma_{ii}, \]

where

\[ \hat{\alpha}_i = \frac{1}{S_{ii}} \quad \text{and} \quad \sum_{i=1}^{m} \hat{\alpha}_i = 1. \]

To reduce the problem to the simplest form, consider the case \( m = 2 \). For this case

\[ (4.6) \]

\[ \hat{\alpha}_1 = \frac{\hat{w}_1}{\hat{w}_1 + \hat{w}_2} \]

\[ = \frac{1}{1 + \frac{\hat{w}_2}{\hat{w}_1}}, \]

where

\[ \hat{w}_i = \frac{1}{S_{ii}} \quad \text{i = 1, 2}. \]

Then

\[ \frac{\hat{w}_2}{\hat{w}_1} = \frac{1/S_{22}}{1/S_{11}} \]

\[ = \frac{\sigma_{11}}{\sigma_{22}} \cdot \frac{S_{11}}{S_{22}} \cdot \frac{S_{11}}{S_{22}} \]

\[ = \rho_{12} F_{12}, \]

with
\[ \rho_{12} = \frac{\sigma_{11}}{\sigma_{22}}, \]

and

\[ F_{12} = \frac{s_{11}}{s_{22}} \frac{\sigma_{11}}{\sigma_{22}} \]

is distributed as a Snedecor F with \((r-1, r-1)\) degrees of freedom. Thus, \( \hat{\alpha}_1 \) is a function of the random variable \( \hat{\omega}_2 / \hat{\omega}_1 \), which is distributed according to a Snedecor F.

The unconditional variance involves evaluating the first and second moments of the random variable \( \hat{\alpha}_1 \). Now,

\[ (4.7) \quad \text{Var}(\hat{\mu}_w) = E[\text{Var}(\hat{\mu}_w)] . \]

Hence, since

\[ (4.8) \quad \text{Var}(\hat{\mu}_w) = \frac{\hat{\alpha}_1^2 \sigma_{11}}{r} + \frac{\hat{\alpha}_2^2 \sigma_{22}}{r} \]

\[ = m_2 \hat{\alpha}_1^2 + m_1 \hat{\alpha}_1 + m_0, \]

with

\[ m_2 = \frac{\sigma_{11} + \sigma_{22}}{r}, \quad m_1 = -2\sigma_{22}/r, \quad \text{and} \quad m_0 = \frac{\sigma_{22}}{r}, \]

our problem is reduced to that of determining the expected values of \( \hat{\alpha}_1^2 \) and \( \hat{\alpha}_1 \).
If $p_{12}$ were known to be unity, the problem could be reduced still further to one of finding the first and second moments of a Beta distribution with parameters $(f, f)$, where $f = r - 1$, since for that case

(4.9) \[ \text{E}(\hat{\alpha}_1)^C = K \int_{F_{12}} \left( \frac{1}{1+F_{12}} \right)^C dG(F_{12}) \]

\[ = K \int_0^\infty \left( \frac{1}{1+F_{12}} \right)^{f-1} \left( \frac{1}{1+F_{12}} \right)^{f} dF_{12} \]

\[ = K \int_0^\infty \left( \frac{F_{12}}{1+F_{12}} \right)^{f} \left( \frac{1}{1+F_{12}} \right)^{f} dF_{12} \]

where $c$ is a known constant and $K_1 = \Gamma(f)/\Gamma^2(f)$. Let $Z_{12} = F_{12}/(1+F_{12})$. Then the Jacobian of this transformation is $|J| = (1-Z_{12})^{-2}$. So we have

(4.10) \[ \text{E}(\hat{\alpha}_1)^C = K \int_{Z_{12}} \left( \frac{1}{Z_{12}} \right)^{f} \left( 1-Z_{12} \right)^{f+c-1} \]

By applying the results of the Beta distribution, the $c^{th}$ moment of $\hat{\alpha}_1$ is found to be

(4.11) \[ \text{E}(\hat{\alpha}_1)^C = K \]

where
\[ K_2 = \left( \frac{Z_{12}}{Z_{12}} \right)^{\frac{f}{2} + c - 1} dZ_{12} \]

\[ = \frac{\Gamma(\frac{f}{2})\Gamma(\frac{f}{2} + c)}{\Gamma(f + c)} = B\left(\frac{f}{2}, \frac{f}{2} + c\right), \]

and

\[ K_1 = \frac{\Gamma(f)}{\Gamma(\frac{f}{2})\Gamma(\frac{f}{2})}. \]

The unconditional variance of the combined estimators for \( m = 2 \) is obtained by substituting for \( E\{\hat{a}_1^c\}, \ c = 1, \text{ and } 2, \)

which gives

\[ V(\hat{\mu}_w) = \delta_1 \frac{\sigma_{11}}{r} + \delta_2 \frac{\sigma_{22}}{r}, \]

where

\[ \delta_1 = \frac{\Gamma(f)\Gamma(\frac{f}{2})\Gamma(\frac{f}{2} + 2)}{\Gamma^2(\frac{f}{2})\Gamma(f + 2)} \]

and

\[ \delta_2 = \delta_1 - \frac{2\Gamma(f)\Gamma(\frac{f}{2})\Gamma(\frac{f}{2} + 1)}{\Gamma^2(\frac{f}{2})\Gamma(f + 1)} + 1. \]

For \( f = 2n \) degrees of freedom the above expression can be evaluated quite easily since \( \Gamma(2n) = (2n - 1)! \) for \( n = 1, 2, \ldots. \)
Also, by using Legendre's duplication formula for gamma functions, we have the identity

\[ \frac{\Gamma(f)}{\Gamma(2f)} = \frac{\Gamma\left(\frac{1}{2}\right)}{2^{2f-1} \Gamma\left(f + \frac{1}{2}\right)}, \]

where \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \).

In general, \( \Gamma(f) \) is expressible as the integral

\[ \Gamma(f) = \int_{x=0}^{\infty} x^{f-1} e^{-x} dx, \]

which can be approximated for large values of \( f \) by Stirling's formula,

\[ \Gamma(f) \approx \sqrt{2\pi f} \left(\frac{f}{e}\right)^f. \]

For even degrees of freedom, we have with \( f = 2n, n \in \mathbb{N}^+ \),

\[ \delta = \delta_1 = \delta_2 = \frac{n+1}{2(2n+1)}, \]

where \( \mathbb{N}^+ \) denotes the set of positive integers.

One may inquire how much precision is lost due to using the random weights \( \alpha_i \) and \( (1 - \alpha_i) \) when the true state for \( \rho_{12} \) is unity. This question can be answered only in terms of loss relative to alternative estimators. One such estimator is
\[ \hat{\mu}_w = \left( \frac{x_1 + x_2}{2} \right) / 2 , \]

which has a variance of \( \sigma^2 / 2(2n + 1) \) when \( \sigma^2 = \sigma_{11} = \sigma_{22} \). The ratio of variance of \( \hat{\mu}_w \) and \( \hat{\mu}_w \) gives an indication of relative precision or relative efficiency of estimation. We have

\[
(4.13) \quad \text{R.E.} = \frac{\text{Var}(\hat{\mu}_w)}{\text{Var}(\hat{\mu}_w)} = \frac{1}{4\delta} = \frac{1 + \frac{2}{n}}{1 + \frac{4}{n}},
\]

which can easily be seen to have a limiting value of unity as \( n \) increases without bound. However, some loss of precision will always occur in practice when random weights of the form \( \gamma_1 \) are used to combine information when the true state is \( \alpha_1 = \frac{1}{2} \).

The problem for \( m = 2 \) takes on a different complexion when it is known that \( \rho_{12} \) is not unity. First and second moments of \( \gamma_1 \) must be obtained but without the aid of the Beta distribution. The use of the moment generating or characteristic function of \( \gamma_1 \) is a possibility, but for the present case repeated use of integration by parts allows one to proceed to evaluate the \( c^{th} \) moment where \( c \) is a positive integer.

Consider the definite integral,

\[
(4.14) \quad \psi = \int_{0}^{\infty} (1 + \rho_{12} x)^{-\alpha} (1 + x)^{-\beta} dx.
\]
This integral, whenever it exists, can be evaluated by repeated double partial integration in the following way. Let

\[ \delta_c = (1 + \rho \cdot x)^{-c} \quad \text{and} \quad dv_{\alpha \beta} = x^{\alpha-1}(1+x)^{-\alpha-\beta} \, dx, \]

then

\[ d\delta_c = -c \rho \cdot (1 + \rho \cdot x)^{-\alpha-1} \, dx \quad \text{and} \quad V_{\alpha \beta} = \int x^{\alpha-1}(1+x)^{-\alpha-\beta} \, dx. \]

We first obtain a pertinent expression for \( V_{\alpha \beta} \). Note that the definite integral

\[ V_{\alpha \beta} = B(\alpha, \beta), \text{ for } \alpha, \beta > 0, \]

where \( B(\alpha, \beta) \) is the well-known beta function defined by

\[ \frac{1}{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \, dx}. \]

In the following developments we take \( \alpha \) and \( \beta \) to be positive integers. The expression for \( V_{\alpha \beta} \) can be evaluated by repeated partial integration. If we let

\[ \theta_\alpha = x^{\alpha-1} \quad \text{and} \quad d\gamma_\alpha = (1+x)^{-\alpha-\beta} \, dx, \]

then

\[ d\theta_\alpha = (\alpha - 1)x^{\alpha-2} \, dx \quad \text{and} \quad \gamma_\alpha = \frac{(-1)}{(\alpha+\beta-1)}(1+x)^{-\alpha-\beta+1}, \]
so

\begin{equation}
\nabla_{\alpha \beta} = \frac{(-1)}{(\alpha+\beta-1)} x^{\alpha-1}(1+x)^{-(\alpha+\beta-1)} - \frac{(-1)(\alpha-1)}{(\alpha+\beta-1)} \nabla_{\alpha-1,\beta},
\end{equation}

where

\[ \nabla_{\alpha-1,\beta} = \int x^{\alpha-2}(1+x)^{-(\alpha+\beta-1)} \, dx. \]

An expression for \( \nabla_{\alpha-1,\beta} \) can be obtained similarly to get

\begin{equation}
\nabla_{\alpha-1,\beta} = \frac{(-1)}{(\alpha+\beta-2)} x^{\alpha-2}(1+x)^{-(\alpha+\beta-2)} - \frac{(-1)(\alpha-2)}{(\alpha+\beta-2)} \nabla_{\alpha-2,\beta}.
\end{equation}

This process can be continued until the exponent of \( x \) in the first term is unity. Then

\[ \nabla_{2,\beta} = \left( \frac{(-1)}{(\beta+1)} x(1+x)^{-(\beta+1)} - \frac{(-1)}{\beta+1} \int (1+x)^{-(\beta+1)} \, dx \right) \]

\[ = \left( \frac{(-1)}{(\beta+1)} x(1+x)^{-(\beta+1)} - \frac{(-1)^2}{(\beta+1)^2} (1+x)^{-\beta} \right), \]

which then yields the sum
\begin{equation}
V_{\alpha \beta} = \frac{(-1)}{\alpha + \beta - 1} (x^{\alpha - 1} (1+x)/(\alpha + \beta - 1) - \frac{(-1)(\alpha - 1)}{\alpha + \beta - 2} x^{\alpha - 2} (1+x)/(\alpha + \beta - 2) - \frac{(-1)}{\alpha + \beta - 3} \{ \ldots \frac{(-1)}{\beta} x^1 (1+x)/(\beta + 1) \ldots \} \}
\end{equation}

The integral expression for \( V_{\alpha \beta} \) evaluated between the limits \([0, \infty)\) can be obtained in the following manner:

\begin{equation}
(4.17) \quad V_{\alpha \beta}^{\infty} = \lim_{N \to \infty} V_{\alpha \beta}^{N} \bigg|_{0}^{N} = \frac{\alpha}{\Sigma \frac{K_{k_1}}{k_1}} \lim_{N \to \infty} \frac{(\alpha - 1)}{\alpha - k_1} \frac{1}{(1 + N)/(1 + \alpha - k_1)} \quad \text{for } \beta, \alpha \geq 1.
\end{equation}

where

\begin{equation}
K_{k_1} = \frac{(-1)^{k_1} (\alpha - 1)^{k_1}/k_1}{(\alpha + \beta - 1)^{k_1}/k_1}.
\end{equation}

Consequently, we have

\begin{equation}
(4.19) \quad V_{\alpha \beta}^{\infty} = -K_{\alpha} = \frac{1/\alpha}{(\alpha + \beta - 1)/\alpha} = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} = B(\alpha, \beta).
\end{equation}
We will use expressions (4.17) and (4.19) to evaluate

\[ (4.20) \quad E_{\alpha}^{C} = \int_{0}^{\infty} (1+\rho_{12} F_{12})^{-\alpha} dG(F_{12}) \]

\[ = K_{1} \int_{0}^{\infty} (1+\rho_{12} F_{12})^{-\alpha} F_{12}^{\frac{f-1}{2}} (1+F_{12})^{-f} dF_{12} , \]

where \( K_{1} = B^{-1}(\frac{f}{2}, \frac{f}{2}) \) is defined under expression (4.11). We consider first even integer values for \( f \). Let \( f = 2n, n \in \mathbb{I}^{+} \) and let

\[ (4.21) \quad \Delta_{C,n} = \int (1+\rho_{12} F_{12})^{-\alpha} F_{12}^{n-1} (1+F_{12})^{-2n} dF_{12} . \]

Integrating (4.21) by parts, we let

\[ u_{C} = (1+\rho_{12} F_{12})^{-\alpha} , \quad dv_{n} = F_{12}^{n-1} (1+F_{12})^{-2n} dF_{12} , \]

then

\[ du_{C} = -\alpha \rho_{12} (1+\rho_{12} F_{12})^{-\alpha-1} dF_{12} , \]

\[ v_{n} = \sum_{k=1}^{n} K_{1}^{k} F_{12}^{k-1} (1+F_{12})^{-2n+k} , \]

where by the expression under (4.18) we have
\[
K_1^l = \frac{(-1)^{2k_1-1} \binom{n-1}{k_1-1}}{\binom{2n-1}{k_1}}.
\]

So

\[\Delta_{c,n} = u_c v_n - (-1)^{(c+1)} \rho_{12}^n \sum_{k_1=1}^{k_1} K_k^l \Delta_{c+1,n-k_1+1},\]

where

\[\Delta_{c+1,n-k_1+1} = \int \left(1 + \rho_{12,12} F_{12}\right)^{-(c+1)} \left(1 + \rho_{12,12} F_{12}\right)^{(n-k_1+1)-1} \left(1 + \rho_{12,12} F_{12}\right)^{-(2n-k_1)} dF_{12}.\]

Expression (4.23) can be evaluated by parts in a similar manner to that used in (4.21). Let

\[u_{(c+1)} = \left(1 + \rho_{12,12} F_{12}\right)^{-(c+1)},\]

\[d v_{n-k_1+1} = F_{12}^{(n-k_1+1)-1} \left(1 + \rho_{12,12} F_{12}\right)^{-(2n-k_1)} dF_{12},\]

then

\[d u_{(c+1)} = -(c+1) \rho_{12} \left(1 + \rho_{12,12} F_{12}\right)^{-(c+2)},\]
\[ v_{n-k_1+1} = \sum_{k_2=1}^{n-k_1+1} K_{2}^{2} F_{12}^{(n-k_1+1)-k_2^2 (1+F_{12}^{1})-(2n-k_1-k_2)}, \]

where

\[ K_{k_2}^{2} = \frac{2k_2-1}{(-1)^{k_1}} \binom{n-k_1}{k_2} \frac{1}{2n-k_1-1} \binom{k_1}{k_2}. \]

Expression (4.21) can now be written

\[ \Delta_{c,n} = u_{c} v_{n} - (-1)^{c+1} \rho_{12} \sum_{k_1=1}^{n} K_{k_1}^{1} [u_{c+1} v_{n-k_1+1} \]

\[ - \rho_{12} \sum_{k_2=1}^{n-k_1+1} K_{2}^{2} \Delta_{c+2,n-k_1-k_2+2}], \]

where

\[ \Delta_{c+2,n-k_1-k_2+2} = \int \left( l + \rho_{12} F_{12} \right)^{-(c+2)} \]

\[ \cdot \frac{(n-k_1-k_2+2)-1}{F_{12}^{(n-k_1-k_2+2)-1} \left( 1+F_{12} \right)^{(2n-k_1-k_2)}} \] \[ dF_{12}. \]

The above process can be continued until the exponent of \( (l+F_{12})^{-1} \) reaches a specified minimum value. Let \( q \) be a positive integer such that the minimum of \( \sum_{i=1}^{q} k_{i} \) is \( n-2 \), where the \( k_{1}, k_{2}, \ldots, k_{q} \) are indices of steps 1, 2, \ldots, \( q \). Since
the minimum of each \( k \) is one, \( q \) must equal \( n-2 \). The maximum value of \( k_1 \) is \( n \), of \( k_2 \) is \( n-k_1 +1 \), and for \( k_q \) the maximum value is \( n-k_1-k_2-...-k_{q-1} + q-1 \) so that the maximum value of \( \sum_{i=1}^{q} k_i \) is \( 2n-3 \) when \( q = n-2 \).

At the end of step \( q \) (\( 1 < q \leq n-2 \)), we have

\[
\Delta_{c,n} = u_{c} v_{n}
\]

\[
= (-1)c\rho_{12} \sum_{k_1=1}^{n} k_1^1 \left( u_{c+1} v_{n-k_1 +1} \right)
\]

\[
- (-1)(c+1)\rho_{12} \sum_{k_2=1}^{n-k_1 +1} k_2^2 \left( u_{c+2} v_{n-k_1 -k_2 +2} \right)
\]

\[
- (-1)(c+2)\rho_{12} \sum_{k_3=1}^{n-k_1 -k_2 +2} k_3^3 \left( u_{c+3} v_{n-k_1 -k_2 -k_3 +3} - ... \right)
\]

\[
- (-1)(c+q-1)\rho_{12} \sum_{k_q=1}^{n-q+1} k_q^q \sum_{i=1}^{q} k_i \Delta_{c+q,n-\sum_{i=1}^{q} k_i +q} \right) \}
\]

where

\[
\Delta_{c+q,n-\sum_{i=1}^{q} k_i +q} = \int \left( 1+\rho_{12} F_{12} \right)^{(c+q)} \left( n-\sum_{i=1}^{q} k_i +q \right) \left( 1+\rho_{12} F_{12} \right)^{-1} dF_{12}.
\]
To evaluate (4.27) by parts, let

\[ u_{c+q} = (1 + \rho_{12} F_{12})^{-(c+q)}, \]

\[ dv = F_{12}^{(n- \sum k_i + q -1) - (2n- \sum k_i)} \]

\[ n- \sum k_i + q \]

\[ i=1 \]

then

\[ du_{c+q} = (-1)(c+q)\rho_{12} (1 + \rho_{12} F_{12})^{-(c+q+1)}, \]

\[ n- \sum k_i + q \]

\[ q+1 \]

\[ i=1 \]

\[ k_{q+1} \]

\[ F_{12} \]

\[ n- \sum k_i + q \]

\[ i=1 \]

\[ (1 + F_{12}) \]

\[ i=1 \]

so that (4.27) can be written

(4.28)

\[ \Delta_{c+q,n- \sum k_i + q} = u_{c+q} v_{c+q,n- \sum k_i + q} \]

\[ n- \sum k_i + q \]

\[ i=1 \]

\[ \sum \]

\[ k_{q+1} \]

\[ F_{12} \]

\[ n- \sum k_i + q \]

\[ i=1 \]

\[ k_{q+1} \]

\[ k_{q+1} \]

\[ \Delta_{c+q+1,n- \sum k_i + q+1} \]

\[ i=1 \]
where

\( (4.29) \)

\[
\Delta_{c+q+1,n} = \sum_{i=1}^{q+1} k_i + q + 1 = \int (1+\rho_{12} F_{12})^{-1} (c+q+1)
\]

\[
\left( n - \sum_{i=1}^{q+1} k_i + q + 1 \right) - \left( 2n - \sum_{i=1}^{q+1} k_i \right)
\]

\[
\cdot \frac{(n-\sum_{i=1}^{q+1} k_i + q + 1)-1}{(1+\rho_{12} F_{12})} \cdot dF_{12}
\]

and

\[
k_{q+1} = \frac{(-1)^{q+1} k_{q+1}}{k_{q+1}}
\]

\[
\cdot \left( n - \sum_{i=1}^{q} k_i + q - 1 \right) / \left( 2n - \sum_{i=1}^{q} k_i - 1 \right)
\]

Substituting for (4.28), we rewrite (4.26) as

\( (4.30) \)

\[
\Delta_{c,n} = u_c v_n
\]

\[
\begin{align*}
&+ (-1)^2 c \rho_{12} \sum_{k_1=1}^{n} k_1 \sum_{k_2=1}^{n-k_1} u_{c+1} v_{n-k_1 + l} \\
&+ (-1)^4 c(c+1) \rho_{12}^2 \sum_{k_1=1}^{n-k_1 + l} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n-k_1-k_2+2} u_{c+2} v_{n-k_1-k_2+2} \\
&+ (-1)^6 c(c+1)(c+2) \rho_{12}^3 \sum_{k_1=1}^{n-k_1} \sum_{k_2=1}^{n-k_1-k_2+2} \sum_{k_3=1}^{n-k_1-k_2-k_3+3} u_{c+3} v_{n-k_1-k_2-k_3+3}
\end{align*}
\]
\[ + \ldots + (-1)^{2q+2}c(c+1)\ldots(c+q)\rho q+1_{\Delta \Sigma k_i+q} \sum_{i=1}^{n-k+1} \sum_{k_1=1}^{q-1} \sum_{k_2=1}^{q} \ldots \sum_{k_q=1}^{q} k_1^q k_2^q \ldots k_q^q \]

\[ u_{c+q} v_{n-\Sigma k_i+q} \]

\[ + R_q, \]

where

\[ (4.31) \quad R_q = (-1)^{2q}c(c+1)\ldots(c+q)\rho q+1_{\Delta \Sigma k_i+q} \sum_{i=1}^{n-k+1} \sum_{k_1=1}^{q-1} \sum_{k_2=1}^{q} \ldots \sum_{k_q=1}^{q} k_1^q k_2^q \ldots k_q^q \Delta^o_{c+q+1,n-\Sigma k_i+q+1} \sum_{i=1}^{q+1} \]

and

\[ (4.32) \quad \Delta^p_{c+q+1,n-\Sigma k_i+q+1} = f \left( 1+\rho_{12} F_{12} \right) -(c+q+1) F_{12} \sum_{i=1}^{q+1} \left( n-\Sigma k_i \right) \sum_{i=1}^{q+1} \left( 1+F_{12} \right) dF_{12}. \]
Procedures developed previously utilizing integration by parts cannot be used for exact evaluation of expression (4.32) because with \( q > n - 2 \) the exponent of \( (1 + \frac{1}{F_{12}}) \) will violate the restriction \( \beta > 1 \) for some combination of values of the \( k_i \)'s on the \((q+1)\)th step. The maximum value for \( q \) and the range of values for the exponent of \( F_{12} \) and \( (1 + \frac{1}{F_{12}}) \) can be obtained from (4.28). To satisfy the requirement \( \alpha, \beta \geq 1 \), \( q \) must satisfy the relationship

\[
\alpha_q = (n - \sum_{i=1}^{q+1} k_i + q + 1) \geq 1 ,
\]

\[
\alpha_q + \beta_q = 2n - \sum_{i=1}^{q+1} k_i \geq 2 .
\]

From (4.33) and the condition that \( \beta > 1 \), we must have

\[
(4.34) \quad 2n - \sum_{i=1}^{q+1} k_i - n + \sum_{i=1}^{q+1} k_i - q - 1 \geq 1
\]

so that \( q \) must not exceed \( n - 2 \). When \( q = n - 2 \), the minimum of \( \sum_{i=1}^{q} k_i \) equals \( n - 2 \) since the minimum of each \( k_i \) is unity. The maximum of \( \sum_{i=1}^{q} k_i \) equals \( 2n - 3 \) occurs when \( k_1 = n \) and each of the remaining \( n - 3 \) \( k_i \)'s is unity. So we have at the \( q \)th step of integration

\[
(4.35) \quad \max_{k_i \text{'s}} \sum_{i=1}^{q} k_i = 2n - 3
\]
whenever $q$ is such that

$$\min_{k_i's} \sum_{i=1}^{q} k_i = n-2.$$  

At the $(q+1)^{th}$ step of integration

$$(4.36)$$

$$\max_{k_i's} \sum_{i=1}^{q+1} k_i = 2n-2,$$

$$\min_{k_i's} \sum_{i=1}^{q+1} k_i = n-1.$$  

For each $q$

$$\max_{k_i's} \sum_{i=1}^{q+1} k_i = n - \min_{k_i's} \sum_{i=1}^{q+1} k_i = n,$$

$$\min_{k_i's} \sum_{i=1}^{q+1} k_i = n - \max_{k_i's} \sum_{i=1}^{q+1} k_i = 1.$$  

Maximum and minimum values of $\beta_q$ are the same for each $q$ but decrease by unity as $q$ increases by unity. So $\beta_q = 2$ when $q = n - 2$. An exact value for $R_q (4.31)$ cannot be obtained. However, an upper bound for the integral (4.32) and thus an upper bound for $R_q$ can be obtained over specified values of $\rho_{12}$. The integral in the remainder $R_q$ after the $(q+1)^{th}$ step (4.29) may be written as
(4.37)  
\[ \Delta_{c+q+1, \alpha_q} = \int \frac{F_{12}^\alpha q - 1}{(1 + F_{12})^{(c+q+1)}} \frac{dF_{12}}{1 + F_{12}} - (\alpha_q + \beta_q) \]

where

\[ \alpha_q = n - \sum_{i=1}^{q+1} k_i + q + 1, \quad \beta_q = n - q - 1. \]

Expression (4.37) can be evaluated easily for certain values of \( \rho_{12} \). For example, if \( \rho_{12} = 0 \), then

(4.38)  
\[ \Delta^0_{c+q+1, \alpha_q} = \int_0^\infty \frac{F_{12}^{\alpha q - 1}}{(1 + F_{12})^{(c+q+1)}} dF_{12} \]
\[ = \frac{\Gamma(\alpha_q) \Gamma(\beta_q)}{\Gamma(\alpha_q + \beta_q)} = B(\alpha_q, \beta_q). \]

If \( \rho_{12} = 1 \), then

(4.39)  
\[ \Delta^1_{c+q+1, \alpha_q} = \int_0^\infty \frac{F_{12}^{\alpha q - 1}}{(1 + F_{12})^{(c+q+\alpha_q + \beta_q + 1)}} dF_{12} \]
\[ = \frac{\Gamma(\alpha_q) \Gamma(\beta_q + q + c + 1)}{\Gamma(\alpha_q + \beta_q + q + c + 1)} = B(\alpha_q, \beta_q + q + c + 1). \]

If \( \rho_{12} > 1 \), then

\[ \left(1 + \rho_{12} F_{12}\right)^{-(c+q+1)} \frac{F_{12}^{\alpha q - 1}}{(1 + F_{12})^{(c+q+\alpha_q + \beta_q + 1)}} \]
\[ \times \frac{dF_{12}}{1 + F_{12}} \]
so that

$$\Delta_{c+q+1}^0, \alpha_q < \Delta_{c+q+1}^1, \alpha_q .$$

On the other hand, if $0 < \rho_{12} < 1$, then

$$\Delta_{c+q+1}^1, \alpha_q < \Delta_{c+q+1}^0, \alpha_q \leq \Delta_{c+q+1}^0, \alpha_q .$$

Therefore, an upper bound of $\Delta_{c+q+1}^0, \alpha_q$ for all $\rho_{12} > 0$ is $\Delta_{c+q+1}^0, \alpha_q$. If it is known, however, that $\rho_{12} > 1$, then a much tighter upper bound for $\Delta_{c+q+1}^0, \alpha_q$ is $\Delta_{c+q+1}^1, \alpha_q$.

Consequently, the bound for $R_q$ can be written

$$(4.40) \quad R_q < T_{c+q+1}^c, \alpha_q$$

if $\rho_{12} > 1$

$< T_{c+q+1}^c, \alpha_q$ if $\rho_{12} > 0$.

where

$$T_q^c = c(c+1)...(c+q) \rho^{q+1} \sum_{k=1}^{n-k+1} \frac{\alpha_q-1}{12} \sum_{k_1=1}^{k_{q+1}} \frac{k_{k_1} k_{k_2} ... k_{k_q+1}}{k_{q+1}^1 k_{q+1}^2 k_{q+1}^3 ... k_{q+1}^q} ,$$

$\Delta_{c+q+1}^0, \alpha_q$ and $\Delta_{c+q+1}^1, \alpha_q$ are defined by (4.38) and (4.39), respectively.

Now, to evaluate an integral expression between the limits zero and infinity of the form $A(x)|_0^\infty$, we utilize the results
(4.41) \[ A(x) \bigg|_0^\infty = \lim_{N \to \infty} A(x) \bigg|_0^N \]
\[ = \lim_{N \to \infty} A(N) - A(0) , \]

and the limits theorem stated as follows.

**Theorem 4.1:** If \( A_N \) and \( B_N \) are two sequences or series such that \( \lim A_N = L_A \) and \( \lim B_N = L_B \neq 0 \), then

\[
\begin{align*}
\lim A_N B_N &= \lim A_N \lim B_N , \\
\lim A_N^m &= (\lim A_N)^m , \\
\lim \left( \frac{A_N}{B_N} \right) &= \left( \lim A_N \right) / \left( \lim B_N \right) , \\
\lim (A_N + B_N) &= \lim A_N + \lim B_N ,
\end{align*}
\]

where \( L_A \) and \( L_B \) are real finite numbers and \( m \) is any nonzero real number.

The first term in expression (4.30) evaluated between the limits zero and infinity is

(4.42) \[ u_C v_n \bigg|_0^\infty = \lim_{N \to \infty} u_C(N) v_n(N) - u_C(0) v_n(0) \]
\[ = \lim_{N \to \infty} (1 + \rho_{12} N)^{1-c} \sum_{k_1=1}^{n-k_1} K_{k_1}^1 \left( 1 - N \right)^{n-k_1}(1-N) - (2n-k_1) - u_C(0) v_n(0) \]
\[ = 0 - K_{n}^1 \]
\[
\frac{(-1)^{2n-l}(n-1)}{n(2n-1)} = \frac{1}{n(2n-1)}.
\]

The second term of (4.30), evaluated similarly to (4.42), is

\[
(4.43)
\]

\[
u_{c+1}v_{n-k_1+1}^{\infty} = \lim_{N \to \infty} u_{c+1}(N)v_{n-k_1+1}(N) - u_{c+1}(0)v_{n-k_1+1}(0)
\]

\[
= \lim_{N \to \infty} (1+\rho_{12}N)^{n-k_1+1} \sum_{k=1}^{N} \frac{K_1^2 N^{n-k_1+1-k_2}}{k_2!}
\]

\[
\cdot (1+N)^{-2(n-k_1-k_2)} - u_{c+1}(0)v_{n-k_1}(0)
\]

\[
= 0 - K_1^2
\]

\[
\frac{2(n-k_1+1)-1}{(n-k_1+1)(n-k_1+1)}
\]

\[
(\frac{-1}{(2n-k_1+1)}
\]

\[
= \frac{1}{(n-k_1+1)(n-k_1+1)}, \quad k = 1, 2, \ldots, n.
\]

The third term of (4.30) is evaluated to be
(4.44) \[ u_{c+2^n-k_{1} - k_{2} + 2} \bigg|_{0}^{\infty} = \frac{1}{(n-k_{1} - k_{2} + 2)(n-k_{1} - k_{2} + 2)}, \]

\[ k_{1} = 1, 2, \ldots, n, \]

\[ k_{2} = 1, 2, \ldots, n-k_{1} + 1. \]

Continuing the above process, the \((q+1)^{th}\) term yields the result

(4.45) \[ u_{c+q^n-k_{1} - k_{2} - \ldots - k_{q} + q} \bigg|_{0}^{\infty} = \lim_{N \to \infty} (1+\rho_{12} N)^{-(c+q)} \]

\[ = \sum_{i=1}^{n-k_{1}+q} k_{i}^{q+1} \]

\[ \cdot \sum_{k_{q+1}=1}^{q+1} k_{q+1} \]

\[ = \sum_{i=1}^{q+1} (n - \sum_{i=1}^{q+1} k_{i}) \]

\[ - (2n - \sum_{i=1}^{q+1} k_{i}) \]

\[ \cdot (1+N) \]

\[ - u_{c+q} (0) v_{n} - \sum_{i=1}^{q} k_{i} + q (0) \]

\[ = 0 - k_{q+1}^{q+1} \]

\[ \frac{\alpha_{q-1}}{\alpha_{q-1}} \]
\[
\left(-1\right)^{2(n-1)-1}\frac{\alpha_{q-2}^{n-1}}{\alpha_{q-2}} = -\frac{\alpha_{q}^{n-1}}{\alpha_{q-1}^{n-1}}.
\]

where \(a_q\), defined by (4.33), satisfy the restriction \(a_q > 1\).

When \(q = n - 2\), expression (4.45) can be written

\[
\sum_{k_1 = 1}^{n} \sum_{k_2 = 1}^{n-k_1 + 1} \cdots \sum_{k_q = 1}^{n-k_1 - \cdots - k_{q-1} + q-1} \frac{\alpha_{n-2}^{k_1-1}(\cdots(\alpha_{n-2}^{k_q-1})}{(\alpha_{n-2}^{k_q-1})}.
\]

where \(a_{n-2}\) varies from 1 to \(n\) while \(\beta_{n-2}\) has a constant value of 2.

Expression (4.30), evaluated between the limits zero and infinity, can be written with \(q = n - 2\).
\[ (4.47) \]
\[ \Delta_{c,n_0}^\infty = \frac{1}{n(2n-1)} + c \rho \sum_{k_1=1}^{n} \sum_{k_2=1}^{n-k+1} \frac{(-1)^{2k_1-1}(n-1)}{k_1(n-k_1+1)} \frac{2^{n-k_1-1}}{\binom{k_1}{1}} \frac{2^{n-k_1}}{\binom{k_1}{2}} \frac{2n-k_1}{n-k_1+1} \]
\[ + c(c+1) \rho^2 \sum_{k_1=1}^{n-k+1} \sum_{k_2=1}^{n-k+1} \frac{(-1)^{2(k_1+k_2)-2}(n-1)}{k_1(n-k_1-1)} \frac{n-k_1}{k_2-1} \frac{2^{n-k_1-1}}{\binom{k_1}{1}} \frac{2^{n-k_1}}{\binom{k_1}{2}} \frac{2n-k_1}{n-k_1+2} \]
\[ + \ldots + c(c+1) \ldots (c+n-2) \rho^{n-1} \sum_{k_1=1}^{n-k+1} \sum_{k_2=1}^{n-k+1} \sum_{k_3=1}^{n-k+3+n-3} \ldots \sum_{k_{n-2}=1}^{n-k+1+n-2} \]
\[ \frac{(-1)^{2(k_1+k_2+\ldots+k_{n-2})}}{k_1 \ldots k_{n-2} \binom{n-k_1-\ldots-k_{n-2}+n-2}{n-k_1-\ldots-k_{n-2}+n-2}} \]
\[ \frac{(n-1)}{k_1(n-k_1-1)} \ldots \frac{n-k_1-\ldots-k_{n-3}+n-4}{k_{n-2}-1} \]
\[ \frac{2^{n-k_1-1}}{\binom{k_1}{1}} \ldots \frac{2^{n-k_1-n-2}}{\binom{k_1}{2}} \frac{2n-k_1-\ldots-k_{n-2}-1}{\binom{k_1}{2}} \frac{2n-k_1-\ldots-k_{n-2}+n-2}{\binom{k_1}{2}} \]
\[ + R_{n-2}, \]

where
for all \( \rho \geq 0 \), and, furthermore,

\[
R_{n-2} \leq c(c+1)\ldots(c+n-2)\rho^{n-1} \sum_{k_1=1}^{n} \ldots \sum_{k_{n-1}=1}^{k_1} \ldots \sum_{k_{n-1}=1}^{k_1} \Delta^0_{c+n-1,\alpha_{n-2}}
\]

for all \( \rho \geq 1 \). Also, from (4.38) and (4.39)

\[
\Delta^0_{c+n-1,\alpha_{n-2}} = \frac{\Gamma(\alpha_{n-2})\Gamma(\beta_{n-2})}{\Gamma(\alpha_{n-2}+\beta_{n-2})},
\]

\[
\Delta^1_{c+n-1,\alpha_{n-2}} = \frac{\Gamma(\alpha_{n-2})\Gamma(\beta_{n-2}+n+c-2)}{\Gamma(\alpha_{n-2}+\beta_{n-2}+n+c-2)}.
\]

Thus, putting together previous results we have the following.

**Theorem 4.2:** For the two uncorrelated random variables \( x_1 \) and \( x_2 \) with distributions \( N(\mu,\sigma_{11}) \) and \( N(\mu,\sigma_{22}) \), respectively, let \( (\bar{x}_1, s_{11}) \) and \( (\bar{x}_2, s_{22}) \) be the unbiased estimates of the population parameters \( (\mu,\sigma_{11}) \) and \( (\mu,\sigma_{22}) \), calculated respectively from a simple random sample of size \( r \) drawn from each of the two populations. Then a combined unbiased estimate

\[
\hat{\mu}_w = \hat{\alpha}_1 \bar{x}_1 + \hat{\alpha}_2 \bar{x}_2, \quad \hat{\alpha}_i = \frac{\frac{1}{s_{11}}}{\frac{1}{s_{11}} + \frac{1}{s_{22}}}, \quad i = 1, 2, \quad \hat{\alpha}_1 + \hat{\alpha}_2 = 1
\]
of the parameter $\mu$ has the unconditional variance

$$\text{Var}(\hat{\mu}_w) = m_2 E\hat{\alpha}^2 + m_1 E\hat{\alpha}^1 + m_0$$

$$= \frac{1}{\omega}[1 + \alpha(\rho_{12}, n)], \quad \omega = \frac{1}{\sigma_{11}} + \frac{1}{\sigma_{22}},$$

where

$$\alpha(\rho_{12}, n) = \rho_{12} \left(1 + \frac{1}{\rho_{12}}\right)^2 E\hat{\alpha}^2 - 2\left(1 + \frac{1}{\rho_{12}}\right) E\hat{\alpha}^1 + \frac{1}{\rho_{12}}$$

$$= \rho_{12} - (\rho_{12} + 1) \left(2\rho_{12} + 1\right) B(n, n)L_1$$

$$+ 2\rho_{12} (\rho_{12} + 1) \left(3\rho_{12} + 1\right) B(n, n)L_2$$

$$- 12\rho_{12} (\rho_{12} + 1) \left(2\rho_{12} + 1\right) B(n, n)L_3$$

$$+ \ldots$$

$$\pm (n-1)!\rho_{12}^{n-2} (\rho_{12} + 1) (n\rho_{12} + n - 2) B(n, n)L_{n-1}$$

$$- \frac{(\rho_{12} + 1)}{\rho_{12}} B(n, n) \left((\rho_{12} + 1) R_{n-2, 2} - 2R_{n-2, 1}\right)$$

and for even degrees of freedom, $r - 1 = 2n$,

$$B^{-1}(n, n)E\hat{\alpha}^c_1 = \frac{r^2(n)}{1(2n)} - \frac{c}{\sigma_{11} L_1} + c(c+1) \frac{\sigma^2_{11}}{\sigma_{22} L_2} - \ldots + c(c+1) \ldots (c+n-2)$$
\[
\frac{\sigma_{n-1}}{\sigma_{n-1}} L_{n-1} + R_{n-2}, c , c = 1, 2, \\
\]

with

\[
L_1 = \sum_{k_1=1}^{n} \binom{n-1}{k_1-1} \frac{\binom{2n-k-1}{k_1}}{(n-k_1+1)\binom{2n-1}{k_1}}
\]

\[
L_2 = \sum_{k_1=1}^{n-k+1} \binom{n-k}{k_1-1} \binom{n-k_1}{k_2-1} \frac{\binom{2n-k-1}{k_1}}{(n-k_1-k_2+2)\binom{2n-1}{k_1}}
\]

\[
L_{n-2} = \sum_{k_1=1}^{n-k+1} \binom{n-k}{k_1-1} \ldots \binom{n-k_{i-1}}{k_{i-1}-1} \frac{1}{k_1 \ldots k_{n-2} (n-\sum_{i=1}^{n-2} k_i + n-2)}
\]

\[
\binom{n-1}{k_1-1} \ldots \binom{n-k_{i-1}}{k_{i-1}-1} \frac{1}{k_1 \ldots k_{n-2} \binom{n-2}{k_{n-2}-1}}
\]

\[
(\binom{2n-1}{k_1} \ldots \binom{2n-k_{i-1}}{k_{i-1}} \ldots \binom{2n-k_{n-3}}{k_{n-3}+1})
\]

and
\[ R_{n-2,c} = c(c+1)\ldots(c+n-2) \frac{1}{\sigma_{n-1}^{c+1-n-1}} T^C_{n-2} \Delta^0_{c+n-1,\alpha_{n-2}} \]

\[ < c(c+1)\ldots(c+n-2) \frac{1}{\sigma_{n-1}^{c+1-n-1}} T^C_{n-2} \Delta^1_{c+n-1,\alpha_{n-2}} \]

for all \( \sigma_{11}, \sigma_{22} \) such that \( \left( \frac{\sigma_{11}}{\sigma_{22}} \right) > 0 \), and further for all \( \sigma_{11}, \sigma_{22} \) such that \( \left( \frac{\sigma_{11}}{\sigma_{22}} \right) > 1 \),

\[ R_{n-2,c} \leq c(c+1)\ldots(c+n-2) \frac{1}{\sigma_{n-1}^{c+1-n-1}} T^C_{n-2} \Delta^1_{c+n-1,\alpha_{n-2}} , \]

where

\[ \Delta^0_{c+n-1,\alpha_{n-2}} = \int_0^\infty \left( 1 + \rho_{12} F_{12} \right)^{-(c+n-1)} \alpha_{n-2}^{c-1} \]

\[ \cdot \left( 1 + F_{12} \right)^{-(\alpha_{n-2} + \beta_{n-2})} dF_{12} , \]

\[ \Delta^1_{c+n-1,\alpha_{n-2}} = \frac{\Gamma(\alpha_{n-2}) \Gamma(\beta_{n-2})}{\Gamma(\alpha_{n-2} + \beta_{n-2})} , \]

\[ \Delta^0_{c+n-1,\alpha_{n-2}} = \frac{\Gamma(\alpha_{n-2}) \Gamma(\beta_{n-2} + n - c - 2)}{\Gamma(\alpha_{n-2} + \beta_{n-2} + n - c - 2)} , \]

and

\[ T^C_{n-2} = \sum_{k_1 = 1}^{n} \sum_{k_2 = 1}^{n-k_1+1} \ldots \sum_{k_{n-1} = 1}^{n-k_1-k_2-\ldots-k_{n-2}} \ldots \sum_{k_n = 1}^{n-k_1-k_2-\ldots-k_{n-2}} \ldots \sum_{k_n = 1}^{n-k_1-k_2-\ldots-k_{n-2}+1} \alpha_{n-2}^{c-1} . \]

A numerical study was undertaken to compare variance expressions of weighted means given by Meier (1953) and Bement and Williams (1969) with the results of Theorem 4.2. Additionally, comparisons between weighted estimates of mean and some possible alternative estimates are made for a wide range of parameter values and sample sizes. An important point of interest is the study of the effect of heterogeneity of population variances from two different sources on the estimate of a common mean.

The two populations under consideration are denoted \( X_1 \sim N(\mu_1, \sigma_1^2) \) and \( X_2 \sim N(\mu_2, \sigma_2^2) \). Samples of size \( n_1 \) and \( n_2 \) are drawn from populations \( X_1 \) and \( X_2 \), respectively, which for this study are taken to be \( n_1 = n_2 = r \). The primary interest is in the estimation of the common parameter \( \mu \). Unbiased means are calculated

\[
\bar{x}_1 = \frac{1}{r} \sum_{i=1}^{r} x_{1i} \quad \text{and} \quad \bar{x}_2 = \frac{1}{r} \sum_{i=1}^{r} x_{2i}
\]

from the two population samples. Some of the possible estimates utilizing either or both \( \bar{x}_1 \) and \( \bar{x}_2 \) are

\[
(4.48) \quad \hat{\mu}_1 = \bar{x}_1,
\]

\[
(4.49) \quad \hat{\mu}_2 = \bar{x}_2,
\]
\[ (4.50) \quad \hat{\mu}_3 = \left( \bar{x}_1 + \bar{x}_2 \right)/2 , \]
\[ (4.51) \quad \hat{\mu}_4 = \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 , \]
where \( \alpha_1 = \frac{1}{\sigma_{11}^2} \) and \( \alpha_2 = \frac{1}{\sigma_{22}^2} \) are known,
\[ (4.52) \quad \hat{\mu}_5 = \hat{\alpha}_1 \bar{x}_1 + \hat{\alpha}_2 \bar{x}_2 , \]
where \( \hat{\alpha}_1 = \text{Est}(\alpha_1) \) and \( \hat{\alpha}_2 = \text{Est}(\alpha_2) \) for cases where \( \alpha_1 \) and \( \alpha_2 \) are not known.

Unbiased estimates for \( \sigma_{11} \) and \( \sigma_{22} \), denoted \( s_{11} \) and \( s_{22} \), are available either from the immediate sample or from a previous sample to form the estimated weights
\[
\hat{\sigma}_1 = \frac{r_{11}}{s_{11} + r} , \quad \hat{\sigma}_2 = \frac{r_{22}}{s_{22} + r} .
\]

The variances for estimators \((4.48)\) to \((4.51)\) are easily obtainable and are written as
\[ (4.53) \quad V(\hat{\mu}_1) = \sigma_{11}/r , \]
\[ (4.54) \quad V(\hat{\mu}_2) = \sigma_{22}/r , \]
A general exact expression for $V(\hat{\mu}_5)$ for an arbitrary $r$ is not available. An expression of an approximation for $V(\hat{\mu}_5)$ using the method of steepest descent, given by Meier (1953), is written as

$$V_2(\hat{\mu}_5) = \frac{1}{\omega} \{1 + 2[\left(\frac{\theta}{n_1}(1 - \theta) + \frac{\theta}{n_2}(1 - \theta)\right)]\},$$

where $\omega = \frac{1}{\sigma_{11}} + \frac{1}{\sigma_{22}}$, $\theta_1 = \frac{1}{\sigma_{11}/\omega}$, $\theta_2 = \frac{1}{\sigma_{22}/\omega}$.

The approximation for $V(\hat{\mu}_5)$ obtained by using a Taylor expansion (Bement and Williams (1969)) is written as

$$V_3(\hat{\mu}_5) = \frac{\sigma_{11}}{n_2 \rho_{12} + n_1} \left\{1 + \frac{n_1 n_2 \rho_{12}}{(n_2 \rho_{12} + n_1)^2} \left[2 \left(\frac{1}{n_1 - 1} + \frac{1}{n_2 - 1}\right)ight]ight.$$  
$$\left. - \frac{16}{n_2 \rho_{12} + n_1} \left(\frac{n_2 \rho_{12}}{(n_1 - 1)^2} + \frac{n_1}{(n_2 - 1)^2}\right) + \frac{12}{n_2 + n_1} \left(\frac{3n_2^2 \rho_{12}^2}{(n_1 - 1)^2}\right)\right\}.$$
\[
\frac{n^2 \rho^2}{(n - 1)^2} + \frac{2}{n_1 n_2} + \frac{3n^2}{n_2 - 1} + \frac{l2n^2 \rho^2}{n_1 - 1}^3
\]

where \( \rho_{12} = \sigma_{11} / \sigma_{22} \).

Expressions (4.57) and (4.58) are compared numerically with the results of Theorem 4.2 for sample sizes \( n_1 = n_2 = r = 2, 4, 6, 8, 10, 12, 14, 16, 18, 20 \) and for \( \rho = .0001, .001, .01, .1 \). The results are tabulated in Table 4.1. For each combination of values for \( r \) and \( \rho \) the values calculated and tabulated in Table 4.1 are

(4.59) \[ C_1 = V(\hat{\mu}_x) = \frac{1}{r \left( \frac{1}{\sigma_{11}} + \frac{1}{\sigma_{22}} \right)} \],

(4.60) \[ \gamma_1 = V(\hat{\mu}_1)/C_1 \],

(4.61) \[ \gamma_2 = V(\hat{\mu}_2)/C_1 \],

(4.62) \[ \gamma_3 = V(\hat{\mu}_3)/C_1 \],

(4.63) \[ \delta_1 = V_1(\hat{\mu}_1)/C_1 \],

(4.64) \[ \delta_2 = V_2(\hat{\mu}_1)/C_1 \],

(4.65) \[ \delta_3 = V_3(\hat{\mu}_1)/C_1 \],
Table 4.1

\[ \rho_{12} = .0001, \quad \gamma_1 = 1.0001, \quad \gamma_2 = 10001. \quad \gamma_3 = 2500. \]

<table>
<thead>
<tr>
<th>d.f.</th>
<th>(C_1)</th>
<th>(\delta_1)</th>
<th>(\delta_2)</th>
<th>(\delta_3)</th>
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\[ \rho_{12} = .001, \quad \gamma_1 = 1.001, \quad \gamma_2 = 1001. \quad \gamma_3 = 250.5 \]

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Table 4.1 (Continued)

\( \rho_{12} = .01, \gamma_1 = 1.01, \gamma_2 = 101.0, \gamma_3 = 25.5 \)

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\( \rho_{12} = .1, \gamma_1 = 1.10, \gamma_2 = 11.0, \gamma_3 = 3.025 \)

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where $V_{1}\left(\hat{\mu}_{5}\right)$ is the sum of the first four terms of the expression given in Theorem 4.2. From the values given in Table 4.1 it can quickly be observed that as the degrees of freedom $(r-1)$ increase for a given $\rho_{12}$, each of $\delta_1$, $\delta_2$, and $\delta_3$ approach unity, as could be expected. It is clear that estimate (4.49), and to a lesser degree estimate (4.50), does poorly compared to (4.48) and (4.52) for all values of $\rho_{12} < .1$. As $\rho_{12}$ gets nearer 1, however, estimate (4.50) will have the smaller variance. The relative accuracies of the variance expressions can be compared with the results of the Monte Carlo study presented in Chapter VI. It appears that $\delta_2$ is an underestimate while $\delta_1$ slightly overestimates the variance ratio $V(\hat{\mu}_{3})/V(\hat{\mu}_{4})$, especially for the larger degrees of freedom and larger values of $\rho_{12}$. This is due, however, to the fact that only the first four terms were used in $\delta_1$. The fifth term would subtract from the tabulated value. The values of $\delta_3$ fall between $\delta_1$ and $\delta_2$ for almost all values for $r$ and $\rho_{12}$.

The results given in Theorem 4.2 are for equal sample size and even degrees of freedom. It is not necessary to make these restrictions, however. The same procedures followed in the development of Theorem 4.2 can be used in evaluating the variance of a combined estimate of two sources having unequal sample sizes. The estimate

$$\hat{\mu}_{w} = \hat{\alpha}_{1} \bar{X}_{1} + \hat{\alpha}_{2} \bar{X}_{2} , \quad \hat{\alpha}_{1} + \hat{\alpha}_{2} = 1$$
has an unconditional variance

\[ V(\hat{\mu}_w) = \frac{1}{n} \left( 1 + \alpha(n_1, n_2, \rho_{12}) \right), \]

where

\[ \alpha(n_1, n_2, \rho_{12}) = \frac{n_2 \rho_{12}}{n_1} \left( 1 + \frac{n_1 \rho_{12} - 1}{n_2} \right)^2 \hat{\alpha}_1^2 - 2 \left( 1 + \frac{n_1 \rho_{12} - 1}{n_2} \right) \hat{\alpha}_1 + \frac{n_1 \rho_{12} - 1}{n_2} \]

and

\[ \beta(n_1, n_2) \hat{\alpha}_1^c = \frac{\Gamma(n_1) \Gamma(n_2)}{\Gamma(n_1 + n_2)} - c \frac{n_2 \rho_{12}}{n_1} L_1 + c(c+1) \left( \frac{n_2 \rho_{12}}{n_1} \right)^2 L_2 - \ldots \]

\[ + c(c+1) \ldots \left( c + \frac{1}{2} \right)^2 - 2 \left( \frac{n_2 \rho_{12}}{n_1} \right)^2 - c \]

\[ + R \left( \frac{n_1 + n_2 - 1}{2} - 2 \right)^2 c \]

for \( c = 1, 2 \).

Expressions \( \hat{\alpha}_1, \hat{\alpha}_2, L_1, L_2, \ldots, L_{n-2} \) and \( R \) are defined as in Theorem 4.2 with the exception that \( n = \frac{n_1 + n_2 - 1}{2}, n_1, n_2 \in \mathbb{I}^+ \) is an integer.
The results of Theorem 4.2 can be applied to estimating the estimable parametric function $\lambda' \tau$ under model (4.2) with $k = 2$. If the estimators for $\lambda' \tau$ drawn from two sources, denoted $(\lambda_1')$ and $(\lambda_2')$, are such that $V(\lambda_1') = \sigma_{11}$ and $V(\lambda_2') = \sigma_{22}$, then the best estimator by simple weighting is

\begin{equation}
\hat{\lambda'} = \omega_1 (\lambda_1') + \omega_2 (\lambda_2'),
\end{equation}

where $\omega_1 + \omega_2 = 1$, $\omega_i = \frac{1}{\sigma_{ii} + \frac{1}{\sigma_{22}}}$, $i = 1, 2$.

Whenever estimators $\hat{\omega}_1$ and $\hat{\omega}_2$ are substituted for $\omega_1$ and $\omega_2$ in (4.66), the resulting variance

\begin{equation}
V(\hat{\lambda'}^{'}) = V(\hat{\omega}_1 (\lambda_1') + \hat{\omega}_2 (\lambda_2') ),
\end{equation}

\begin{equation}
\hat{\omega}_1 + \hat{\omega}_2 = 1, \hat{\omega}_i = \frac{1}{\frac{1}{\sigma_{11}} + \frac{1}{\sigma_{22}}}, i = 1, 2.
\end{equation}

may be expressed in the form of Theorem 4.2 and evaluated.
V. APPLICATION OF COMBINABILITY TO INCOMPLETE BLOCK DESIGNS

A. Statement of Problem and General Considerations

A discussion of pertinent definitions and estimating procedures of combining information obtained from incomplete block experimental designs with special emphasis given to the application of computing technique on a high-speed digital computer is presented in this chapter. The contention of this author is that no single tool does all jobs equally well and that no single computing procedure or technique will do all analyses of statistical data with equal precision, accuracy, and efficiency. An algorithm for computing analysis of variance and covariance of balanced complete block classification will generally not be efficiently applicable in solving a problem of partially balanced or nonorthogonal classification designs. On the other hand, various levels of inefficiency will exist for obtaining an analysis of balanced complete classification data from an algorithm general enough to obtain analysis of the less restrictive incomplete block and non-orthogonal experimental designs. With this in mind, we proceed to present general computing techniques of analysis that give asymptotically efficient unbiased combined estimators of parameters in a subclass of incomplete block designs.

The general model considered can be written

\[ Y_i = X_i \delta + \varepsilon_i, \ i = 1, 2, \ldots, r, \]
where $Y_i$ is an $(n_i \times 1)$ vector of observations, $X_i$ is an $(n_i \times p)$ known design matrix, $\delta$ is a $(p \times 1)$ vector of unknown parameters, and $\varepsilon_i$ is an $(n_i \times 1)$ vector of errors such that $E\varepsilon_i\varepsilon_i' = \sigma_{ii}I$, $E\varepsilon_i\varepsilon_j' = 0$, $i \neq j$, where the $V_i$'s are known $(n \times n)$ nonsingular matrices and the $\sigma_{ii}$'s are unknown scalars. In what follows we study an estimator of the form

\begin{equation}
\hat{\tau}_w^* = \left[ \sum_{i=1}^{r} (X_i'V_i^{-1}X_i) \right]^{-1} \sum_{i=1}^{r} X_i'V_i^{-1}Y_i,
\end{equation}

where $S_i = s_{ii}V_i$ is such that $E s_{ii} = \sigma_{ii}$ so that (5.1) can be written in terms of the known $(n \times n)$ $V_i$'s as

\begin{equation}
\hat{\tau}_w^* = \left[ \sum_{i=1}^{r} s_{ii}(X_i'V_i^{-1}X_i)^{-1} \right]^{-1} \sum_{i=1}^{r} s_{ii}X_i'V_i^{-1}Y_i.
\end{equation}

We consider first the case $r = 2$ of combining inter- and intra-block information for a class of incomplete block experimental designs and associated computer algorithms.

B. Algorithm for Estimation of Incomplete Block Parameters

Pertinent definitions and estimating procedures, with emphasis placed on the application for implementation on a high-speed digital computer, are presented in this section.
A distinguishing characteristic of the class of incomplete block designs under consideration from balanced complete and nonorthogonal classification designs is the combination of occurrences of treatments within blocks. The amount of inter-block information available is dependent on the particular design and the amount of intra-block variability. A similar statement can be made regarding estimation of treatment effects on an intra-block basis. The definition of a class of incomplete block designs with regard to treatment combinations within blocks that will satisfy the property of connectedness and maximality of rank may be stated as follows (Bose and Nair (1939)).

**Definition 5.1:** A given design $\mathcal{D}$ is called a partially balanced incomplete block (pbib) design if it satisfies the following. Let

- $v =$ number of treatments,
- $b =$ number of blocks,
- $r =$ number of replicates of each treatment, and
- $k =$ number of plots in each block.

The set consisting of $n_1$ treatments, each of which occurs in combination with a given treatment $\lambda_1$ times, is called the $i^{th}$ associate class of the given treatment for $i = 0, 1, 2, \ldots, m$, where $m \geq 1$ is the number of such classes. Each treatment occurs no more than once in a given block. With
$n_0 = 1$ (i.e., each treatment is its own 0th associate) and $\lambda_0 = r$, the following conditions must be satisfied.

(5.3) \[ b_k = rv, \]
\[ n_0 + n_1 + \ldots + n_m = v, \]
\[ \lambda_0 n_0 + \lambda_1 n_1 + \ldots + \lambda_m n_m = rk, \text{ and} \]
\[ r = \lambda_0 > \lambda_1 > \ldots > \lambda_m > 0. \]

Unique integers $p_{ij}^k$ denote the number of treatments in common with the $i^{th}$ and $j^{th}$ associates of a pair of treatments, which are themselves $k^{th}$ associates, where $i, j = 0, 1, \ldots, m$ for each $k, k = 0, 1, \ldots, m$. The following conditions regarding the $p_{ij}^k$'s must hold

(5.4) \[ p_{ij}^0 = n_i \text{ for } i = j \]
\[ = 0 \text{ for } i \neq j, \ i, j = 0, 1, \ldots, m, \]
\[ p_{ij}^k = p_{ji}^k, \ i \neq j, \ i, j, k = 0, 1, \ldots, m, \]
\[ \Sigma_{j=0}^{m} p_{ij}^k = n_i, \ \Sigma_{i=0}^{m} p_{ij}^k = n_j, \ k = 0, 1, \ldots, m, \text{ and} \]
\[ n_k p_{ij}^{ik} = n_i p_{jk}^i = n_j p_{ik}^j, \ i, j, k = 0, 1, \ldots, m. \]

This definition of a general class of partially balanced incomplete block (PBIB) designs is presented here for completeness and for future reference. Slight generalizations of definition (5.1) are possible by relaxing the restriction that
the design be connected; i.e., not requiring associate classes to be formed by $\lambda_i$'s alone but by a combination of $\lambda_i$'s and $p_{ij}^k$'s and by relaxing the requirement that the $p_{ij}^k$'s be integers. The nonconnected designs, however, will not have maximal rank so that all treatment differences are not estimable. Our interest will generally be confined to the class of connected designs. Many designs which satisfy definition (5.1) have been shown to exist and have been tabulated. They are classified by several criteria including method of design construction, dimension, rank of design matrix, number of associate classes, values of characteristic roots of the design matrix, and both the magnitude and relative values of the design parameters. Included in this class of pbib designs are designs known as balanced incomplete block designs, Youden squares, balanced and some partially balanced lattices, triangular, group divisible, cyclic, and $L_r$ designs.

Analysis of incomplete block designs can be a lengthy and tedious task without the use of a high-speed computer. It is imperative, then, that a method of analysis be incorporated in a system that is reasonably efficient and as general as possible in application. Some considerations of such a system must include the size of the problem, which is a combination of number of treatments, blocks, plots, and replicates, as well as the number of associate classes, and the efficiency of the system with regard to required memory space and amount of time
to calculate an analysis. A general computer program, GAVIAL (1969), was developed to handle a wide class of experimental designs, among which are those satisfying definition (5.1). Some of the more important features of the algorithms and their implementation to a high-speed computer are presented in this section.

The first part of a general algorithm should include a reasonably efficient method for determining the association scheme of the given design, defined by definition (5.1). Also, as will be shown later, the association parameters can be used effectively in estimating model parameters and in calculating the analysis of variance.

Two methods of calculating association parameters are available. In Method 1 the association parameters of the given design are assumed to satisfy definition (5.1) so that it will be necessary to find only those association parameters essential for further analysis of data. Method 2 provides steps for a complete check of the given design to see if all conditions of definition (5.1) are satisfied. The design subscripts of blocks and treatment combinations, as well as the number of blocks, treatments, plots, and replicates, are assumed to be input.

Method 1: An arbitrary treatment number, say $t_0$, is chosen from the input design. The block numbers in which $t_0$ occurs are found by comparing $t_0$ with the input treatment numbers
and by picking off the input block number whenever a match is made. Counters for all treatments are set up to find the frequency of all treatments occurring with $t_0$ for all the previously found blocks. The $(v-1)$ treatments are then ordered by frequency of occurrence with $t_0$. This is done by placing treatments of frequency $\lambda^0_i$ in the $(i+1)^{th}$ row of an $((m+1) \times v)$ temporary storage Array II, where $m$ is the maximum number of such sets. The result is transferred to the first row of an $((m+1) \times v)$ temporary storage Array I2 and is denoted $[t_0, C^0_1, C^0_2, ..., C^0_m]$, where $C^0_i$ is the set of treatments that occurred in blocks with $t_0$, $\lambda^0_i$ times, $\lambda^0_i > \lambda^0_{i+1}$, $i = 0, 1, ..., m-1$.

An arbitrary treatment is selected sequentially from each of the sets $C^0_1, C^0_2, ..., C^0_m$ and the selected treatments are denoted by $t^0_1, t^0_2, ..., t^0_m$, respectively. Each in turn replaces $t_0$ in Array II. Associate classes are in turn found for each and the results transferred to Array I2. At the completion of the $(m+1)^{th}$ step, we have the complete set of associate classes, frequency of occurrence, and the number of treatments occurring in each class, which may be displayed as follows.
<table>
<thead>
<tr>
<th>Classes (I2)</th>
<th>Frequency</th>
<th>Number/Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_0, C^0_1, C^0_2, \ldots, C^0_m )</td>
<td>( l, \lambda^0_1, \lambda^0_2, \ldots, \lambda^0_m )</td>
<td>( l, n^0_1, n^0_2, \ldots, n^0_m )</td>
</tr>
<tr>
<td>( t_1, C^1_1, C^1_2, \ldots, C^1_m )</td>
<td>( l, \lambda^1_1, \lambda^1_2, \ldots, \lambda^1_m )</td>
<td>( l, n^1_1, n^1_2, \ldots, n^1_m )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( t_m, C^m_1, C^m_2, \ldots, C^m_m )</td>
<td>( l, \lambda^m_1, \lambda^m_2, \ldots, \lambda^m_m )</td>
<td>( l, n^m_1, n^m_2, \ldots, n^m_m )</td>
</tr>
</tbody>
</table>

A quick check of frequency and number per class should lead to the results \( \lambda_i = \lambda^k_i, n_i = n^k_i \), \( i = 1, 2, \ldots, m, k = 0, 1, \ldots, m \), where the \( \lambda_i \)'s are such that \( \lambda_1 > \lambda_2 > \ldots > \lambda_m > 0 \).

The \( p^k_{ij} \)'s are determined next from the association classes stored in Array I2. The position of the first element in Class \( C^k_j \) is \( 1 + \sum_{i=0}^{j-1} n_i = I^k_{ij} \), while the last element of \( C^k_j \) is in position \( \sum_{i=0}^{j} n_i = I^k_{ij} \) of row \( k \), where \( n_0 = 1 \) and \( n_{-1} = 0 \) for \( j, k = 0, 1, \ldots, m \). It is now a simple matter to make all possible comparisons of pairs of association classes of \( t_0 \) with association classes of \( t_j \), \( j = 1, 2, \ldots, m \), which will determine the values of the complete set of \( p^k_{ij} \)'s. For example, the number of treatments in the intersection of Classes \( C^0_i \) and \( C^k_j \) is the value for \( p^k_{ij} \) for any \( i, j, k \). The results, stored in a three-dimensioned array \( IP \), may be expressed as
where \( t_i \) is treated as Class \( C^i_0 \). The number of treatments in \( C^i_0 \) is one.

If, in addition to determining the above required parameters, one wishes a check for the properties of definition (5.1), a printout of a partial set of the association parameters is available. The printout consists of associate classes, \( n_i \)'s, and \( \lambda_i \)'s for each treatment 1, 2, ..., \( v \) and a set of \( P_j \)'s corresponding to the set of treatments that appears in Class \( C^i_0 \) with treatment \( t_i \), \( j = 1, 2, ..., n_i \), \( i = 1, 2, ..., m \). This provides for the calculation of a \( P_i \) for each of the \( n_i \) \( i^{th} \) associates of \( t_0 \) for each \( i \). A quick visual inspection should reveal that the \( P_i \)'s of all \( i^{th} \) associates of \( t_0 \) are symmetric and identical. A complete check of design characteristics would include a similar evaluation for each of the \( (v - 1) \) other treatments with complete cross-checking of \( P_i \)'s for all combinations of pairs of treatments that are \( i^{th} \) associates, \( i = 0, 1, ..., m \). This method works well for designs that have a small to moderate number of treatments and associate classes but requires
increasingly more computer time for larger designs. This approach requires minimum computer memory space without the necessity of using storage file space on a disk or a tape.

**Method 2:** This method provides for a complete check of design characteristics with greater efficiency relative to Method 1 regarding computing time, especially for a moderate to a large number of treatments. More main core or memory will be required than was necessary for Method 1. In addition, it may be necessary to use temporary storage space on a tape or disk.

The input treatment-block subscripts are used to form the \((v \times b)\) incidence matrix \(N\) by inserting a 1 in the \((i,j)^{th}\) position of \(N\) corresponding to the input treatment-block and zeros elsewhere so that:

\[
N = (\delta_{ij}) \text{ where } \delta_{ij} = \begin{cases} 
1 & \text{if treatment } i \text{ occurred in block } j \\
0 & \text{otherwise}
\end{cases}
\]

In the following steps, any condition checked that does not satisfy definition (5.1) will cause termination of execution so that no further checks will be made. The first check is made on the input parameters \(v, b, r,\) and \(k\) to see if \(vr = bk\). A check for common block size of the \(b\) blocks is made by comparing each element in the \((1 \times b)\) vector \(N^T_{ij} = (k_i)\) to see if \(k_i = k, i = 1, 2, \ldots, b\). A common replicate for all treatments is checked for similarly by comparing each element
of \( NJ^1_{b}(v \times l) = \{r_i\} \) to see if \( r_i = r \), \( i = 1, 2, \ldots, v \), where \( J^b_a \) is an \((a \times b)\) matrix of all ones.

For the next step, form the matrix product \( NN' = W \) where \( W(v \times v) \) is symmetric and has rank that does not exceed \((v - 1)\). The matrix \( W \) is expressible as the sum of the \((v \times v)\) symmetric matrices \( G_0, G_1, \ldots, G_m \), formed on the basis of identical nonzero elements. If \( m \) exceeds \( v - 1 \), the given design cannot satisfy definition \((5.1)\), thus terminating further checks.

There will be \( n_k > 0 \) nonzero identical elements in every row and column of \( G_k \). Denoting the nonzero elements of \( G_k \) by \( \lambda_k \), we can write \( G_k = \lambda_k B_k \) so that

\[
NN' = \sum_{i=0}^{m} \lambda_i B_i \text{ where } B_k = (\delta_{ij}^k)
\]

and

\[
\delta_{ij}^k = 1 \text{ if row } i \text{ and block } j \text{ of } G_k \text{ is nonzero}
\]

\[
\delta_{ij}^k = 0, \text{ otherwise.}
\]

In order to distinguish between the \( G_k \)'s, we must have

\[
\lambda_0^* > \lambda_1^* > \ldots > \lambda_m^*.
\]

Additionally, \( \sum_{k=0}^{m} B_k^* = J^v \) where \( B_0^* = I_v \), \( \lambda_0^* = r \), and \( \lambda_m^* > 0 \).

Common associate class \( n_i \) is checked by scanning for identical diagonal elements of the \((v \times v)\) matrix product \( B_i^* B_i^* \) for each \( i, i = 0, 1, \ldots, m \). The general product \( B_i^* B_j^* \) is expressible as the linear combination (Bose and Mesner (1959))
if definition (5.1) is satisfied. Symmetry of the $P_j$'s may be checked by comparing elements of the product $B_i^*B_j^*$ with $B_j^*B_i^*$, $i \neq j = 0, 1, \ldots, m$.

The final step required is to check for common $p_{ij}^k$'s for all pairs of $k^{th}$ associates. The association matrices $B_0^*$, $B_1^*$, ..., $B_m^*$ are retained from a previous step for this check. For large designs where it is necessary to use disk, tape, or bulk storage, it will only be necessary to have in main core one row of the product $C_{ij} = B_i^*B_j^*$ and a corresponding row of each of $B_0^*$, $B_1^*$, ..., $B_m^*$ at any given time during this check. For an initial step, the value in position $(i',j')$ of $C_{ij}$ is stored in the $(k+1)^{th}$ position of temporary storage vector $IK$, corresponding to the first nonzero element encountered in the $(i')^{th}$ row of $B_k^*$, for each $k = 0, 1, \ldots, m$. After the $k^{th}$ initial element is found, it is used to compare all elements in row $i'$ of $C_{ij}$, corresponding to each nonzero element in $B_k^*$, to find out if there is a common $p_{ij}^k$, $k = 0, 1, \ldots, m$. This process is continued until $i' = v$, at which
time a common nonnegative integer value for $p_{ij}^k$, $k = 0, l, \ldots$, $m$ will be established.

This process is continued for all $C_{ij}$, $i \geq j = 0, l, \ldots$, $m$. The results may be expressed in lower triangular form as

$$(5.6)$$

$$
\begin{pmatrix}
C_{00} \\
C_{10} & C_{11} \\
& \cdots \\
C_{m_0} & C_{m_1} & \cdots & C_{mm}
\end{pmatrix}
= 
\begin{pmatrix}
\begin{array}{c}
m \\
\sum_{k=0}^m \Sigma \text{ } p_{00}^k \text{ } B_k^k
\end{array} \\
\begin{array}{c}
m \\
\sum_{k=0}^m \Sigma \text{ } p_{10}^k \text{ } B_k^k \text{ } \Sigma \text{ } p_{11}^k \text{ } B_k^k
\end{array} \\
& \cdots \\
\begin{array}{c}
m \\
\sum_{k=0}^m \Sigma \text{ } p_{m_0}^k \text{ } B_k^k \text{ } \Sigma \text{ } p_{m_1}^k \text{ } B_k^k \text{ } \cdots \text{ } \Sigma \text{ } p_{mm}^k \text{ } B_k^k
\end{array}
\end{pmatrix}
= 
\begin{pmatrix}
p_{00}^k \\
p_{10}^k & p_{11}^k \\
& \cdots \\
p_{m_0}^k & p_{m_1}^k & \cdots & p_{mm}^k
\end{pmatrix}
= 
\sum_{k=0}^m p^k \otimes B_k^k,
$$

where

$$
p^k = 
\begin{pmatrix}
p_{00}^k \\
n_{10}^k & p_{11}^k \\
& \cdots \\
p_{m_0}^k & p_{m_1}^k & \cdots & p_{mm}^k
\end{pmatrix}, \text{ } k = 0, l, \ldots, m.
$$
In both Methods 1 and 2 it is apparent that numerous computations are necessary. It is, therefore, advisable to save design parameters in a form that can be input with further problems having the same experimental design.

Experimental designs such as the simple, triple, and quadruple rectangular lattices do not satisfy definition (5.1), due in part to the fact that there may be equal $\lambda_i$ parameters. An association scheme among treatments exists, however, and analysis of the rectangular lattice and similar type designs can proceed in many cases similarly to designs satisfying definition (5.1). When equal $\lambda_i$ design parameters are encountered, it becomes necessary to partition further the association classes formed initially by the number of times each treatment occurs in blocks with a given treatment. The partitions are made on the basis of equal $p_{ij}^k$'s for all treatments within a subclass. This point can be illustrated nicely by the following $2 \times 3$ simple rectangular lattice.

<table>
<thead>
<tr>
<th>Block</th>
<th>Block</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
</tbody>
</table>

Replicate 1
Replicate 2
Initial association classes \( (C_{0i}) \) determined on the basis of the \( \lambda_{0i} \)'s are the following.

<table>
<thead>
<tr>
<th></th>
<th>( C_{00} )</th>
<th>( C_{01} )</th>
<th>( C_{02} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24</td>
<td>356</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>346</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>46</td>
<td>125</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>13</td>
<td>256</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>26</td>
<td>134</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>35</td>
<td>124</td>
<td></td>
</tr>
</tbody>
</table>

The \( \lambda_{0i} \) and \( n_{0i} \) design parameters for the initial classes are

\[
\begin{array}{c|ccc}
\lambda_{0i} & C_{00} & C_{01} & C_{02} \\
\hline
2 & 1 & 0 \\
1 & 2 & 3 \\
\end{array}
\]

The initial set of \( B_{0i} \) arrays corresponding to classes \( C_{00} \), \( C_{01} \), and \( C_{02} \) are, respectively,

\[
B_{00} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
B_{01} = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix}, \quad
B_{02} = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0
\end{pmatrix},
\]
with $P_{0i}$ arrays

$$P_{00} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad P_{01} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad P_{02}^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad P_{02}^{**} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$ 

It will be noted there are two different $P_{02}$ arrays for treatments in Class $C_{02}$, thus violating the condition of definition (5.1) that states, in effect, that there should be only one $P_{0i}$ for each $i$.

Treatments in Class $C_{02}$ can be partitioned according to whether or not they have arrays $P_{02}^*$ or $P_{02}^{**}$ associated with it. The set of association classes $\{C_{1i}\}$ generated by both $\lambda_{0i}$'s and $P_{0i}$'s is

<table>
<thead>
<tr>
<th>$C_{10}$</th>
<th>$C_{11}$</th>
<th>$C_{12}$</th>
<th>$C_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24</td>
<td>35</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>46</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>46</td>
<td>15</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
<td>26</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>26</td>
<td>13</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>35</td>
<td>24</td>
<td>1</td>
</tr>
</tbody>
</table>

with $\lambda_{1i}$ and $n_{1i}$ design parameters
The set of arrays corresponding to the classes $C_{1i}$, $i = 0, 1, 2, 3$ is

$$B_{10} = B_{00}, \quad B_{11} = B_{01},$$

where $B_{00}$ and $B_{01}$ are previously defined.

The set of $P_{1i}$ arrays is formed by factoring constants from array products as follows. We have

$$B_{12} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad B_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $B_{00}$ and $B_{01}$ are previously defined.
For designs that have less balance of treatments within blocks, more than two sequences of partitioning classes of treatments may be necessary before a final set of design elements is determined. An arbitrary unbalanced design will often generate the same number of association classes and $P_i$ arrays as there are treatments since the lack of association among treatments will cause each treatment to be placed in a class by itself. The point of diminishing returns sets in as the number of classes increases toward the number of treatments. The computing time for determining the design parameters increases as the number of classes increases, while the numerical precision of model parameter estimation gained in utilizing design parameters decreases. The reason is that as the number of classes increases, the size of the system of equations that must be solved also increases.

After establishing values for the $n_i$'s, $\lambda_i$'s, $B_i$'s, and $p_{ij}^k$'s for all $i,j,k$, the computing problem of model parameter estimation and analysis of variance is greatly enhanced. For
describing observations, a mixed additive classification model of the following form is used:

\[ Y = \mu J^1_\omega + \psi \tau + \phi \beta + \epsilon, \]

where \( Y \) is an \((\omega \times 1)\) vector of observations, \( \mu (1 \times 1) \) and \( \tau (\nu \times 1) \) are unknown mean and treatment parameters, respectively, \( \beta (b \times 1) \) and \( \epsilon (\omega \times 1) \) are uncorrelated random vectors such that \( E \beta = 0, E \epsilon = 0, E\beta\beta' = \sigma^2 I_b, E\epsilon\epsilon' = \sigma^2 I_\omega, \text{cov}(\beta, \epsilon) = 0, \psi(\omega \times \nu) \) and \( \phi(\omega \times b) \) are known matrices such that \( \psi'\psi = rI_\nu, \phi'\phi = kI_b, \phi'J^1_\omega = kJ^1_b, \psi'J^1_\omega = rJ^1_b \), where \( \omega = bk \). The expected value and variance of \( Y \) is, respectively,

\[ EY = \mu J^1_\omega + \psi \tau \quad \text{and} \quad \text{Var}(Y) = \phi'\phi\sigma^2_\beta + \sigma^2 I_\omega. \]

Under model conditions (5.7), the best linear unbiased estimate of any estimable function of \( \tau \) on an intra-block basis is known to be obtainable from any solution to the least squares normal equations of the form \( X'_1X_1 \lambda = X'_1Y \) where \( X_1 = (J^1_\omega, \psi, \phi) \) and \( \lambda' = (\hat{\mu}, \hat{\tau}', \hat{\beta}') \) where \( \beta (b \times 1) \) is treated as fixed for intra-block estimation of \( \tau \). The complete set of equations is
By applying the restrictions \( \tau'J_v^1 = \hat{\beta}'J_b^1 = 0 \), the estimate of the overall mean \( \mu \) is found to be \( \hat{\mu} = (J_w^1)'(J_w^1)^{-1}(J_w^1)'Y \) where \((J_w^1)'J_w^1 = \omega \). Substituting for \( \hat{\mu} \), the set of equations (5.8) can be reduced to the equivalent set

\[
(5.9) \begin{pmatrix}
\psi' \\
\psi' \\
\phi'
\end{pmatrix}
\begin{pmatrix}
\tau \\
\beta
\end{pmatrix}
= \begin{pmatrix}
\psi' Y - \omega^{-1}J_w^1 Y \\
\phi' Y - \omega^{-1}J_w^1 Y
\end{pmatrix}
= \begin{pmatrix}
R_1 \\
R_2
\end{pmatrix},
\]

which can be written in the form

\[
(5.10a) \quad \psi'\hat{\tau} + \psi'\hat{\phi}\hat{\beta} = R_1,
\]
\[
(5.10b) \quad \phi'\hat{\tau} + \phi'\hat{\phi}\hat{\beta} = R_2.
\]

From (5.10b) we have

\[
(5.11) \quad \hat{\beta} = (\phi'\phi)^{-1}(R_2 - \phi'\hat{\tau}).
\]

The estimate \( \hat{\beta} \) is effectively eliminated by the substitution of (5.11) for \( \hat{\beta} \) in (5.10a). The resulting set of equations involving the vector \( \hat{\tau} \) alone may be written as
(5.12) \((\psi' \psi - \psi' \phi' \phi^{-1} \phi' \psi) \hat{\tau} = (\psi' - \psi' \phi' \phi^{-1} \phi') Y\)

and is commonly referred to as the "reduced set" or the "error" set of normal equations. Substituting \(\psi' \psi = rI_v\) and \(\phi' \phi = kI_b\) in (5.12), we have the intra-block system of equations,

(5.13) \[A \hat{\tau} = Q,\]

where

\[A = rI_v - k^{-1} \psi' \phi \phi' \psi\]

and

\[Q = \psi'(I_\omega - k^{-1} \phi' \phi)^{-1} Y.\]

If the design were such that blocks and treatments are orthogonal, as would be the case for completely randomized block designs, then \(\phi' \psi\) would be a scalar multiple of \(J_v^b\). The matrix \(A\) would be \(r(I_v - v^{-1} J_v^v)\) and \(Q\) would be equal to \(\psi'(I_\omega - \omega^{-1} J_\omega^\omega) Y\). The numerical solution for treatments would be easily obtainable by a computer to be \(\hat{\tau} = \frac{1}{r} \psi'(I_\omega - \omega^{-1} J_\omega^\omega) Y\) under the restriction \(\hat{J}_v^v \hat{\tau} = 0\).
For general nonorthogonal two-way classification designs having unequal and disproportionate subclass cell frequency, the numerical problem becomes a little more difficult. The matrices \( \Phi'\Phi \) and \( \Psi'\Psi \) in equation (5.8) will no longer be a scalar multiple of an identity matrix. About all that could be said in general about the \((v \times v)\) matrix \( A \) is that it is symmetric and has rank \( v - 1 \) under the assumption of maximal rank and that the sum of the rows of \( A \) is the \((v \times 1)\) vector zero. A least squares solution for \( \hat{\tau} \), under some appropriate constraint, can be obtained by inverting a matrix of at least an order of \((v - 1)\). Many "good" inversion routines are available, but computing time and required memory become increasingly large as \( v \) increases. Even more important is the loss of accuracy of estimates due to rounding errors.

For a two-way classification design in which treatments are arranged in blocks in such a way that will satisfy properties of definition (5.1), intra-block treatment estimates can be obtained by applying special properties of the association relationships of treatments with blocks. The following theorems (Bose and Mesner (1959) and B. V. Shah (1959)) can be used to determine the association scheme of treatments, which in turn can be used as a computing technique to solve a system of equations of the form (5.13).
Theorem 5.1: Necessary and sufficient conditions that \( m + 1 \) association matrices \( B_0, B_1, \ldots, B_m \) determine an association scheme for an \( m \) associate class pbib design are:

1) \( B_0 = I_v, \sum_{j=0}^{m} B_j = J_v \),

2) \( B_i B_j = \sum_{k=0}^{m} p_{ij} B_k \) for \( i, j = 0, 1, \ldots, m \).

Under the conditions of Theorem 5.1, the \( B_k \)'s are such that \( \sum_{i=0}^{m} B_i = J_v \) and \( B_i B_j = B_j B_i, i \neq j = 0, 1, \ldots, m \). The set \( B_0, B_1, \ldots, B_m \) is linearly independent with respect to the field of real numbers and forms a linear associative and commutative algebra. Since the set is closed under scalar multiplication, the following known matrix theorem can be applied to obtain a formulation for a conditional inverse of the \((v \times v)\) matrix \( A \) in (5.13).

Theorem 5.2: If \( M \) is a \((v \times v)\) positive definite matrix such that all nonnegative integral powers of \( M \) are expressible as linear combinations of the form

\[
M^p = \sum_{i=0}^{m} c_i B_i,
\]
where the $c_i$'s are scalar constants and the $B_i$'s are fixed $(v \times v)$ matrices, then the inverse of $A$ will be expressible as a linear combination of the $B_i$'s.

Since the coefficient matrix $A$ in (5.13) is symmetric, there exists an orthogonal matrix $P(v \times v)$ such that

\[(5.14) \quad P'AP = D_v = \text{Diag}(\theta_1, \theta_2, \ldots, \theta_v),\]

where $P = (h_1, h_2, \ldots, h_v)$ are $v$ eigenvectors of $A$ corresponding to eigenvalues $\theta_1, \theta_2, \ldots, \theta_v$, respectively. The spectral decomposition of $A$ can be written

\[(5.15) \quad A = PD_vP' = \sum_{i=1}^{v} \theta_i h_i h_i'.\]

Under the assumption of maximal rank, $A$ has rank $v - 1$ so that one and only one of the eigenvalues is zero. Let $\theta_1$ be the zero eigenvalue. Consider the matrix

\[A + aJ^v = \sum_{i=2}^{v} \theta_i h_i h_i' + aJ^v J^v.\]

This matrix is nonsingular for all $a \in \mathbb{R}$, $a \neq 0$.

**Lemma 5.1**: If a design with coefficient matrix $A(v \times v)$ is connected, then the matrix $A + aJ^v$ is nonsingular for $a \in \mathbb{R}$, $a \neq 0$, $A$ symmetric, and $\hat{r} = (A + aJ^v)^{-1}Q$ is a solution to (5.13).
Since $\theta_i \neq 0$, $i = 2, \ldots, v$ for connected designs, the inverse of $(A + aJ_v^V)$ is expressible as

\[(5.16) \quad (A + aJ_v^V)^{-1} = \sum_{i=2}^{v} \theta_i^{-1} h_i h_i^t + \frac{1}{a} J_v^t J_v^1.
\]

Premultiplying by $A$ yields the result

\[(5.17) \quad A(A + aJ_v^V)^{-1} = \sum_{i=2}^{v} h_i h_i^t = I_v - v^{-1} J_v^V.
\]

If the matrix $P$ consisting of eigenvectors of $A$ were known or could be readily calculated with sufficient precision and efficiency on a high-speed computer, then the numerical problem for finding solutions for equation of the type (5.13) would essentially be solved. If, in addition, zero characteristic roots could readily be identifiable by a computer, then solutions for estimable parameters could easily be obtained by the above method for any classification design whether or not it has maximal rank. Although several algorithms are available for determining characteristic roots, none has been shown to perform adequately with regard to precision and efficiency for arbitrary matrices.
The computing problem for solving the system of equation (5.13) for the general class of incomplete block designs satisfying definition (5.1) is greatly reduced by applying Theorem 5.2 to the set of the design association matrices, thus effectively reducing the number of equations needed to solve for intra-block estimates.

Corollary 5.1: If there exist \( m+1 \) matrices \( B_0, B_1, \ldots, B_m \) of order \( v \) such that \( I_v, J_v^v \) and all positive integral powers of the coefficient matrix \( A \) of a connected design are expressible as linear combinations of \( B_0, B_1, \ldots, B_m \), then there exists a solution \( \hat{r} = CQ \) of the system of equations \( A\hat{r} = Q \), where \( C \) is a linear combination of \( B_0, B_1, \ldots, B_m \) such that

\[
AC = CA = I_v - v^{-1}J_v^v.
\]

The matrix \( \psi'\phi'\psi \), of a design satisfying definition (5.1), expressed as the linear combination

\[
(5.18) \quad \psi'\phi'\psi = \lambda_0 B_0 + \lambda_1 B_1 + \ldots + \lambda_m B_m,
\]

can be used to express the coefficient matrix \( A \) as

\[
(5.19) \quad A = rI_v - \frac{1}{k}\psi'\phi'\psi
\]

\[
= \lambda_0 B_0 - \frac{1}{k} \sum_{i=0}^{m} \lambda_i B_i
\]
\[ = r(l - \frac{1}{k})B_0 - \sum_{i=1}^{m} \frac{\lambda_i}{k} B_i \]

\[ = \sum_{i=0}^{m} a_i B_i, \]

where

\[ a_0 = r(l - k^{-1}) \]

and

\[ a_i = \frac{\lambda_i}{k}, \quad i = 1, 2, \ldots, m. \]

If \( C = \sum_{i=0}^{m} c_i B_i \) is such that \( AC = CA = I_v - v^{-1}J_v \), then the \( c_i \) coefficients must satisfy the following equations:

(5.20) \[ AC = \sum_{i=0}^{m} a_i B_i \sum_{j=0}^{m} c_j B_j \]

\[ = \sum_{i=0}^{m} \sum_{j=0}^{m} a_i c_j \sum_{k=0}^{m} p_{ij} B_k \]

\[ = \sum_{k=0}^{m} \left( \sum_{i=0}^{m} \sum_{j=0}^{m} a_i p_{ij} c_j \right) B_k, \]

which equals \( I_v - v^{-1}J_v \) if the set of \( c_j \)'s satisfies the relation
$$\sum_{i=0}^{m} \sum_{j=0}^{m} a_{i} p_{ij} c_{j} = 1 - \frac{1}{\nu} \quad \text{for } k = 0$$

$$= -\frac{1}{\nu} \quad \text{for } k = 1, 2, \ldots, m.$$  

In matrix notation (5.21) may be written as $D \delta = \eta$ where $D$ is an $(m+1) \times (m+1)$ matrix of coefficients

$$D = \begin{pmatrix}
\sum_{i=0}^{m} a_{i} p_{i0} & \sum_{i=0}^{m} a_{i} p_{i1} & \cdots & \sum_{i=0}^{m} a_{i} p_{im} \\
\sum_{i=0}^{m} a_{i} p_{i0} & \sum_{i=0}^{m} a_{i} p_{i1} & \cdots & \sum_{i=0}^{m} a_{i} p_{im} \\
\sum_{i=0}^{m} a_{i} p_{i0} & \sum_{i=0}^{m} a_{i} p_{i1} & \cdots & \sum_{i=0}^{m} a_{i} p_{im} \\
\sum_{i=0}^{m} a_{i} p_{i0} & \sum_{i=0}^{m} a_{i} p_{i1} & \cdots & \sum_{i=0}^{m} a_{i} p_{im}
\end{pmatrix}$$

$$= \sum_{i=0}^{m} a_{i} \begin{pmatrix}
p_{i0} & p_{i1} & \cdots & p_{im} \\
p_{i0} & p_{i1} & \cdots & p_{im} \\
p_{i0} & p_{i1} & \cdots & p_{im} \\
p_{i0} & p_{i1} & \cdots & p_{im}
\end{pmatrix}$$

$$= \sum_{i=0}^{m} a_{i} p_{i} ,$$

$\eta' = (1 - \nu^{-1}, -\nu^{-1}, \ldots, -\nu^{-1})$ is an $(m+1 \times 1)$ vector and $\delta' = \{c_{0}, c_{1}, \ldots, c_{m}\}$.

For connected designs the matrix $D$ has rank $m$. Thus, one permissible restriction must be imposed to allow for a unique solution for $\delta$. Any restriction imposed on $\delta$ will result in
a corresponding restriction being imposed on the \((v \times 1)\) vector \(\tau\) of equation (5.13). The restriction imposed on \(\delta\) that will yield the corresponding solution for \(\hat{\tau}\) in Lemma 5.1 is \(\delta'J^1_{m+1} = 0\). This means that for connected designs the \((m + 1) \times (m + 1)\) matrix \(D + aJ^m_{m+1}\), where \(a \neq 0\) is nonsingular and \((D + aJ^m_{m+1})^{-1}\eta\) is a solution for \(\delta\). The proof is similar to that of Lemma 5.1 and is not given here.

The solution for \(\hat{\tau}\) of the system of equation (5.13) may be written as

\[(5.23) \quad \hat{\tau} = CQ \]

\[= \left( \sum_{i=0}^{m} c_i B_i \right) Q ,\]

where the \(c_i\)'s is a solution to the system \(D\delta = \eta\) under the constraint \(\delta'J^1_{m+1} = 0\). By computing the estimates sequentially, it is necessary to have only one row from each of \(B_0\), \(B_1\), ..., \(B_m\), the \(c_i\)'s, and \(Q\) in main core at one time.

C. Utilization of Inter-block Information

Inter-block information, as its name suggests, is obtained from block totals of an incomplete block design. As was shown by Kempthorne (1956), all parameters may not be estimable on an inter-block basis. Some of the more efficient incomplete block designs allow for complete independent estimation of all
parameters on both inter- and intra-block bases. For a given
design the inter-block model may be expressed as the \((b \times 1)\)
vector

\[(5.24) \quad \phi'Y = \phi'J^1_\omega \mu + \phi'\psi \tau + \phi'\phi \beta + \phi'\epsilon,\]

where \(\phi'Y\) is the \((b \times 1)\) vector of block totals and has an
expectation and variance-covariance matrix given by

\[E\phi'Y = \phi'J^1_\omega \mu + \phi'\psi \tau\]

and

\[\text{Var}(\phi'Y) = \kappa(k\sigma^2_\beta + \sigma^2)I_B,\]

respectively. Since all covariances in the variance matrix
are zero, the least squares equation that leads to the esti-
mation of all estimable treatment effects on an inter-block
basis alone is written as

\[(5.25) \quad \begin{pmatrix} J^1_\omega' \phi \phi' J^1_\omega & J^1_\omega' \phi \phi' \psi \\ \psi' \phi \phi' J^1_\omega & \psi' \phi \phi' \psi \end{pmatrix} \begin{pmatrix} \tilde{\mu} \\ \tilde{\tau} \end{pmatrix} = \begin{pmatrix} J^1_\omega' \phi \phi' Y \\ \psi' \phi \phi' Y \end{pmatrix}.\]

An estimate \(\tilde{\mu}\) of \(\mu\) can be obtained from the system \((5.25)\)
under the constraint \(J^1_\omega' \phi \phi' \psi \tilde{\tau} = J^1_\psi \tilde{\tau} = 0\). Substituting for \(\tilde{\mu}\)
in \((5.25)\) leads to the reduced system of equations involving
only treatment estimates on an inter-block basis alone,
Solutions for this system of equations provide an estimate for treatment effects which are independent of the intra-block estimates (5.23) but which have different variances. The problem is to find the best estimator of treatment effects utilizing all available information. In order to set the problem into the generalized Gauss-Markov framework, we express the two sources of information in terms of the transformed models

\begin{align*}
Z_1 &= \Phi Y - \Phi^T \beta_1 + \eta_1, \\
Z_2 &= \Phi' Y - \Phi' \beta_2 + \eta_2, \quad \text{cov} \{\eta_1, \eta_2\} = 0,
\end{align*}

where \( E \eta_1 = 0, E \eta_2 = 0 \). The matrix \( H = I_\omega - k^{-1} \Phi' \) is an \((\omega \times \omega)\) symmetric, idempotent matrix with rank \( \omega - b \) such that \( H \phi' = 0 \) and \( P \) is an \((\omega - b) \times \omega\) orthogonal matrix such that \( \Phi H (\phi' \sigma^2 + \sigma^2 I_\omega) H' P' = I_{\omega-b} \sigma^2 \). So

\[
\text{Var} \left( \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \right) = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix},
\]

where \( V_1 = I_{\omega-b} \sigma^2 \) and \( V_2 = k(\sigma^2 + k\sigma^2_b) I_b \) are scalar multiples of identity matrices with dimension \( \omega - b \) and \( b \), respectively. If \( V_1 \) and \( V_2 \) are known, then by the Gauss-Markov theorem the
best linear unbiased estimate for the set of estimable parameters is any solution to the generalized normal equations, written as

\[(5.28) \hspace{1cm} \begin{pmatrix} \phi' J_1^1 \phi' \psi \\ 0 \end{pmatrix} \begin{pmatrix} V_1^{-1} & 0 \\ 0 & V_2^{-1} \end{pmatrix} \begin{pmatrix} \phi' J_1^1 \phi' \psi \\ 0 \end{pmatrix} \begin{pmatrix} \mu \\ \tau \end{pmatrix} = \begin{pmatrix} \phi' J_1^1 \phi' \psi \\ 0 \end{pmatrix} \begin{pmatrix} V_1^{-1} & 0 \\ 0 & V_2^{-1} \end{pmatrix} \begin{pmatrix} \mu \\ \tau \end{pmatrix} \]

or

\[(5.29) \hspace{1cm} \begin{pmatrix} J_1^1 \phi V_2^{-1} \phi' J_1^1 \\ \psi' V_2^{-1} \phi' J_1^1 \\ J_1^1 \phi V_2^{-1} \phi' \psi \\ \psi' V_2^{-1} \phi' \psi \end{pmatrix} \begin{pmatrix} \mu \\ \tau \end{pmatrix} = \begin{pmatrix} J_1^1 \phi V_2^{-1} \phi' \psi \\ \psi' V_2^{-1} \phi' \psi \end{pmatrix}, \]

where \( H_j^1 = 0 \). A reduced system of equations involving treatment parameters alone is upon substituting for \( V_1^{-1} \) and \( V_2^{-1} \) in (5.28) the following.

\[(5.29) \hspace{1cm} \begin{pmatrix} \frac{1}{\sigma^2} \psi' H \psi + \frac{1}{\kappa (\sigma^2 + k\sigma_2^2)} \phi' \phi' \psi \end{pmatrix} \begin{pmatrix} \tau \end{pmatrix} = \frac{1}{\sigma^2} \psi' H \psi + \frac{1}{\kappa (\sigma^2 + k\sigma_2^2)} \psi' \phi' \psi, \]

\[\begin{pmatrix} \psi' H \psi + \rho^{-1} k^{-1} \psi' \phi' \psi \end{pmatrix} \begin{pmatrix} \tau \end{pmatrix} = \psi' H \psi + \rho^{-1} k^{-1} \psi' \phi' \psi, \]
or

$$A_w \overline{t} = \psi'HY + \rho^{-1}k^{-1}\psi'\phi'Y,$$

where

$$\rho = (\sigma^2 + k\sigma_B^2)/\sigma^2.$$

The coefficient matrix $A_w$, expressed in terms of the association matrices, is in the case of a pbib design

$$A_w = \psi'(I_w - k^{-1}\phi\phi')\psi + \rho^{-1}k^{-1}\psi'\phi'\psi$$

$$= rB_0 - rk^{-1}B_0 - k^{-1} \sum_{i=1}^{m} \lambda_i B_i + \rho^{-1}k^{-1}(rB_0 + \sum_{i=1}^{m} \lambda_i B_i)$$

$$= r[1 - k^{-1}(1 - \rho^{-1})]B_0 - \sum_{i=1}^{m} k^{-1}(1 - \rho^{-1})\lambda_i B_i$$

$$= \sum_{i=0}^{m} a^w_i B_i,$$

where

$$a^w_0 = r[1 - k^{-1}(1 - \rho^{-1})]$$

and

$$a^w_i = k^{-1}(1 - \rho^{-1})\lambda_i, \quad i = 1, 2, \ldots, m.$$
Now, following the procedure of Corollary 5.1, the best combined estimator of $\tau$ can be written as

$$\bar{\tau} = C_w Q_w,$$

where $C_w = \sum_{i=0}^{m} c_i^w B_i$ is such that $A_w C_w = C_w A_w = I_v - v^{-1} J_v$. The coefficients $c_0^w, c_1^w, \ldots, c_m^w$ are determined from the following system of equations. If

$$A_w C_w = \sum_{i=0}^{m} \sum_{j=0}^{m} a_{ij}^w c_i^w B_i B_j$$

$$= \sum_{k=0}^{m} \left( \sum_{i=0}^{m} \sum_{j=0}^{m} a_{ij}^w c_i^w c_j^w \right) B_k$$

$$= I_v - v^{-1} J_v,$$

then the $c_i^w$'s must satisfy the relation

$$\sum_{i=0}^{m} \sum_{j=0}^{m} a_{ij}^w c_i^w c_j^w = 1 - \frac{1}{v} \text{ if } k = 0$$

$$= -\frac{1}{v} \text{ if } k = 1, 2, \ldots, m.$$

For given $a_i^w$'s in a connected design, the system of equations (5.30) has rank $m$ so that one restriction on the $c_i^w$'s may be imposed to obtain a unique solution. Just as was the case
with the intra-block estimates, the restriction imposed will result in a corresponding restriction on the treatment estimates.

For cases where the weight \( p \) is not known, an estimate of \( p \) can be used to obtain a combined estimator. The substitution of an unbiased estimate in the system (5.29) will lead to an asymptotically efficient combined estimator for treatment effects.

One of the difficulties with incomplete block design analysis is that if one utilizes least squares analysis of variance procedures, there is no unique way to partition the total source of variation into an additive sum of orthogonal components. This leads to the problem of having more estimators than parameters to be estimated, which in turn leads one to search for a minimal set of sufficient statistics that contains all the information about the parameters. Two partitions of the total variation into orthogonal components are tabulated in Tables 5.1 and 5.2.

The expected mean squares in analysis of variance, Tables 5.1 and 5.2, were obtained under model (5.7) by applying formulations of expectations of the general quadratic form \( Y'AY \) as follows: If \( Y(\omega \times 1) \) is such that \( EY = \mu \) and \( \text{Var}(Y) = V \), then for a symmetric \( (\omega \times \omega) \) matrix \( G \) we have
For the quadratic form \((v - 1)T_e\) of treatments eliminating block sum of squares, we have the expected value

\[
(5.32) \quad (v - 1)ET_e = E[\hat{\tau}'Q]
\]

\[
= E[Q'CQ]
\]

\[
= E[Y'H'\psi C\psi'HY]
\]

\[
= [(EY)'H'\psi C\psi'H(EY) + tr H'\psi C\psi'H \text{Var}(Y)]
\]

\[
= \left( (J_1^T \mu + \psi \tau)'H'\psi C\psi'H(J_1^T \mu + \psi \tau) \\
+ \sigma^2 \text{tr } H'\psi C\psi'H\phi\phi' + \sigma^2 \text{tr } H'\psi C\psi'H \right)
\]

\[
= \tau'\psi'H'\psi C\psi'H\psi \tau + (v - 1)\sigma^2 \\
= \tau'\psi'H\psi \tau + (v - 1)\sigma^2 ,
\]

where \(H = I_\omega - k^{-1}\phi\phi'\).

Under the null hypothesis of no treatment effects, the mean square of treatments eliminating blocks is distributed as a central chi-square times \(\sigma^2\) with \(v - 1\) degrees of freedom and the ratio \(T_e/s_e\) is distributed as a central F with \(v - 1\) and \(bk - v - b + 1\) degrees of freedom, thus providing for an
### Table 5.1

<table>
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<th>d.f.</th>
<th>m.s.</th>
<th>E(m.s.)</th>
</tr>
</thead>
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<td>(bk\mu^2 + \frac{\tau'}{v} J'_v \tau + k\sigma^2 + \sigma^2)</td>
</tr>
<tr>
<td>Blocks Ignoring Treatments</td>
<td>b-1</td>
<td>BLI</td>
<td>((b-1)^{-1} \tau' (k^{-1} \psi \phi' \psi - \frac{\tau'}{v} J'_v \tau + k\sigma^2 + \sigma^2))</td>
</tr>
<tr>
<td>Treatments Eliminating Blocks</td>
<td>v-1</td>
<td>Te</td>
<td>((v-1)^{-1} \tau' (rI_v - k^{-1} \psi \phi' \psi) \tau + \sigma^2)</td>
</tr>
<tr>
<td>Intra-block Error</td>
<td>bk-v-b+1</td>
<td>se</td>
<td>(\sigma^2)</td>
</tr>
<tr>
<td>Total</td>
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### Table 5.1 (Continued)

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<th>m.s.</th>
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<tr>
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<td>(\hat{\tau}'Q)</td>
<td>Te</td>
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<tr>
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<td>Y'Y-k^{-1} (\phi'Y)'(\phi'Y) - (\hat{\tau}'Q\sigma^2)</td>
<td>se</td>
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Table 5.2

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<td></td>
<td></td>
</tr>
<tr>
<td>Blocks Eliminating Treatments</td>
<td>b-l</td>
<td>BL_e</td>
<td>$b k u^2 + \Sigma_1 Y^T J V^T + k \sigma^2 + \sigma^2$</td>
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<tr>
<td>Treatments Ignoring Blocks</td>
<td>v-l</td>
<td>$T_I$</td>
<td>$\frac{r}{v-1} (I_v - V^{-1} J V) \tau + \hat{V} - k \sigma^2 + \sigma^2$</td>
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<tr>
<td>Intra-block Error</td>
<td>bk-v-b-l</td>
<td>$s_e$</td>
<td>$\sigma^2$</td>
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<tr>
<td>Total</td>
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<td>BL_e</td>
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<td>Treatments Ignoring Blocks</td>
<td>v-l</td>
<td>$r^{-1}(\psi' Y)(\psi' Y) - CT$</td>
<td>$T_I$</td>
</tr>
<tr>
<td>Intra-block Error</td>
<td>bk-v-b-l</td>
<td>$Y'Y - k^{-1}(\phi' Y)(\phi' Y) - \hat{Q}$</td>
<td>$s_e$</td>
</tr>
<tr>
<td>Total</td>
<td>bk-l</td>
<td>$Y'Y - CT$</td>
<td></td>
</tr>
</tbody>
</table>
unbiased test, in the sense of Yates, of the null hypothesis of no overall treatment differences utilizing intra-block information alone.

An estimate \( \hat{\rho} \) for the weight \( \rho \) can be obtained from the analysis of variance, Table 5.2, by substituting estimates \( s_e \) and \( M_b \) for \( \sigma^2 \) and \( \sigma^2_b \).

\[
(5.33) \quad M_b = \frac{b-1}{b-2} (B_{Le} - s_e) \text{ for } B_{Le} - s_e > 0,
\]

and

\[
s_b = s_e + kM_b.
\]

An unfortunate consequence of using (5.33) is that negative values will be admitted as estimates for the nonnegative parameter \( \sigma^2_b \). To avoid this problem, one can set \( M_b = 0 \) whenever \( B_{Le} - s_e \leq 0 \) so that

\[
(5.34) \quad \hat{\rho} = \frac{s_b}{s_e} \text{ if } M_b > 0
\]

\[
= 1 \quad \text{if } M_b \leq 0.
\]

D. Combining Information from R Incomplete Block Designs

Various possibilities encountered in combining R incomplete block designs include those designs having a common set of treatments in R replicates of a common design satisfying
definition (5.1) or from R designs that independently satisfy definition (5.1). The treatments that appear in each design may contain a partial set of treatments that occurs together in common with all R designs, while the remaining treatments are entirely different from design to design. For a given treatment arrangement and set of designs, two common methods exist for combining inter- and intra-block information across R designs. One method is the combining of inter- and intra-block information for each design, then combining across the R designs. In the second method, combined inter- and intra-block information for each of the R designs is obtained first; then the combined inter- and intra-block treatment estimates are obtained. A general additive model that can be used to describe data in R incomplete block designs having a common set of treatments can be written as

\[ Y_i = \mu \omega_i^1 + J_i \alpha_i + \psi_i \tau + \phi_i \beta_i + \varepsilon_i, \quad i = 1, 2, \ldots, R, \]

where \( Y_i \) is an \( (w_i \times 1) \) vector of observations with \( \omega_i = r_i \cdot v = k_i \cdot b_i \). \( \alpha_i \) is the fixed effect of the \( i \)th design, \( \psi_i \) and \( \phi_i \) are \( (w_i \times v) \) and \( (w_i \times b_i) \) known matrices of design coefficients, \( \beta_i \) and \( \varepsilon_i \) are uncorrelated random block and residual effects such that \( E\beta_i = 0, E\varepsilon_i = 0, E\beta_i \beta_i' = \sigma^2 \cdot I_{b_i}, E\varepsilon_i \varepsilon_i' = \sigma^2 \cdot I_{\omega_i}, \)

\( E\beta_i \beta_j = 0, E\varepsilon_i \varepsilon_j = 0, \) and \( E\beta_i \varepsilon_i = 0, i \neq j. \) By applying the
results of the previous section, the intra-block treatment estimate for the \( i^{th} \) design can be written as

\[
\tau_i = \left( \sum_{j=0}^{m_i} c_i^j B_i^j \right) Q_i,
\]

where \( B_i^0, B_i^1, \ldots, B_i^{m_i} \) are the \((v \times v)\) association matrices of the \( i^{th} \) design. A combined estimate of treatment effects of intra-block information alone over \( R \) designs can be obtained in the following manner. Consider the transformed model of intra-block contrasts, written as

\[
Z_{1i} = P_i H_i Y_i = \mu P_i H_i J_i^{\top} + P_i H_i J_i^{\top} \omega_i^{\top} + P_i H_i \psi_i \tau + \eta_{1i},
\]

with variance \( \text{Var}(Z_{1i}) = \sigma_i^2 \omega_i - b_i \omega_i^{\top} = V_i \) where \( H_i = I - \omega_i - \phi_i \phi_i^{\top} \) and \( P_i \) is an orthonormal matrix such that \( P_i H_i \text{Var}(Y_i) H_i^{\top} P_i = I - \omega_i - b_i \sigma_i^2, i = 1, 2, \ldots, R \). If the \( \sigma_i^2 \)'s were known, then the best linear unbiased estimate of \( \tau \) on an intra-block basis is any solution to the system of equations

\[
\sum_{i=1}^{R} \psi_i H_i \Psi_i^{\top} P_i H_i \Psi_i^{\top} \hat{\tau} = \sum_{i=1}^{R} \psi_i H_i \Psi_i^{\top} P_i H_i Y_i.
\]

Since the \( H_i \)'s are idempotent and the \( P_i \)'s are orthonormal, the system (5.38) can be written in the form
where the rank of $\psi_i^! H_i \psi_i$ is $v - 1$ for each $i$. A unique solution, obtained by imposing the restriction $J_{v}^1 \tau = 0$, is written as

\[ \tau = \left( \sum_{i=1}^{R} \left( \frac{1}{\sigma_i^2} \psi_i^! H_i \psi_i \right) + v^{-1} I_v \right)^{-1} \sum_{i=1}^{R} \frac{1}{\sigma_i^2} \psi_i^! H_i Y_i. \]

For cases where the $\sigma_i^2$'s are not known, an unbiased estimate $s_i^2$ can be substituted for $\sigma_i^2$ in (5.40) that will give an asymptotically efficient combined intra-block estimator. A preliminary test for homogeneity of intra-block variance should always precede the use of estimated weights due to their contribution to the overall variance of the combined estimator.

An alternative combined estimator to (5.40), in which the design association parameters can be utilized, is written as

\[ \tau = \sum_{i=1}^{R} \hat{w}_i \hat{\tau}_i, \]

where

\[ \hat{w}_i = \frac{r_i/s_i^2}{\sum_{j=1}^{R} r_j/s_j^2}, \]
and \( \hat{\tau}_i \) is given by (5.36) with \( r_i \) equals the number of treatment replicates in the \( i^{th} \) design. This estimator, however, is not asymptotically efficient because the weighting scheme used does not reflect the full variance-covariance structure of \( \hat{\tau}_i \). For cases where \( R \) arbitrary incomplete block designs containing a common set of \( v \) treatments are such that definition (5.1) is not satisfied, the design association structure cannot be utilized in the calculation of an efficient estimator of the form (5.40). It will, therefore, be necessary to treat the set of \( R \) incomplete block designs as a general nonorthogonal design, in which case the entire coefficient matrix of order \( v \) must be inverted to obtain the combined estimator of intra-block information. A similar situation also exists for the combined inter- and intra-block estimates across the \( R \) designs. The inter-block model for each of the \( R \) designs can be written, from (5.35), as

\[
(5.42) \\
Z_{2i} = \phi_i Y_i = \mu \phi_i J_i^1 \omega_i + \phi_i J_i^1 \alpha_i + \phi_i \psi_i \tau + \eta_{2i}, \quad i = 1, 2, \ldots, R,
\]

where \( \eta_{2i}(b_i \times 1) \) is such that

\[
E_{\eta_{2i}} = 0, \quad E_{\eta_{2i} \eta_{2i}'} = k(\sigma_i^2 + k\sigma_\beta^2)I_{b_i} = V_{2i},
\]

and is uncorrelated with \( \eta_{1i} \) of intra-block model (5.37), as well as \( \sigma_i \), and \( \tau \). If the intra- and inter-block variances
were known for each of the R designs, the best linear unbiased estimator for \( \tau \) utilizing all available information described by models (5.37) and (5.42) is any solution to the combined equations

\[
\begin{align*}
\sum_{i=1}^{R} \frac{1}{\sigma_i^2 + k_i \sigma_i^2} \left( \psi_i^2 H_i^2 \psi_i + \psi_i^2 \phi_i^2 \phi_i^2 \psi_i \right) \Gamma_{w} &= \sum_{i=1}^{R} \frac{1}{\sigma_i^2 + k_i \sigma_i^2} \psi_i \phi_i^2 \phi_i^2 Y_i \\
\text{or} \quad A_{wc} \bar{\Gamma}_w &= Q_{wc},
\end{align*}
\]

where

\[
A_{wc} = \sum_{i=1}^{R} \left( \psi_i^2 H_i^2 \psi_i + \rho_i^{-1} k_i^{-1} \psi_i \phi_i^2 \phi_i^2 \psi_i \right)
\]

\[
= \sum_{i=1}^{R} \left[ \psi_i^2 \psi_i - k_i^{-1} (1 - \rho_i^{-1}) \psi_i \phi_i^2 \phi_i^2 \psi_i \right]
\]

\[
= \sum_{i=1}^{m_i} a_{ik} \bar{B}_k
\]

and

\[
Q_{wc} = \sum_{i=1}^{R} \psi_i \left( \omega_i - k_i^{-1} (1 - \rho_i^{-1}) \phi_i^2 \phi_i^2 \right) Y_i.
\]
The set of $a_{ik}^W$'s is such that

$$a_{ik}^W = r_i [1 - k_i (1 - \rho_i^{-1})] \text{ for } k = 0$$

$$= k_i^{-1} (1 - \rho_i^{-1}) \lambda_i \text{ for } k = 1, 2, \ldots, m_i, i = 1, 2, \ldots, R,$$

where $\rho_i = \left(\sigma_i^2 + k \sigma_i^2\right) / \sigma_i^2$.

In the case of pbib designs there will be $(v \times v)$ matrices $B^i_0, B^i_1, \ldots, B^i_{m_i}$ for the $i$th design, $i = 1, 2, \ldots, R$.

Unfortunately, the computing problem for obtaining a solution for $\tau_\infty^W$ in (5.43) is not eased by knowing the association matrices unless definite relationships exist between the designs. If all of the designs are connected, the $(v \times v)$ matrix of weighted coefficients has rank $(v - 1)$ so that only one restriction, $J_v^T \tau_\infty^W = 0$, needs to be imposed to calculate a unique solution,

(5.44) \[ \tau_\infty^W = (A_{wc} + v^{-1} J_v^v)^{-1} Q_{wc}, \]

obtained by inverting the full $(v \times v)$ matrix $A_{wc} + v^{-1} J_v^v$.

For cases where the $\rho_i$ parameters are not known, an estimate $\hat{\rho}_i$ of the form (5.34) can be substituted for the $\rho_i$'s in (5.44) to give an asymptotically efficient combined estimator of the form

(5.45) \[ \hat{\tau}_\infty^W = (A_{wc}^\hat{\rho} + v^{-1} J_v^v)^{-1} Q_{wc}^\hat{\rho}. \]
The set of $p_i$'s can be obtained from the analysis of variance (Table 5.2) of each of the $R$ individual designs.

It is apparent that there is no computational advantage in using $R$ incomplete block designs having the number of treatments the only additional characteristic in common. There are, however, definite computational advantages to using a set of incomplete block designs that have related association schemes and design parameters.

Theorem 5.3: Given $R$ incomplete block designs with incidence matrices $\psi_1^i\phi_1^j$, $\psi_2^i\phi_2^j$, ..., $\psi_R^i\phi_R^j$, respectively, for each design

\[ v = \text{number of treatments}, \]
\[ b = \text{number of blocks}, \]
\[ k = \text{number of plots per block}, \]
\[ r = \text{number of replicates of each treatment}, \]
\[ m + 1 = \text{number of association classes}. \]

If there exist $(v \times v)$ matrices $B_0$, $B_1$, ..., $B_m$ such that each of the given designs satisfy properties of definition (5.1) and

\[ \psi_1^i\phi_1^j\psi_j = \sum_{k=0}^{m} \lambda_{i,k}^j B_k \]

holds for some set of real numbers $\lambda_{0}^{ij}$, $\lambda_{1}^{ij}$, ..., $\lambda_{m}^{ij}$, $i,j = 1, 2, \ldots, R$, then the incomplete block design having
incidence matrix \( \Psi' \Phi \) satisfying all properties of definition (5.1) was

\[
\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_R \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_R \end{pmatrix}
\]

and

\[
v = \text{number of treatments}, \\
Rb = \text{number of blocks}, \\
k = \text{number of plots per block}, \\
Rr = \text{number of replicates of each treatment}, \\
m + 1 = \text{number of associate classes}.
\]

**Proof:** If there is a common set of association matrices \( B_0, B_1, \ldots, B_m \) for the \( R \) designs, a necessary condition for any pair of treatments to be \( k^{th} \) associates in the \( i^{th} \) design is that they are \( k^{th} \) associates in design \( j \), for all \( i \neq j = 1, 2, \ldots, R \), and \( k = 0, 1, \ldots, m \). This condition requires that there be a common set of \( n_i \)'s and \( p_{ij}^k \) association parameters since

\[
B_i B_j = \sum_{k=0}^{m} p_{ij}^k B_k, \quad i,j = 0, 1, \ldots, m, \quad \text{for each of the} \quad R \text{ designs. If} \quad \psi_i^1 \phi_i^1 \psi_j^1 = \sum_{k=0}^{m} \lambda_{ij}^k B_k, \quad \text{we can write} 
\]
\[
\psi'\phi'\psi = \sum_{i=1}^{R} \sum_{j=1}^{R} \psi_i^t \phi_i^t \psi_j^t \\
= \sum_{i=1}^{R} \sum_{j=1}^{R} \sum_{k=0}^{m} \lambda_{i,j,k}^t B_k \\
= \sum_{k=0}^{m} \left( \sum_{i=1}^{R} \sum_{j=1}^{R} \lambda_{i,j,k}^t \right) B_k \\
= \sum_{k=0}^{m} \lambda_k^t B_k,
\]

where

\[
\lambda_k^t = \sum_{i=1}^{R} \sum_{j=1}^{R} \lambda_{i,j,k}^t, \quad k = 0, 1, \ldots, m.
\]

Also,

\[
\psi'\psi = \sum_{i=1}^{R} \psi_i^t \psi_i = \sum_{i=1}^{R} r_{i,i} = Rr_{I_{v}} = Rr_{I_{v}},
\]

where \( Rr \) is the total number of treatment replicates, and the proof is completed.

It is easy to see for cases of \( R \) identical replicates of a given design such that \( \psi_i = \psi \) and \( \phi_i = \phi \) for all \( i \), then
\[ \Psi' \Phi' \Psi = \sum_{i=1}^{R} \sum_{j=1}^{R} \psi_i \phi_i \psi_j \]

\[ = R^2 \psi' \phi' \psi \]

\[ = R \sum_{k=0}^{m} R \lambda_k B_k \]

where the \( \lambda_k \)'s are such that

\[ \psi_i \phi_i \psi_i = \sum_{k=0}^{m} \lambda_k B_k, \quad i = 1, 2, \ldots, R. \]

A combined intra-block estimate for \( \tau \) under the general model (5.37) is any solution to the system of equations of the form (5.39), written as

\[ \hat{A}_c \hat{\tau}_c = Q_c, \]

where

\[ \hat{A}_c = \sum_{i=1}^{R} \frac{1}{\sigma_i^2} \psi_i H_i \psi_i \] and \[ Q_c = \sum_{i=1}^{R} \frac{1}{\sigma_i^2} \psi_i H_i Y_i. \]

Under the assumption of homogeneous and uncorrelated intra-block variances and a common design, we can write


\[
A_{c_1} = \frac{1}{\sigma^2} \sum_{i=1}^{R} \left(\psi_i'\psi_i - k^{-1}\psi_i'\phi_i'\phi_i\psi_i\right)
\]

\[
= \frac{R}{\sigma^2} \left(\rho I_v - k^{-1}\psi'\phi'\psi\right)
\]

\[
= \frac{1}{\sigma^2} \sum_{i=0}^{m} R\lambda_i B_i
\]

and

\[
Q_{c_1} = \frac{1}{\sigma^2} \sum_{i=1}^{R} \psi_i'\left(I_{\omega} - k^{-1}\phi_i'\phi_i\right)Y_i
\]

\[
= \frac{1}{\sigma^2} \psi'\left(I_{\omega} - k^{-1}\phi'\right)\sum_{i=1}^{R} Y_i.
\]

In solving this system of \(R\) replicates, it will only be necessary to find the association parameters for just one of the designs. For a set of \(R\) designs that are not identical but which satisfy Theorem 5.3, the combined system of intra-block equations has the coefficient matrix

\[
A_{c_2} = R\left(\rho I_v - k^{-1} \sum_{i=1}^{R} \psi_i'\phi_i'\phi_i\psi_i\right)
\]

\[
= \sum_{i=0}^{R} R\lambda_i B_i
\]

and
Q_{C_2} = \sum_{i=1}^{R} \psi_i^!(I_\omega - k^{-1}\phi_i^!\phi_i^!)Y_i,

so that only the calculation of $Q_{C_2}$ differs from that of an identical set of $R$ designs.

The set of $v$ treatments in $R$ incomplete block designs can be considered as being nested or crossed within a balanced or partially balanced classification sample structure. For crossed classifications, the parameter $\alpha_i$ in model (5.35) can be expressed in terms of a main effect or interaction of $f$ factors with levels $h_1, h_2, \ldots, h_f$, respectively, as

(5.47) $\alpha_i = a(x_1^i, x_2^i, \ldots, x_f^i),$

where

$x_j^i \in GF(h_j), j = 1, 2, \ldots, f, i = 1, 2, \ldots, R,$

and $R$ is a scalar multiple of $\prod_{j=1}^{f} h_j$. The notation $GF(h_j)$ denotes the Galois Field, comprised of integers 0, 1, ..., $h_j$.

A breakdown of the total variation into orthogonal components, including terms for interaction between the balanced classification factors and the partially balanced set of treatments, is displayed in Table 5.3. In the classification sum of squares calculations, the $i^{th}$ design total is expressed in
terms of the classification factor as \( Y_i = Z_i x_{i_1} x_{i_2} \ldots x_{i_f} \).

The total for the \( x_{j_i} \)th level of factor \( j \) is denoted
\[ Z = x_{j_1} x_{j_2} \ldots x_{j_1} x_{j_2+1} \ldots x_f, \]
while the grand total across the \( R \) designs is \( Z = x_{1_1} x_{1_2} \ldots x_f \). The sum of squares for designs denoted SSD in Table 5.3 is calculated in the usual way as

\[
(5.48) \quad SSD = \omega^{-1} R \sum_{i=1}^{R} z_i^2 \prod_{j=1}^{f} x_{j_i} - CT,
\]

where \( CT = \omega^{-1} R^{-1} Z^2 \).

Calculation for the set of \( f \) main effect classification sum of squares also proceeds in the usual way. For the \( k \)th main effect, for example, the sum of squares denoted by \( SSC_k \) can be written

\[
(5.49) \quad SSC_k = R \omega^{-2} \sum_{x_{k_1} = 0}^{h_k} z_i^2 x_{k_1} \ldots x_{k-1} x_{k+1} \ldots x_f - CT.
\]

The residual \( D \) contains all two-factor and higher order interaction of the classification factors and is calculated by subtraction as

\[
(5.50) \quad SSRD = SSD - \sum_{i=1}^{f} SSC_i.
\]
Table 5.3

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>s.s.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1</td>
<td>CT</td>
</tr>
<tr>
<td>Designs</td>
<td>R-1</td>
<td>SSD</td>
</tr>
<tr>
<td>Class 1</td>
<td>h -1</td>
<td>SSC _1</td>
</tr>
<tr>
<td>Class 2</td>
<td>h -1</td>
<td>SSC _2</td>
</tr>
<tr>
<td>:</td>
<td></td>
<td>:</td>
</tr>
<tr>
<td>Class f</td>
<td>h_f-1</td>
<td>SSC_f</td>
</tr>
<tr>
<td>Residual D</td>
<td>R- Σ h_i+f-1</td>
<td>SSRD</td>
</tr>
</tbody>
</table>

Blocks Ignoring Treatments/Designs | R(b-1) | SSBIT |
Treatments Eliminating Blocks       | v-1    | SSTEB |
Intra-block Error                   | R(bk-v-b+1) | SSE  |
Total                               | Rbk-1  | SSTOT |

Blocks ignoring treatments within design sum of squares, denoted by SSBIT, is calculated by

\[
SSBIT = \sum_{i=1}^{R} \left( k^{-1}(\phi_i^{T}Y_i)(\phi_i^{T}Y_i) - \omega^{-1}x_i x_i^{T} \right).
\]

Treatments eliminating blocks sum of squares SSTEB with v - 1 degrees of freedom is obtained by applying Theorem 5.3 to the R designs so that
\[(5.52)\]

\[\text{SSTEB} = \hat{\tau}'Q\]
\[= Q'CQ\]
\[= \sum_{i=1}^{R} (Y_i'H_i\psi_i)^C(\sum_{j=1}^{R} \psi_j'H_jy_j)\]
\[= \sum_{i=1}^{R} \sum_{j=1}^{R} Y_i'H_i\psi_i'C\psi_j'H_jy_j,\]

where

\[C = \sum_{i=0}^{m} c_i B_i \text{ and } H_i = I_w - k^{-1}\phi_i\phi_i'.\]

The total sum of squares is calculated in the usual way, and the pooled intra-block error within designs is obtained by subtraction. An overall test for no differences between the set of partially balanced treatments is comparing the ratio \(\text{MSTEB}/\text{MSE}\) with a tabulated Snedecor F value with \((v - 1)\) and \(R(bk - v - b + 1)\) degrees of freedom, where \(\text{MSTEB} = \text{SSTEB}/(v - 1)\) and \(\text{MSE} = \text{SSE}/R(bk - v - b + 1)\). Under the assumption of homogeneous intra-block error, MSE has expectation \(\sigma^2\) and is distributed under the assumption of normality as \(\chi^2\sigma^2\) with \(R(bk - v - b + 1)\) degrees of freedom. An estimate for the inter-block variance can be obtained from the sum of squares of blocks eliminating treatments denoted \(\text{SSBET}\) and calculated by
The mean square $\text{MSBET} = \frac{\text{SSBET}}{R(b - k + v - b + 1)}$ is distributed as a chi-square times $\sigma^2 + \frac{R(v - b)\sigma^2_\beta}{b - 1}$ with $R(b - 1)$ degrees of freedom and can be used in the estimation of $\sigma^2 + k\sigma^2_\beta$ similar to the method previously described for the case of a single design.

An unbiased combined inter- and intra-block estimator for the partially balanced treatment parameters can be obtained by substituting $\hat{\rho} = (\sigma^2 + k\sigma^2_\beta)/\hat{\sigma}^2$ for all $\rho_i$ in (5.43) to get an estimator of the form

\[(5.54) \quad \hat{\tau}_w^\wedge = \left( \sum_{i=0}^{m} c_i B_i \right) Q_w^\wedge,
\]

where

\[Q_w^\wedge = R^{-1} \sum_{i=1}^{R} \psi_i^\wedge (I_{\omega} - k^{-1}(1 - \hat{\rho}^{-1})\phi_i^\wedge \phi_i^\wedge) Y_i
\]

\[= R^{-1} \psi^\wedge (I_{\omega} - k^{-1}(1 - \hat{\rho}^{-1})\phi \phi^\prime)^\wedge \sum_{i=1}^{R} Y_i.
\]

The set $\hat{c}_0$, $\hat{c}_1$, ..., $\hat{c}_m$ are solutions to the system of equations

\[\sum_{i=0}^{m} \sum_{j=0}^{m} a_{ij} \hat{c}_k \hat{c}_j = 1 - v^{-1} \text{ if } k = 0
\]

\[= -v^{-1} \text{ if } k = 1, 2, ..., m,
\]
where

\[ a_i = r[l - k^{-1}(1 - \rho^{-1})] \text{ if } i = 0 \]
\[ = k^{-1}(1 - \rho^{-1}) \lambda_i \text{ if } i = 1, 2, \ldots, m. \]

So (5.54) can be written

\[
(5.55) \hat{\tau}_w = \left[ \sum_{i=0}^{m} \hat{c}_i \hat{B}_i \psi'(I_\omega - k^{-1} \phi' \phi') + \sum_{i=0}^{m} \hat{c}_i \hat{\rho}^{-1} \psi'(k^{-1} \phi') \right] \hat{Y} \\
= \sum_{i=0}^{m} \left( \hat{c}_i \hat{B}_i \psi'H + k^{-1} \hat{c}_i \hat{\rho}^{-1} \psi' \phi' \right) \hat{Y},
\]

where \( H = I_\omega - k^{-1} \phi' \phi' \) is such that \( \psi'H(\phi') = 0. \)
VI. EMPIRICAL RESULTS
A. Introduction

It is apparent from previous chapters that exact mathematical formulations of the unconditional variance of combined estimators utilizing random weights are difficult, if not impossible, to express depending on the particular sampling techniques and model assumptions made. Variances of estimators of \( \tau \) are generally somewhat elusive due to the fact that infinite series are sometimes involved and there is a limitation for distributions, such as Snedecor F, of the number of moments that can be estimated. For many cases, an unbiased estimate of an approximate mathematical expression for the true variance is the best that can be done. This introduces uncertainty into mathematical comparisons of estimators so that numerical results are helpful, and in some cases necessary, to use as a guide in making estimator recommendations regarding criteria, such as relative asymptotic efficiency and effectiveness of estimation. A Monte Carlo study was undertaken to investigate the effectiveness of using random weighting for combining information under a wide variety of population conditions.
B. Computational Procedures

Pseudo random uniform $U(0,1)$ numbers, drawn by the generating procedure RANDU, are transformed into independent random normal $N(0,1)$ numbers by the transformations

\[
\begin{align*}
    x_1 &= \left(-2 \ln u_1\right)^{\frac{1}{2}} \sin 2\pi u_2, \\
    x_2 &= \left(-2 \ln u_1\right)^{\frac{1}{2}} \cos 2\pi u_2,
\end{align*}
\]

where $u_1$ and $u_2$ are independent $U(0,1)$ random variables. The set of independent $N(0,1)$ numbers generated by (6.1) is transformed to satisfy given population conditions which are systematically varied throughout the study to give a wide coverage of possible values so that more general conclusions are possible.

The general model for the $M^{th}$ population selected for this study is denoted

\[
(6.2) \quad y_{i}^{M} = X\mu + \varepsilon_{i}^{M}, \quad i = 1, 2, \ldots, N_{REP},
\]

where $y_{i}^{M}(\omega \times 1)$ is such that $E\varepsilon_{i}^{M} = 0$, and $y_{i}^{M}(\omega \times 1)$ is the $i^{th}$ generated vector of observations for the $M^{th}$ selected population. The design matrix $X$ was fixed for this study to be the $(\omega \times 1)$ vector of ones, denoted $J_{\omega}^{1}$, so that an overall population mean $\mu$ is the $(1 \times 1)$ unknown parameter to be
estimated. Design matrices other than $J^1_w$ can easily be handled by the computer program developed for this study. To compare estimators of a vector, ratios of generalized variances, average variances, or variances of some fixed linear combination of the estimates can be used. For this study, utilizing design matrix $X = J^1_w$, ratios of scalar variances are used to make comparisons of the set of estimators of the scalar parameter $\mu$.

The variance-covariance matrix for the random vector $Y_i^M$ associated with the $M^{th}$ population $\{(m, \beta) \rightarrow M\}$ has the form

$$
V_M = \begin{pmatrix}
\sigma_{11}^m & \rho_1 \sigma_{12}^m & \ldots & \rho_1 \sigma_{1\omega}^m \\
\rho_1 \sigma_{21}^m & \sigma_{22}^m & \ldots & \rho_1 \sigma_{2\omega}^m \\
\vdots & \vdots & \ddots & \vdots \\
\rho_1 \sigma_{\omega 1}^m & \rho_1 \sigma_{\omega 2}^m & \ldots & \sigma_{\omega \omega}^m
\end{pmatrix}
$$

(6.3)

where $\sigma_{ii}^m = \sqrt{\sigma_{ii}^m}$, $i = 1, 2, \ldots, \omega$ and is generated in the following manner. First, values for $\sigma_{11}^m, \sigma_{22}^m, \ldots, \sigma_{\omega \omega}^m$ are generated by the linear equation

(6.4) $\sigma_{ii}^m = 1$ if $m = 1$

$= i$ if $m = 2, i = 1, 2, \ldots, \omega$

$= 1 - 2^{m-1} + 2^{m-1} \sigma_{ii}^2$ if $m = 3, 4, \ldots, N^G$

$= i, 2, \ldots, \omega$, $i = 1, 2, \ldots, \omega$, $i = 1, 2, \ldots, \omega$,
thus providing a wide range of values. With the range of \( m \) fixed at 20, a partial tabulation of values for \( \omega \leq 100 \) is presented in Table 6.1.

Table 6.1

<table>
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<tr>
<th>( m )</th>
<th>( \sigma_{11}^m )</th>
<th>( \sigma_{22}^m )</th>
<th>( \sigma_{33}^m )</th>
<th>( \sigma_{44}^m )</th>
<th>( \sigma_{55}^m )</th>
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<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( 10^6 )</td>
</tr>
</tbody>
</table>

To transform a random vector with the variance-covariance matrix \( I_{\omega} \) to the random vector \( Y_i^M \) with \( \text{Var}(Y_i^M) = V_M \) for a specified \( M \), we first calculate the lower triangular matrix \( A_M(\omega \times \omega) \) where \( A_M \) is such that \( A_M A_M' = V_M \). Denote \( A_M = (a_{ij}^M) \) and \( V_M = (v_{ij}^M) \). Then with \( a_{11}^M = \sqrt{v_{11}^M} \), values for \( A_M \) are expressible in terms of elements of \( V_M \) and previously computed elements of \( A_M \) as

\[
a_{ii}^M = \sqrt{v_{ii}^M} - \sum_{l=1}^{i-1} a_{il}^M
\]
and

\[ a_{ij} = v_{ij} - \sum_{k=1}^{j-1} a_{ik}a_{kj}/a_{jj}, \]

\( j < i, \ i = 2, 3, \ldots, \omega. \)

Since \( V_M \) is symmetric and \( A_M \) is a lower triangular matrix, required computer memory space can be reduced by storing only the lower triangular part of \( V_M \) and \( A_M \) in one-dimensioned arrays, expressible as

\[ V_M = \begin{pmatrix} \sigma_1^m & \rho_\beta \sigma_2^m \sigma_2^m & \rho_\beta \sigma_3^m \sigma_2^m \sigma_3^m & \cdots & \rho_\beta \sigma_\omega^m \sigma_\omega^m \sigma_\omega^m \end{pmatrix}, \]

(6.6)

and

\[ A_M = \begin{pmatrix} a_{11}^M & a_{21}^M & a_{31}^M & a_{41}^M & \cdots & a_{\omega 1}^M \\ a_{21}^M & a_{22}^M & a_{32}^M & a_{42}^M & \cdots & a_{\omega 2}^M \\ a_{31}^M & a_{32}^M & a_{33}^M & a_{43}^M & \cdots & a_{\omega 3}^M \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{\omega 1}^M & a_{\omega 2}^M & a_{\omega 3}^M & a_{\omega 4}^M & \cdots & a_{\omega \omega}^M \end{pmatrix}, \]

where the correlation coefficient \( \rho_\beta \) is chosen at fixed points covering the closed interval \([-1,1]\). Numerical manipulations are made on lower triangular arrays stored in one-dimensioned arrays by utilizing a subscript transformation. The mapping of the \((i,j)\)\(^{th}\) element of a two-dimensioned array into element \( \ell \) of the one-dimensioned array is given by

\[ \ell = NS(j) + i \text{ if } i > j, \]

\[ \ell = NS(i) + j \text{ if } j > i, \ i = 2, 3, \ldots, \omega, \]

(6.7)
where the one-dimensioned array \( NS \) is initially computed as

\[
\begin{array}{cccccccc}
  j & 1 & 2 & 3 & 4 & \ldots & p & \ldots & w \\
  NS(j) & 0 & 1 & 3 & 6 & \ldots & \sum_{q=1}^{p-1} q & \ldots & \frac{(w-1)w}{2}
\end{array}
\]

In the computations to follow, corresponding to a given population choice of a set of variances indexed by \( m \), and of a correlation coefficient indexed by \( \beta \), where \((m, \beta) \rightarrow M\), a set of vector replicates \( Y_1^M, Y_2^M, \ldots, Y_{N_{REP}}^M \) will be obtained. Estimates of the scalar \( \mu \) and of the matrix \( V_M \) will then be calculated. This process will then be repeated \( NA \) times, each time with a newly generated set of observed vectors. The generations sampled will be indexed by \( k \), \( k = 1, 2, \ldots, NA \).

The computer program developed for this study allows complete flexibility in choice of number of generations (\( NA \)) for each combination of correlation coefficients, denoted \((RY(I), I = 1, IR)\), selected rows of \( V_Y \) generated variance structures \((LLY(J), J = 1, ILL)\), number of different dimensions for population \( M \), denoted by \((NDY(K), K = 1, IND)\), and the number of replicates sampled from a selected population \((NREPY(L), L = 1, IREP)\). For each \( I, J, K, \) and \( L \) combination input, values for the above are denoted \( R = RY(I), LL = LLY(J), ND = NDY(K) \) and \( NREP = NREPY(L) \). The number of parameters (\( NP \)) is equal to one. For each combination \( R, LL, \)
and ND, \( k^{th} \) generated values for the basic set of estimators of the parameter \( \mu \) are computed as

\begin{align*}
T(k,1) &= (X'X)^{-1}X'\tilde{y}_k^k, \\
T(k,2) &= (X'V^{-1}X)^{-1}X'V^{-1}\tilde{y}_k^k, \\
T(k,3) &= (X'S_D^{-1}X)^{-1}X'S_D^{-1}\tilde{y}_k^k, \\
T(k,4) &= (X'S^{-1}X)^{-1}X'S^{-1}\tilde{y}_k^k,
\end{align*}

where \( S \) and \( S_D \), respectively, are computed as

\begin{align*}
S &= \sum_{i=1}^{NREP} (y_i^k - \bar{y}_k^k)(y_i^k - \bar{y}_k^k)'/(NREP - 1), \\
S_D &= \text{matrix of diagonal elements of } S \\
&= \text{Diag}(s_{11}, s_{22}, \ldots, s_{ND,ND}),
\end{align*}

and the mean \( \bar{y}_k^k \) for the \( k^{th} \) generation is computed as

\begin{align*}
\bar{y}_k^k &= \frac{NREP}{\sum_{i=1}^{NREP} y_i^k/NREP}.
\end{align*}

An inverse for the symmetric (\( ND \times ND \)) matrix \( S \) is found by the technique known as Bordering for cases where \( NREP > ND \). If any minor determinant of \( S \) of \( |S| \) itself is found to be zero, default values of zero are set for \( S^{-1} \) and for all quantities
in which $S^{-1}$ appears. The problem of a singular matrix does not occur for the diagonal $(ND \times ND)$ matrix $S_D$ except for a very small probability that one or more of the generated $s_{ii}$'s is zero.

For each combination $R$, $LL$, and $ND$, population variances are computed in Array TV as

\begin{align*}
TV(1) &= (X'X)^{-1}(X'VX)(X'X)^{-1}, \\
TV(2) &= (X'V^{-1}X)^{-1}, \\
TV(3) &= (1+\alpha)(X'V^{-1}X)^{-1}.
\end{align*}

A set of estimated values for $TV(I)$, $I = 1, 2, 3$, is computed utilizing matrices $S$ and $S_D$ of (6.10). An overall average scalar variance computed across both dimension (ND) and replication (NREP) is denoted

\begin{equation}
VARY = \sum_{i=1}^{NREP} \frac{\sum_{k} y_{ik}^k y_{ik}^k - NDR(GM)^2/(NDR - 1),}
\end{equation}

where

\begin{align*}
NDR &= (ND)(NREP), \\
GM &= \frac{\sum_{k} y_{ik}^k j_{ik}^k}{ND}.
\end{align*}

The $k^{th}$ generation of selected estimates for TV is computed in Array ST as
\begin{align*}
(6.14) & \\
& ST(k,1) = (X'X)^{-1}\text{VARY/NREP}, \\
& ST(k,2) = (X'X)^{-1}(X'SX)(X'X)^{-1}/\text{NREP}, \\
& ST(k,3) = (X'S^{-1}X)^{-1}/\text{NREP}, \\
& ST(k,4) = (1+\alpha)(X'S^{-1}X)^{-1}/\text{NREP}, \\
& ST(k,5) = (X'X)^{-1}(X'S_DX)(X'X)^{-1}/\text{NREP}, \\
& ST(k,6) = (X'S_D^{-1}X)^{-1}/\text{NREP}, \\
& ST(k,7) = (1+\alpha)(X'S_D^{-1}X)^{-1}/\text{NREP},
\end{align*}

The scalar \((1+\alpha)\) is defined by (3.70) and is computed for given values of \(ND, NP,\) and \(NREP\) as

\begin{align*}
(6.15) & \\
1 + \alpha(ND, NP, NREP) &= 1 + \frac{ND - NP}{NREP - (ND - NP) - 2} \\
& \text{if } ND - NP \geq 1 \text{ and } NREP > ND - NP - 2 \\
& = 0, \text{ otherwise.}
\end{align*}

Variance-covariance matrices, denoted \(XVI\) and \(XV2\), are computed for \(ST\) and \(T\), respectively, in lower triangular form and are expressed as

\begin{align*}
(6.16) & \\
XVI &= (ST)'(ST) - CT_1, \\
XV2 &= T'T - CT_2,
\end{align*}

where \(CT_1\) and \(CT_2\) are correction terms for the mean of \(ST\) and \(T\), respectively. By placing a one in array positions
T(k,5) and ST(k,8) for each k and by letting the accumulations in (6.16) include these two columns, overall means across NA generation for each of the selected variances and for each of the selected estimates of $\mu$ appear in row 8 and row 5 of XVI and XV2, respectively. The set of average variances provides estimates for unconditional variances of estimators of the type (6.9). The diagonal elements of XV2, on the other hand, are the actual variance of estimators (6.12), as computed across the set of NA generations under the selected population conditions. Sample correlation matrices, denoted VU and VW, are computed from XVI and XV2, respectively, to provide a measure of how closely the estimated means vary together and also how closely the estimated variances vary together. Finally, each column of ST and T are ordered and their relative distributions, checked.

C. Numerical Comparisons

The main criterion used in this study for comparing the effectiveness of selected estimation procedures is the relative increase or decrease for each of the generated variances to a fixed standard, calculated from the selected population under study. The population mean $\mu$ was set to zero for the entire study. A numerical estimate for the unconditional variance for each of the selected estimates of $\mu$ is obtained by averaging the total number of generations of
individual variance estimates. These average estimators of variances are compared to variances of estimators of $\mu$ obtained from the entire set of generations. An initial standard for population variance comparisons is computed

$$C_1 = \frac{TV(1)}{TV(2)}, \quad C_2 = \frac{TV(3)}{TV(2)},$$

where $C_1 > 1$ indicates the increase of the variance of the least squares relative to the variance of the best linear unbiased estimator and $C_2 = (1 + \alpha)$ is a scalar multiple indicating the increase in variance over the best linear unbiased estimator due to using the selected set of random weights in combining information. Values for $C_2$ covering the range of $ND$, $NP$, and $NREP$ included in the study are presented in Table 6.2. Values for $C_2$ are not computed for combinations of $NP$, $ND$, and $NREP$ such that $ND - NP < 1$ and/or $NREP - (ND - NP) < 2$, as indicated by (6.15).

Table 6.2

| $C_2$ ($NP = 1$, $ND$, $NREP$), Dimension ($ND$) |
|----------|----------|----------|----------|
| NREP   | 2        | 4        | 8        |
| 3       | -        | -        | -        |
| 5       | 1.500    | -        | -        |
| 7       | 1.250    | 2.500    | -        |
| 11      | 1.125    | 1.500    | 4.500    |
| 15      | 1.084    | 1.300    | 2.167    |
| 21      | 1.055    | 1.187    | 1.583    |
Ratios, defined by

\[ \eta_j = \frac{\sum_{k=1}^{\text{NA}} ST(k,j)}{(\text{NA})TV(2)}, \quad j = 1, 2, \ldots, 7 \]

and

\[ \gamma_\ell = \frac{\sum_{k=1}^{\text{NA}} [T(k,\ell) - \bar{T}(\ell)]^2}{(\text{NA} - 1)TV(2)}, \quad \ell = 1, 2, 3, 4, \]

for \( \text{NA} = 100 \), \( \text{NP} = 1 \), and selected combinations of \( R \), \( LM \), \( ND \), and \( \text{NREP} \), are presented in Table 6.3. The number of generations (\( \text{NA} \)) set for this study at 100 is sufficiently large to provide very good numerical results for the range of parameters under study. Quantities \( T \), \( TV \), and \( ST \), used in (6.18), are defined by (6.9), (6.12), and (6.14), respectively.

While the main objective of this study is to examine and compare variances, it is noteworthy to mention that the observed averages of \( T \) over the \( \text{NA} \) generations indicated that each of the four selected estimators is unbiased for the population mean \( \mu = 0 \). The largest deviations from zero were observed in the study to be for least squares estimates \( T_1 \).

For homogeneous cases (\( LL = 1 \)) estimators \( T_3 \) and \( T_4 \) using random weights \( S_D \) and \( S \), respectively, had slightly larger variances than the simple least squares or best linear unbiased estimators \( T_1 \) and \( T_2 \), respectively, indicated by the relative magnitudes of \( \gamma_3 \) and \( \gamma_4 \) over \( \gamma_1 \) and \( \gamma_2 \), respectively, in
Table 6.3. The least squares estimator suffered greatly in losses of precision compared to estimators $T_3$ and $T_4$ in heterogeneous cases ($LL = 5, 10$) for both uncorrelated ($R = 0.0$) and correlated ($R = 0.5$) population variance-covariance structures, again indicated by $\hat{\gamma}_1$ over $\hat{\gamma}_3$ and $\hat{\gamma}_4$. Values for $\hat{\gamma}_2$ were as small or smaller for all cases studied than were values for $\hat{\gamma}_1, \hat{\gamma}_3,$ or $\hat{\gamma}_4$. This, of course, was expected to be the performance of the best linear unbiased estimator $T_2$. The variance of estimator $T_4$ using $S$ as the estimated variance-covariance matrix generally exceeded the variance of $T_3$ using the diagonal matrix $S_D$ as random weights for all cases of diagonal ($R = 0.0$) population variance-covariance structures, while the reverse was true for non-diagonal ($R = 0.5$) population structures.

The average of the variances over NA generations for $ST_1$, defined by (6.14), calculated

$$\overline{ST}_i = \frac{\sum_{k=1}^{NA} ST(k,i)}{NA}, i = 1, 2, \ldots, 7,$$

would appear to be an approximation for the unconditional variance of selected estimators. Quantities $\overline{ST}_4$ and $\overline{ST}_7$ can be studied as an estimator of the expected value over weights $S$ and $S_D$, respectively, of the variance of estimator types $T_4$ and $T_3$, respectively. Comparisons are made between average variance $\overline{ST}_4$ and $\overline{ST}_7$ to variances, denoted $\text{Var}(T_4)$ and
Var(\hat{T}_3), respectively, to determine the effectiveness of the scalar parameter C_2 (NP, ND, NREP) in estimating the increase in variance due to using estimated weights, where

\[(6.20) \quad \text{Var}(\hat{T}_i) = \frac{\sum_{k=1}^{NA} [T(k,i) - \overline{T}(i)]^2}{(NA-1)}, \quad i = 1, 2, 3, 4.\]

It can readily be ascertained by observing relative magnitude of values of \(\hat{\eta}_4\) to \(\hat{\gamma}_2\) in Table 6.3 that \(\overline{ST}_4\) approaches \(\text{Var}(\hat{T}_2)\), which in turn approaches \(TV(2) = (X'V^{-1}X)^{-1}\) as NREP increases. This result occurred for almost all combinations of R and ND under study. On the other hand, \(\overline{ST}_7\), utilizing random weight S_D, is considerably less than \(\text{Var}(\hat{T}_3)\) for nondiagonal (R = 0.5) population covariance structures, while the reverse is true for diagonal (R = 0.0) structures. While the scalar parameter C_2 was not derived for specific use when diagonal weight S_D was utilized, results indicate that C_2 does provide a good approximation for the increase in variance due to using estimated weights of Type S_D.
Table 6.3

\( R = 0.0, \ LL = 1, \ ND = 2, \ C_1 = 1 \)

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<th>d.f.</th>
<th>( \hat{\eta}_1 )</th>
<th>( \hat{\eta}_2 )</th>
<th>( \hat{\eta}_3 )</th>
<th>( \hat{\eta}_4 )</th>
<th>( \hat{\eta}_5 )</th>
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\( R = 0.0, \ LL = 1, \ ND = 4, \ C_1 = 1 \)

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Table 6.3 (Continued)

\[ R = 0.0, \ LL = 5, \ ND = 2, \ C_1 = 4.765 \]

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Table 6.3 (Continued)

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Table 6.3 (Continued)

\[ R = 0.5, \, LL = 10, \, ND = 4, \, C_1 = 574.5 \]

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It can readily be observed from $\hat{\eta}_1$, $\hat{\eta}_2$, $\hat{\eta}_5$, and $\hat{\gamma}_1$ of Table 6.3 that the average variance of unweighted least squares estimators of Type $T_1$ increases rapidly as the dimension and/or heterogeneity increases in magnitude so that large gains in estimating efficiency due to using random weights in combining information are possible.

For essentially all population cases studied, $\hat{\eta}_3$ and $\hat{\eta}_6$ of Table 6.3 indicate clearly the gross underestimation of variances realized whenever estimate $S$ or $S_D$ is substituted in the variance formula $(X'V^{-1}X)^{-1}$ disregarding the additional variance caused by using estimated weights. However, both $\hat{\eta}_3$ and $\hat{\eta}_6$ approach unity as NREP increases, as could be expected since greater precision is used in estimating the set of random weights. One of the questions that may be raised is how many replicates one needs to estimate weights of Type $S$ or $S_D$. For cases utilizing $S$, a minimal number of replicates that can be used must exceed the dimension $ND$. Otherwise, $S$ frequently will be singular or nearly singular. In the latter case, large numerical errors may be introduced in estimators due to the selected inversion routine. For both $S$ and $S_D$, the magnitude of heterogeneity of the population structure under study determines in part the potential gain in efficiency utilizing random weights over possible alternative estimators such as simple least squares.
Table 6.4

\( R = 0.0, LL = 1, ND = 2, C_1 = 1 \)

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\( R = 0.0, LL = 1, ND = 4, C_1 = 1 \)

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\[ R = 0.0, \ LL = 10, ND = 8, C_1 = 225.3 \]

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Table 6.4 (Continued)  
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\[ R = 0.5, \text{ LL } = 5, \text{ ND } = 8, C_1 = 39.46 \]

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\[ R = 0.5, \text{ LL } = 10, \text{ ND } = 2, C_1 = 171.3 \]

<table>
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Table 6.4 (Continued)

$R = 0.5$, $LL = 10$, $ND = 4$, $C_1 = 574.5$

<table>
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<td>0.801</td>
<td>0.731</td>
<td>0.616</td>
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<tr>
<td>10</td>
<td>0.065</td>
<td>0.700</td>
<td>0.051</td>
<td>0.754</td>
<td>0.825</td>
<td>0.616</td>
</tr>
</tbody>
</table>
Results of simple correlations between estimators $T_1$, $T_2$, $T_3$, and $T_4$, calculated as

$$
\hat{\rho}_{ij} = \frac{\sum_{k=1}^{NA} [T(k,i) - \overline{T}(i)][T(k,j) - \overline{T}(j)]}{(NA - 1)\sqrt{\text{Var}(\hat{T}_i) \cdot \text{Var}(\hat{T}_j)}}, \quad j \ i = 1, 2, 3, 4,
$$

are tabulated in Table 6.4. Estimators $T_1$ and $T_2$ are highly correlated for diagonal ($R = 0.0$) cases, but this correlation decreases rapidly for nondiagonal ($R = 0.5$) cases as dimension (ND) and heterogeneity (LL) increase. Correlations between $T_3$ and $T_4$, utilizing random weights $S_D$ and $S$, respectively, increase rapidly toward unity as NREP increases for diagonal populations ($R = 0.0$). For nondiagonal ($R = 0.5$) cases, correlations are somewhat smaller, indicating the randomness of the weights used.
In the course of this dissertation, we have examined in great detail, both mathematically and numerically, the problem of combining information under a wide variety of population model assumptions. Numerical investigations were conducted to study, and for some cases to verify, relative effectiveness of various proposed linear and nonlinear estimation techniques. This research illustrates the potential of a high-speed digital computer in solving problems that have previously been extremely difficult or unwieldy to solve with exact mathematical formulation. Techniques used here are not limited to cases presented in this study but may be readily applied to other experimental designs, models, sampling, and distributional conditions that may be proposed to simulate a particular physical situation.

For our study, we consider basically three potential estimators of a vector parameter $\tau(p \times 1)$ from a general linear model $Y_j = X\tau + \epsilon_j$, where $Y_j$ is an $(n \times 1)$ vector of observations, $X$ is an $(n \times p)$ matrix of known coefficients with rank $p \leq n$, $\tau$ is a $(p \times 1)$ vector of unknown parameters, and $\epsilon_j$ is an $(n \times 1)$ vector of residuals such that $E\epsilon_j = 0$ and $E\epsilon_j\epsilon_j' = V$, $j = 1, 2, \ldots, r$. The three basic estimators of the parameter vector $\tau$ are written as
(7.1) Simple least squares estimator:
\[
\hat{\tau} = (X'X)^{-1}X'Y,
\]

(7.2) Weighted least squares estimator with \( V \) known:
\[
\hat{\tau}_W = (X'V^{-1}X)^{-1}X'V^{-1}Y,
\]

(7.3) Weighted least squares estimator with \( V \) unknown, but an unbiased estimate of \( V \), denoted \( S \), is available:
\[
\hat{\tau}_W = (X'S^{-1}X)^{-1}X'S^{-1}Y,
\]

where \( Y = \frac{1}{r} \sum_{j=1}^{r} Y_j \) and \( Y_j = (y_{ij}) \) has \( i \) as a running subscript.

The matrix \( S \) takes the form of the maximum likelihood estimate of \( V \) corrected for bias under normality where \( V \) is positive definite, symmetric, and nonsingular. Studies were conducted for cases where \( S \) is calculated as

(7.4) \[
S = \sum_{j=1}^{r} (Y_j - \overline{Y})(Y_j - \overline{Y})'/(r-1),
\]

(7.5) \[
S = \text{Diag}(s_{11}, s_{22}, \ldots, s_{nn}).
\]

where \( s_{ii} = \frac{1}{r} \sum_{j=1}^{r} (y_{ij} - \overline{y}_i)^2/(r-1) \), \( i = 1, 2, \ldots, n \). The minimum number of replicates of observation vector \( Y \) needed to allow calculation of estimator (7.3) utilizing the
variance-covariance estimators (7.5) and (7.4) is two. In studies conducted utilizing a Bordering technique of inversion, however, almost all of the estimates (7.4) calculated from \( r \leq n \) random generated vectors of \( Y \) drawn under a variety of population conditions were found to be singular or nearly singular, thus introducing numerical errors in estimator type (7.3). The need for several replicates is a big disadvantage for using estimator of type (7.3) over alternative estimators of type (7.1), which requires only one replicate. For cases of homogeneous diagonal population variance-covariance structures and for cases of \( V \) where it can be shown that the simple least squares estimators are, in fact, best linear unbiased, then estimators of type (7.3) should not be used. Otherwise, the variance of the estimator will be increased needlessly due to random variation of weights. The true weights \( 1/n \) should be used for homogeneous cases. Distributional formulations for estimators of type (7.3) are difficult to express in exact mathematical terms. At least the first moment is needed to evaluate the unconditional variance expressed in terms of the conditional expectation

\[
(7.6) \quad \text{Var}(\hat{\tau}_w) = E_w[\text{Var}(\hat{\tau}_w/\hat{w})].
\]

A first approximation for an estimate of (7.6), which in effect ignores the contribution to the overall variance due
to variation of \( \hat{w} \), is obtained by substituting \( S \) for \( V \) in \( \text{Var}(\hat{\tau}_w^\wedge) \) and is written as

\[
(7.7) \quad \text{Var}(\hat{\tau}_w^\wedge) \approx \text{Var}(\hat{\tau}_w).
\]

Studies conducted in this research indicate that the unconditional variance is grossly underestimated by expression (7.7). A scalar multiple \( \beta > 1 \) was derived directly by evaluating (7.6) under conditions of positive definite, symmetric, nonsingular variance-covariance for a normally distributed population. Under these conditions, distributions for Wishart, chi-square, and Snedecor F are utilized in evaluating moments for expressions involving estimated dispersion ratios of quadratic forms.

Combinability of information by random weighting was discussed for a wide class of incomplete block experimental designs. A working definition was given for this class of designs, around which algorithms were evaluated and discussed for implementation on a high-speed computer to solve general estimation and analysis problems for both single and combined experiments. Two methods were discussed for obtaining design association parameters and for checking the set of conditions of the given definition to determine the feasibility of utilizing the ensuing algorithms for estimation and analysis of variance. It is felt that the implemented algorithms
developed in the course of this research offer greater efficiency and accuracy for a wide class of incomplete block designs than alternative approaches used in the analysis of general nonorthogonal experimental designs.

Mathematical formulations for the variance of several estimators of a single parameter \( \mu \) were given in Chapter IV. A detailed development for finding the variance of a weighted estimator by the method of integration by parts was presented for cases of normally distributed heterogeneous populations. A table of values for approximate variance expressions of a combined mean from two sources developed by the methods of steepest descent, Taylor's expansion, and integration by parts was presented for a wide range of sample sizes and population values.

The general problem of combining information was studied numerically under a variety of simulated conditions, and results are presented in Chapter VI. Comparisons were made between estimators of types (7.1), (7.2), and (7.3), their variances, and between several proposed techniques for estimating variances. Criteria for studying the effectiveness of estimating techniques are presented. Applications are not limited to the present study. Many physical situations for which data are too expensive or cannot be collected can be studied similarly for almost any distributional population conditions.
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Zyskind, G. and F. B. Martin  
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