AN IMPROVED PUBLIC KEY CRYPTOSYSTEM BASED ON CHEBYSHEV CHAOTIC MAP OVER FINITE FIELD

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Abstract: Public key cryptosystem based on a chaotic map over finite field, depends on what is called Chebyshev map, that has a semi-group property and ability to provide a digital signature. The cryptosystem suggested by Ljupco Kocarev has a weakness in its security since it suffers from the state of repetition, resulting into more than one decryption keys (Multiple keys problem).

This paper suggests an improved public key cryptosystem which does not suffer from the repetition, so as to complicate the process of breaking the system since there is only one key that can decrypt the system. This is done by adding a large prime number to the system. Computer simulation results show that the proposed algorithm succeeds to remove the repetition problem completely.

Keywords: *Public Key, Cryptosystem, Chaotic Map, multi-key problem.*

1. Introduction

The first ideas of security of communication was presented by (C. E. Shannon) [1, 2]. (W. Diffie and M.E. Hellman) showed for the first time the ability to transfer information in a secret manner without the need to transfer the secret key [3].

Among the most prominent researchers who used chaos in encryption was Ljupco Kocarev [4]. In 2003, Ljupco Kocarev published a paper [5] which suggests the use of chaos in public

encryption based on Chebyshev map, in which he used real numbers $[-1, 1]$, his idea was then analyzed by Bergamo et al [6]. Then Kocarev repeated the same idea, but using chaotic map over finite field [7]. In 2010 Xiaofeng Liao, Fei Chen, and Kwok- Wo Wong provided an attack on the public key algorithm based on the polynomial Chebyshev over Finite Field Z_N . They showed that the sequences resulting from Chebyshev polynomials on the Z_N boundary domain have somehow cyclic distribution [8]. In 2013, Jinhui Sun, Geng Zhao and Xufei Li used the multiply K_i factor method to improve the Kocarev algorithm, where K_i is secret, and only shared with participants, to overcome the encrypted text attack only as they indicated [9].

Some researchers use some known traditional functions with chaotic systems to avoid the problem by proposing hybrid systems. In 2018 (Nidal Tahat and Emad Abdullah) used the hash functions in addition to Chebyshev map [10].

In this paper, a new way to solve multiple keys problem is suggested by a modification to Kocarev algorithm.

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The paper is arranged as follows: The second section gives basic information about Chebyshev map and properties. The third section explains the Kocarev algorithm to show repetition problem with a numerical example. Section four introduces the proposed algorithm with the conditions of choosing its parameter and an analysis of the performance of it. Section five gives comparisons between the traditional Kocarev algorithm and the proposed algorithm. Section six gives some notes on the proposed method. Finally in section seven, a conclusion is given.

2. PRELIMINARIE

In this section, a Chebyshev map and some of its important characteristics are briefly explained.

2.1. Chebyshev Map

Chebyshev polynomial map is defined as:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad (1)$$

Where (n) is the order for the polynomial and

(x) is the variable, with initials

$T_0(x) = 1$ and $T_1(x) = x$, some next orders are:

$$\text{for } n = 2, \quad T_2(x) = 2x^2 - 1$$

$$\text{for } n = 3, \quad T_3(x) = 4x^3 - 3x \quad \text{and so on}$$

Some Chebyshev polynomial map properties

1. Semigroup property i.e.

$$T_r(T_s(x)) = T_{rs}(x)$$

2. Commute under composition i.e.

$$T_s(T_r(x)) = T_r(T_s(x))$$

for $x \in [-1, 1]$ then $T_n(x) \in [-1, 1]$

& for $x \in \mathbb{Z}$ then $T_n(x) \in \mathbb{Z}$.

2.2. Modular Chebyshev Polynomials Map

In order to deal with integers, a simple modification of the basic map is made by adding modN to handle a specific field where the definition of the modular map becomes: -

$$Y = T_n(x) \bmod N \quad (2)$$

Where x and Y are less than N.

Some Modular Chebyshev Polynomials map properties

1. Semigroup property i.e.

$$T_s(T_r(x) \bmod N) \bmod N = T_r(T_s(x) \bmod N) \bmod N$$

2. Commute under composition i.e.

$$T_s(T_r(x) \bmod N) \bmod N = T_{sr}(x) \bmod N$$

3. In this case $Y=1$

a) If order of polynomial is zero.

b) If $(x=1)$ or $(x=N-1 \text{ \& } n \text{ is even})$.

4. May be $Y=0$, that depends on values of

(n, x, N) used in algorithm.

3. Traditional algorithm

In this section, the public key cryptosystem based on Chebyshev map over finite field proposed by Ljupco Kocarev, is explained.

3.1. Kocarev's algorithm over finite field

The algorithm has three stages to send a message from Bob to Alice, these are:

i) Keys Generation \ to create the keys, Alice will do the following:

1. Choose a large prime number N and integer number x less than N.

2. Compute $A = T_s(x) \bmod N$ where s is a random integer number less than N .

3. The public key for Alice is (x, N, A) and the private key is s .

ii) Encryption process \ to send the encrypted text, Bob will:

1. Use the public key for Alice (x, N, A)

2. Choose a random integer r less than N .

3. Represent $m \in [0, N-1]$.

4. Calculate $B = T_r(x) \bmod N$ and

$$X = mT_r(A) \bmod N$$

5. Send the cipher text (B, X) to Alice.

iii) Decryption \ to restore the original message, Alice will:

1. use cipher text (B, N) to calculate $T_s(B) \bmod N$.

2. Finally, compute $m = (X/T_s(B) \bmod N) \bmod N$

3.2. Analysis of Kocarev's algorithm and multi-key problem

The cryptosystem based on Chebyshev map is analyzed by the analysis of the map itself. In order to analyze the map, a table is generated for the results produced from the Chebyshev map. Since the value of x is fixed in the algorithm we will compare the results produced from the same value of x together i.e. we will compare the results in each row in the table. Table (1) shows the $T_n(x) \bmod N$ values for $N=19$.

Table 1. Chebyshev Map $T_n(x) \bmod N$ with $N=19$

$n \backslash x$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	1	0	18	0	1	0	18	0	1	0	18	0	1	0	18	0	1	0	18
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	7	7	2	1	2	7	7	2	1	2	7	7	2	1	2	7	7
3	1	3	17	4	7	0	12	15	2	16	18	16	2	15	12	0	7	4	17
4	1	4	12	16	2	0	17	3	7	15	18	15	7	3	17	0	2	16	12
5	1	5	11	10	13	6	9	8	14	18	14	8	9	6	13	10	11	5	1
6	1	6	14	10	11	8	9	5	13	18	13	5	9	8	11	10	14	6	1
7	1	7	2	2	7	1	7	2	2	7	1	7	2	2	7	1	7	2	2
8	1	8	13	10	14	5	9	6	11	18	11	6	9	5	14	10	13	8	1
9	1	9	9	1	9	9	1	9	9	1	9	9	1	9	9	1	9	9	1
10	1	10	9	18	9	10	1	10	9	18	9	10	1	10	9	18	9	10	1
11	1	11	13	9	14	14	9	13	11	1	11	13	9	14	14	9	13	11	1
12	1	12	2	17	7	18	7	17	2	12	1	12	2	17	7	18	7	17	2
13	1	13	14	9	11	11	9	14	13	1	13	14	9	11	11	9	14	13	1
14	1	14	11	9	13	13	9	11	14	1	14	11	9	13	13	9	11	14	1
15	1	15	12	3	2	0	17	16	7	4	18	4	7	16	17	0	2	3	12
16	1	16	17	15	7	0	12	4	2	3	18	3	2	4	12	0	7	15	17
17	1	17	7	12	2	18	2	12	7	17	1	17	7	12	2	18	2	12	7
18	1	18	1	18	1	18	1	18	1	18	1	18	1	18	1	18	1	18	1

For ($x=6$) for example then we will see that for $n= (3,15)$ the result is 10 and for $n= (8,10)$ the result is 13 and so on other repetitions are shown in figure (1).

Figure 1. Chebyshev Map $T_n(x) \bmod N$ repetitions with $N = 19$ & $x=6$

Similarly, for $x= 11$ then we will see that for $n= (1, 8, 10, 17)$ the result is (11) and for $n= (3, 6, 12, 15)$ the result is (9) and so on as shown in figure (2).

Figure 2. Chebyshev Map $T_n(x) \bmod N$ repetition with $N = 19$ & $x=11$

The number of n that produces the same result from the map is related with the choice of x . For example, if $x=6$, then two values of n produce the same result; while for $x=11$ four values of n produce the same results. Therefore, the choice of $x=11$ is worse than the choice of $x=6$.

The important thing is that the repetition problem exists for every values of (x) from $x=0$ to $x=18$ and not only for $N=19$ but for any other values for (N), This repetition will produce multiple keys for the same value of x .

3.3. Encryption/ Decryption example for Kocarev's algorithm

If Bob sends $m = 10$ message to Alice, then the three stages of the algorithm are;

Generating keys \ Suppose Alice chooses $N = 19$, $s = 8$, $x = 11$, and then she calculated $A = T_s(x) \bmod N = T_8(11) \bmod 19 = 11$, the public key for Alice is $(11, 19, 11)$ and the private key is (8) .

Encryption process \ to send the encrypted text, Bob will use the public key for Alice $(11, 19, 11)$ then he Chooses $r=16$, to calculate $B = T_r(x) \bmod N = T_{16}(11) \bmod 19 = 13$ and $X = m \times T_r(A) \bmod N = 10 \times T_{16}(11) \bmod 19 = 10 \times 13 = 130$, finally Bob sends the cipher text to Alice $(B, X) = (13, 130)$.

Decryption \ message recovery process; Alice uses cipher text $(13, 130)$ to calculate $T_s(B) \bmod N = T_8(13) \bmod 19 = 13$, then computes $m = (X / T_s(B) \bmod N) \bmod N = (130 / 13) \bmod 19 = 10 = \text{original message } m$.

Because of the problem of multiple keys, the third party can attack the system easily. Where the value of (s) can be any of the following values $(1, 8, 10, 17)$ and the value of (r) can be any of the following values $(2, 7, 11, 16)$ as shown in figure (2).

4. The proposed improved public key cryptosystem based on Chebyshev map over finite field

4.1. The proposed improved algorithm over finite field

In the improved algorithm, a large prime number ($P > N$) is generated; where N is not necessarily prime number and where $(s, r, m) \in [0, N-1]$. The algorithm has also three stages, these are:

i) Keys Generation \ to create the keys, Alice will do the following:

1. Choose a large integer number N and an appropriate integer number $x < N$.
2. Choose a large prime number $P > N$.
3. Compute $A = T_s(x) \bmod P$, where s is a random integer number less than N .
4. The public key for Alice is (x, N, P, A) and the private key is s .

ii) Encryption process \ to send the encrypted text, Bob will:

1. use the public key for Alice (x, N, P, A) .
2. Choose a random integer r less than N .
3. Represent $m \in [0, N-1]$.
4. Calculate $B = T_r(x) \bmod P$ and $X = m \times T_r(A) \bmod P$.
5. Send the cipher text to Alice (B, X) .

iii) Decryption \ to restore the original message, Alice will:

1. use cipher text (B, X) to calculate $T_s(B) \bmod P$.
2. Finally compute $m = (X / T_s(B) \bmod P) \bmod P$

4.2. How to choose an appropriate value for (P)

It is important to choose an appropriate value for (P) to solve the problem of multiple keys. To achieve that, (P) is chosen as a large prime number and the following conditions are satisfied:

1-The obtained outputs are not similar to each other for a specific value of (x) for all possible values of (n) since the value of (x) is fixed in the algorithm, hence $x \in [2, N-1]$.

2- The obtained outputs are not equal to (1) such that the cipher text is not equal to plain text. This will exclude $n=0$ & $x=1$.

3- The obtained outputs are not equal to (P-1) because the value of $(T_n(P-1) \bmod P = 1)$ for even value of (n) and hence the cipher text equals to plain text ($X = m \times T_r(A) \bmod P = m$).

4- The obtained outputs are not equal to (0) because the value of $(T_n(0) \bmod P)$ equals to one of three output (0,1, P-1) for any value of (n) and hence if the output is (0) the plain text cannot be restored.

The first condition makes the range of possible value of $x \in [2, N-1]$ and the second condition makes the range of possible value of $n \in [1, N-1]$.

As a conclusion from above, the value of (x) must produce (N-1) different outputs and does not include any output equal to (0, 1, and P-1) from the map $(T_n(x) \bmod P)$ for a certain value of (P) and for all orders $n = [1, N-1]$.

For any value of (P) there are certain values of (x) that can be used to satisfy the conditions. The number of (x) values that satisfies the conditions is shown to be related with the ratio (P/N). Where increasing the ratio of (P/N) leading to increase the number of (x) that can meet the required conditions. More values of possible (x) may satisfy the conditions when the ratio (P/N) is large enough.

4.3. Analysis of the performance of the proposed improved algorithm

To analyze the map, the results produced from the modified Chebyshev map is generated in a table since the value of (x) is fixed in algorithm. Table (2) shows the results for $N=19$, $P=41$.

Table 2. Modified Chebyshev Map $T_n(x) \bmod N$ with $N=19$, $P=41$

$\begin{matrix} n \\ X \end{matrix}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	1	0	40	0	1	0	40	0	1	0	40	0	1	0	40	0	1	0	40
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	7	26	15	34	39	40	39	34	15	26	7	2	1	2	7	26	15
3	1	3	17	17	3	1	3	17	17	3	1	3	17	17	3	1	3	17	17
4	1	4	31	39	35	36	7	20	30	15	8	8	15	30	20	7	36	35	39
5	1	5	8	34	4	6	15	21	31	2	30	11	39	10	20	26	35	37	7
6	1	6	30	26	36	37	39	21	8	34	31	10	7	33	20	2	4	5	15
7	1	7	15	39	39	15	7	1	7	15	39	39	15	7	1	7	15	39	39
8	1	8	4	15	31	30	39	20	35	7	36	36	7	35	20	39	30	31	15
9	1	9	38	19	17	0	24	22	3	32	40	32	3	22	24	0	17	19	38
10	1	10	35	34	30	33	15	21	36	2	4	37	39	5	20	26	8	11	7
11	1	11	36	2	8	10	7	21	4	26	35	6	15	37	20	34	31	33	39
12	1	12	0	29	40	29	0	12	1	12	0	29	40	29	0	12	1	12	0
13	1	13	9	16	38	29	19	14	17	18	0	23	24	27	22	12	3	25	32
14	1	14	22	28	24	29	9	18	3	25	0	16	38	23	32	12	17	13	19
15	1	15	39	7	7	39	15	1	15	39	7	7	39	15	1	15	39	7	7
16	1	16	19	18	24	12	32	28	3	27	0	14	38	13	9	29	17	23	22
17	1	17	3	3	17	1	17	3	3	17	1	17	3	3	17	1	17	3	3
18	1	18	32	27	38	29	22	25	17	13	0	28	24	16	19	12	3	14	9

When we notice the outputs from the map for ($x = 6$) in table (2), we get (18) different outputs from the modified Chebyshev map corresponding to $n=1 \rightarrow 18$ as shown in figure (3).

Figure 3. Modified Chebyshev Map $T_n(x) \bmod P$ with $P=41$, $N = 19$, $x=6$

The same thing will be observed for values of $x = \{5, 10, 11\}$, the map produces (18) different outputs. Figure (4) shows the outputs of modified Chebyshev map when $x=11$.

Figure 4. Modified Chebyshev Map $T_n(x) \bmod P$ with $P=41$, $N = 19$, $x=11$

But, if we notice the outputs from modified Chebyshev map for $x = 9$ we get only (10) different outputs i.e. repetition state still exists

and the output includes (0, 40) as shown in Figure (5).

Figure 5. Modified Chebyshev Map $T_n(x) \bmod P$ with $P=41$, $N = 19$, $x=9$

The same thing will be observed for values of $x = \{0, 1, 2, 3, 4, 7, 8, 12, 13, 14, 15, 16, 17, 18\}$ the problem of repetition still exists.

Thus ($T_n(x) \bmod P$) can overcome the problem of repetition (multiple keys) for $x = \{5, 6, 10, 11\}$, but it fails for $x = \{0, 1, 2, 3, 4, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18\}$.

As shown above, choosing the value of x is very important and in order to increase the number of (x) values satisfying the conditions, we increase the ratio (P/N) say ($P/N > 20$). For example, if we choosing $N = 19$, $P = 383$ and computing ($T_n(x) \bmod P$), table (3) is produced.

Table 3. Modified Chebyshev Map $T_n(x) \bmod N$ with $N=19$, $P=383$

$\begin{matrix} n \\ X \end{matrix}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	1	0	382	0	1	0	382	0	1	0	382	0	1	0	382	0	1	0	382
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	7	26	97	362	202	63	50	137	115	323	28	172	277	170	20	293	3
3	1	3	17	99	194	299	68	109	203	343	323	63	55	267	15	206	72	226	135
4	1	4	31	244	6	187	341	243	71	325	231	374	80	266	133	32	123	186	216
5	1	5	49	102	205	33	125	68	172	120	262	202	226	143	55	24	185	294	74
6	1	6	71	80	123	247	160	141	0	242	223	136	260	303	312	377	382	377	312
7	1	7	97	202	50	115	28	277	20	3	22	305	35	185	257	349	33	113	17
8	1	8	127	109	85	102	15	138	278	97	125	371	66	302	170	120	218	304	50
9	1	9	161	208	136	325	352	266	223	301	216	140	6	351	184	280	260	187	42
10	1	10	199	140	303	175	133	187	160	332	352	197	141	325	231	82	260	139	222
11	1	11	241	312	112	237	123	171	192	223	118	75	0	308	265	160	191	212	260
12	1	12	287	365	47	380	264	211	204	89	17	319	362	326	185	284	120	298	138
13	1	13	337	323	18	145	305	125	264	228	302	347	294	20	226	111	362	109	174
14	1	14	8	210	127	282	109	89	85	376	102	182	15	238	138	179	278	328	97
15	1	15	66	50	285	74	20	143	57	35	227	264	33	343	299	201	369	145	151
16	1	16	128	250	212	23	141	276	265	161	291	342	312	67	300	341	271	288	136
17	1	17	194	68	203	323	55	15	72	135	305	277	304	101	66	228	26	273	64
18	1	18	264	294	362	99	138	273	115	37	68	113	170	262	70	343	22	66	56

In table (3), all x values are usable because they meet the required conditions except $x = 6$ and $x=11$ in addition to zero and one that does not satisfy the conditions. For $x=6$ the map produces (16) different outputs and for $x=11$ the map produces (18) different output but one of these output is zero. The outputs for $x=6$ and $x=11$ are shown in figure (6).

Figure 6. Modified Chebyshev Map $T_n(x) \bmod P$ with $P=383$, $N = 19$, $x=6,11$

Consequently, the improved algorithm can overcome the problem of multiple keys, whether by using the first or second ratio, but the difference between them is the number of x

values satisfying the conditions, where in the $((P/N)>20)$ ratio (15) values of (x) satisfy the conditions but, for $((P/N)>2)$ only (4) values of (x) satisfy the conditions.

To clarify the previous relationships between the number of (x) values satisfying the conditions and the ratio (P/N) let us assume that:

- (M) represents the number of (x) values that meet the required conditions.
- (Z) represents the number of all possible values of (x) which is equal to $(N-2)$.
- (Q) Represents the ratio of the number of x values that fulfill the conditions to the number of all possible x value, i.e. $Q = (M/Z) 100\%$.

The values of (Z, P, M, Q) are calculated in the table (4) for the ratio (P/N) greater than (2, 20, 200) and for the range of values of $N=[10, 40]$ to show the effect of the ratio (P/N) on the number of (x) values that meet the condition (M).

Table 4. The values of (Z, P, M, Q) with N= [10,40]

N	Z	P [(P/N)>2]	M [(P/N)>2]	Q % [(P/N)>2]	P [(P/N)>20]	M [(P/N)>20]	Q % [(P/N)>20]	P [(P/N)>200]	M [(P/N)>200]	Q % [(P/N)>200]
10	8	23	1	12.5	211	7	87.5	2003	8	100
11	9	23	1	11.11	223	8	88.88	2203	9	100
12	10	29	1	10	241	6	60	2411	10	100
13	11	29	1	9.09	263	11	100	2609	11	100
14	12	29	1	8.33	281	11	91.66	2801	12	100
15	13	31	2	15.38	307	10	76.92	3001	13	100
16	14	37	4	28.57	331	14	100	3203	14	100
17	15	37	4	26.66	347	14	93.33	3407	15	100
18	16	37	4	25	367	15	93.75	3607	16	100
19	17	41	4	23.52	383	15	88.23	3803	17	100
20	18	41	4	22.22	401	17	94.44	4001	18	100
21	19	43	4	21.05	421	17	89.47	4201	19	100
22	20	47	3	15	443	17	85	4409	20	100
23	21	47	3	14.28	461	18	85.71	4603	21	100
24	22	53	4	18.18	487	21	95.45	4801	21	95.45
25	23	53	5	21.79	503	20	86.95	5003	22	95.65
26	24	53	5	20.83	521	19	79.16	5209	24	100
27	25	59	9	36	541	24	96	5407	24	96
28	26	59	10	38.46	563	26	100	5623	26	100
29	27	59	11	40.74	587	23	85.18	5801	27	100
30	28	61	7	25	601	21	75	6007	28	100
31	29	67	6	20.68	631	28	96.55	6203	29	100
32	30	67	6	20	641	25	83.33	6421	29	96.66
33	31	67	6	19.35	661	27	87.09	6607	30	96.77
34	32	71	4	12.5	683	28	87.5	6803	31	96.87
35	33	71	5	15.15	701	29	87.87	7001	33	100
36	34	73	9	26.47	727	32	94.11	7207	34	100
37	35	79	5	14.28	743	33	94.28	7411	34	97.14
38	36	79	5	13.88	761	33	91.66	7603	36	100
39	37	79	6	16.21	787	37	100	7817	37	100
40	38	83	14	36.84	809	35	92.1	8009	38	100
Average ratio				20.61%					89.26%	99.17%

From table (4) we find that increasing the ratio (P/N) leads to increase the value of (M) and hence the value of (Q) will increase.

The Q results of table (4) are plotted in figure 7.

Figure 7. The relationship between (Q, N)

4.4. Example of the proposed algorithm

Suppose Bob wants to send $m = 10$ message to Alice, where the three stages of the algorithm are:

Generating keys \ Suppose Alice chooses $N = 19$, $P=41$, $s = 8$, $x = 11$, and then calculated $A = T_s(x) \bmod P = T_8(11) \bmod 41 = 4$, the public key for Alice is $(11, 19, 41, 4)$ and private key is (8) .

Encryption process \ to send the encrypted text, Bob will use the public key for Alice is $(11, 19, 41, 4)$ then Choose $r=16$, $m=10$ and Calculate $B = T_r(x) \bmod P = T_{16}(11) \bmod 41 = 31$ and $X = m \times T_r(A) \bmod P = 10 \times T_{16}(4) \bmod 41 = 10 \times 36 = 360$, finally Bob sends the cipher text to Alice $(31, 360)$.

Decryption \ to restore the original message. Alice will use cipher text $(31, 360)$ to calculate $T_s(B) \bmod P = T_8(31) \bmod 41 = 36$, then computes $m = (X/T_s(B) \bmod P) \bmod P = (360/36) \bmod 41 = 10$ equal to original message. There is only one value for (s, r) in order to decrypt the system the third party must know. This makes the attack more difficult.

5. Comparison between the traditional algorithm and the proposed algorithm

In this section, the traditional Kocarev algorithm is compared with the proposed algorithm, where we will use three ratios (P/N) with the proposed algorithm ($(P/N)>2$, $(P/N)>20$, $(P/N)>200$). The comparison is made based on the ratio of the number of x values that fulfill the conditions to the number of all possible x values that can be used, hence:

-Take the values of N as those prime number in the range $5 \rightarrow 97$ since this condition is required in the traditional algorithm. Table (5) shows a comparison between the proposed & traditional algorithms in term of (Z, P, M, Q) .

When observing the above table, the value of (M) of the traditional algorithm equal to zero for all values of (N) . This indicates that the traditional Kocarev algorithm suffers from the problem of multiple keys of all values of (N) .

However, the values of (Q) of the proposed algorithm indicated that the proposed method solves the problem of multiple keys. In the proposed algorithm, the ratio $(P/N)>2$ succeeds to solve the problem of multiple keys for 23.51% of the values of (x) . The ratio $(P/N)>20$ succeeds to solve the problem for 91.91% of the values of (x) . The ratio $(P/N)>200$ succeeds to solve the problem for 99.23% of the values of (x) . The Q values are plotted against N as shown in Figure (8).

Table 5. Comparison between the proposed & traditional algorithms

N	Z	M for traditional algorithm	Q for traditional algorithm	P where $(P/N)>2$	M For proposed algorithm Where $(P/N)>2$	Q For proposed algorithm Where $(P/N)>2$	P where $(P/N)>20$	M For proposed algorithm where $(P/N)>20$	Q For proposed algorithm where $(P/N)>20$	P where $(P/N)>200$	M For proposed algorithm Where $(P/N)>200$	Q For proposed algorithm where $(P/N)>200$
5	3	0	00.00%	11	2	66.66%	101	3	100%	1009	3	100%
7	5	0	00.00%	17	2	40%	149	5	100%	1409	5	100%
11	9	0	00.00%	23	1	11.11%	223	8	88.88%	2203	9	100%
13	11	0	00.00%	29	1	9.09%	263	11	100%	2609	11	100%
17	15	0	00.00%	37	4	26.66%	347	14	93.33%	3407	15	100%
19	17	0	00.00%	41	4	23.52%	383	15	88.23%	3803	17	100%
23	21	0	00.00%	47	3	14.28%	461	18	85.71%	4603	21	100%
29	27	0	00.00%	59	11	40.74%	587	23	85.18%	5801	27	100%
31	29	0	00.00%	67	6	20.68%	631	28	96.55%	6203	29	100%
37	35	0	00.00%	79	5	14.28%	743	33	94.28%	7411	34	97.14%
41	39	0	00.00%	83	14	35.89%	821	38	97.43%	8209	39	100%
43	41	0	00.00%	89	4	9.75%	863	37	90.24%	8609	40	97.56%
47	45	0	00.00%	97	10	22.22%	941	45	100%	9403	45	100%
53	51	0	00.00%	107	17	33.33%	1061	50	98.03%	10601	50	98.03%
59	57	0	00.00%	127	9	15.78%	1181	55	96.49%	11801	57	100%
61	59	0	00.00%	127	9	15.25%	1223	49	83.05%	12203	58	98.30%
67	65	0	00.00%	137	13	20%	1361	57	87.69%	13411	65	100%
71	69	0	00.00%	149	11	15.94%	1423	65	94.2%	14207	66	95.65%
73	71	0	00.00%	149	12	16.9%	1471	61	85.91%	14621	71	100%
79	77	0	00.00%	163	17	22.07%	1583	64	83.11%	15803	77	100%
83	81	0	00.00%	167	15	18.51%	1663	69	85.15%	16603	81	100%
89	87	0	00.00%	179	29	33.33%	1783	82	94.25%	17807	86	98.85%
97	95	0	00.00%	197	14	14.74%	1949	82	86.31%	19403	92	96.84%
Average ratio			00.00%			23.51%			91.91%			99.23%

Figure 8. The relationship between (Q, N)

6. Extra Notes

1-In the Chebyshev map $(T_n(x) \bmod N)$ any value for N, which is Prime can be used in the Kocarev algorithm but this gives multiple keys problem as [8] said "Even if N is properly selected, the algorithm is still insecure from a practical viewpoint".

2- When $(n=0)$ then $(T_n(x) \bmod H) = 1$ for all values of x and that makes cipher text equal to plain text so we don't choose (s) or (r) equal to zero, where H is N or P.

3- When $(x=1)$ then $(T_n(x) \bmod H) = 1$ for all values of n and that makes cipher text equal to plain text so we don't choose $(x=1)$, where H is N or P.

4- When $(x=0)$ then the outputs of map $(T_n(x) \bmod H)$ will be one of these values (0, 1, H-1) of all values of (n) so we don't choose $(x=0)$, where H is N or P.

5- If $(T_n(x) \bmod H) = 0$ for one value of (n) or more we must not choose this value of (n) as (s or r) because:

A) This causes $T_s(T_r(x) \bmod H) \bmod H = T_r(T_s(x) \bmod H) \bmod H = 0$. That makes cipher

text equal to zero and we cannot restore the plain text (in both algorithms), where H is N or P.

B) If the result of $T_s(T_r(x) \bmod H) \bmod H = T_r(T_s(x) \bmod H) \bmod H = 1$ That makes cipher text equal to plain text (in both algorithms), where H is N or P.

6-If $T_n(x) \bmod H = 1$ for one value of (n) or more we must not choose this value of (n) as (s or r), because that causes $T_s(T_r(x) \bmod H) \bmod H = T_r(T_s(x) \bmod H) \bmod H = 1$ That makes cipher text equal to plain text (in both algorithms), where H is N or P.

7-If $T_n(x) \bmod H = H-1$ for one value of (n) or more we must not choose this value of (n) as (s or r), because that may cause $T_s(T_r(x) \bmod H) \bmod H = T_r(T_s(x) \bmod H) \bmod H = 1$. That makes cipher text equals to plain text (in both algorithms), where H is N or P.

7. Conclusions

In this paper, we introduce an improved public key cryptosystem based on Chebyshev map over finite field with ability to overcome the multiple keys problem i.e. ((there is only one key that can decrypt the system)). The proposed algorithm modified Kocarev algorithm by adding a new parameter (P) that must be prime and it's value is greater than N and replacing (modP) instead of (modN) in the proposed algorithm keeping (s, r) less than N. The results show that the proposed algorithm succeeds to remove the repetition problem completely as the ratio (P/N) increases and makes the system have only one key for decryption. This makes it more difficult to attack by third party.

Conflict of interest

The authors confirm that the publication of this article cause no conflict of interest.

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