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NOTES ON SOME SIMPSON TYPE INTEGRAL INEQUALITIES FOR s-GEOMETRICALLY CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In the paper, the authors present several integral inequalities of the Hermite-Hadamard type for s-geometrically convex functions and apply these new integral inequalities to correct several errors appeared in [2].

Keywords: integral inequality, Hermite-Hadamard type, s-geometrically convex function, error, correction, Hölder's integral inequality.

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1. Introduction

The concept of s-geometrically functions was introduced in [6, Definition 1.9].

Definition 1 ([6, Definition 1.9])**.** A function $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}_+$ is said to be an s-geometrically convex function for some $s \in (0,1]$, if the inequality

$$f(x^t y^{1-t}) \le [f(x)]^{t^s} [f(y)]^{(1-t)^s}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Remark 1. Let $s \in (0,1]$ and let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be an s-geometrically convex function.

(1) If $s = 1$, the s-geometrically convex function becomes a geometrically convex function on $\mathbb{R}_+$.

(2) If $s \in (0,1)$, then $f(x) \ge 1$ is valid for all $x \in I$.

In the paper [3], an integral identity was created as follows.

Lemma 1 ([3,Lemma 1])**.** Let $I \subseteq \mathbb{R}$ and let $f: I \to \mathbb{R}$ be differentiable on I° such that $f' \in L_1([a,b])$, where $a, b \in I$ with $a < b$. Then

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$
$$=\frac{b-a}{2}\int_0^1\left(\frac{t}{2}-\frac{1}{3}\right)\left[f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]\mathrm{d}t.$$

In view of Lemma 1, the authors of the paper [2] established the following Hermite-Hadamard type inequalities for s-geometrically convex functions.

Theorem 1 ([2, Theorems 2.2 and 2.4])**.** Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1([a,b])$, where $a, b \in I^\circ$ with $a < b$. If $|f'(x)|^q$ is s-geometrically convex and decreasing on $[a,b]$ for $s \in (0,1]$ and $q \ge 1$, then

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-1/q}\left\{\left[h_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{1/q}+\left[h_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{1/q}\right\}$$
$$\times\begin{cases} |f'(a)f'(b)|^{s/2}, & |f'(a)| \le 1; \\ |f'(a)|^{1-s/2}|f'(b)|^{s/2}, & |f'(b)| \le 1 \le |f'(a)|; \\ |f'(a)f'(b)|^{1-s/2}, & |f'(b)| \ge 1, \end{cases}$$

where $\alpha(u,v) = \dfrac{|f'(b)|^v}{|f'(a)|^u}$ for $u, v > 0$,

$$h_1(\alpha) = \begin{cases} \dfrac{5}{36}, & \alpha = 1; \\ \dfrac{6\alpha^{2/3}+(\alpha-2)\ln\alpha-3\alpha-3}{6(\ln\alpha)^2}, & \alpha \ne 1, \end{cases}$$

and

$$h_2(\alpha) = \begin{cases} \dfrac{5}{36}, & \alpha = 1; \\ \dfrac{6/\alpha^{2/3}+(2-1/\alpha)\ln\alpha-3/\alpha-3}{6(\ln\alpha)^2}, & \alpha \ne 1, \end{cases}$$

Theorem 2 ([2, Theorem 2.3])**.** Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1([a,b])$, where $a,b \in I^\circ$ with $a<b$. If $|f'(x)|^q$ is s-geometrically convex and decreasing on $[a,b]$ for $s \in (0,1]$ and $q>1$ with $\frac{1}{q}+\frac{1}{p}=1$, then

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le \frac{b-a}{2}\left(\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right)^{1/p}\left\{\left[h_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{1/q}+\left[h_4\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{1/q}\right\}$$

$$\times\begin{cases}|f'(a)f'(b)|^{s/2}, & |f'(a)|\le 1;\\ |f'(a)|^{1-s/2}|f'(b)|^{s/2}, & |f'(b)|\le 1\le |f'(a)|,\\ |f'(a)f'(b)|^{1-s/2}, & |f'(b)|\ge 1,\end{cases}$$

where $\alpha(u,v)=\dfrac{|f'(b)|^v}{|f'(a)|^u}$ for $u,v>0$,

$$h_3(\alpha)=\begin{cases}1, & \alpha=1;\\ \dfrac{\alpha-1}{\ln\alpha}, & \alpha\ne 1,\end{cases} \quad \textit{and} \quad h_4(\alpha)=\begin{cases}1, & \alpha=1;\\ \dfrac{1-1/\alpha}{\ln\alpha}, & \alpha\ne 1.\end{cases}$$

Remark 2. Under the conditions of Theorems 1 and 2,

(1) if $q=1$, Theorem 1 is just [2, Theorem 2.2];

(2) if $q>1$, Theorem 1 is equivalent to [2, Theorem 2.4];

(3) for $\alpha>0$, the relations $h_2(a)=h_1\left(\frac{1}{a}\right)$ and $h_4(a)=h_3\left(\frac{1}{a}\right)$ are valid;

We claim that there existed heavy errors and serious mistakes not only in Theorems 1 and 2 but also in other propositions in the paper [2].

In this paper, we will correct, as done in the papers [4, 5], those heavy errors and serious mistakes appeared in Theorems 1 and 2 and other propositions in the paper [2], by establishing several new integral inequalities of the Hermite-Hadamard type for s-geometrically convex functions.

2. Corrected versions of Theorems 1 and 2 in the paper [2]

Now we start out to correct the errors and mistakes in Theorems 1 and 2 by establishing several new integral inequalities of the Hermite-Hadamard type for s-geometrically convex functions.

Theorem 3 (Corrected version of Theorem 1)**.** Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I°, let $a,b \in I^\circ$ with $a<b$, and let $f' \in L_1([a,b])$. If $|f'(x)|^q$ is s-geometrically convex and decreasing on $[a,b]$ for $q \ge 1$ and $s \in (0,1]$, then

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{2}\left[\frac{5}{36}\right]^{1-1/q} \times\left\{\left[h_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{1/q}+\left[h_1\left(\frac{1}{\alpha(sq/2,sq/2)}\right)\right]^{1/q}\right\}|f'(a)f'(b)|^{1-s/2}, \tag{1}$$

where $\alpha(u,v)$ *and* $h_1(\alpha)$ are defined as in Theorem 1.

Proof. From Lemma 1 and Hölder's integral inequality, we obtain

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le \frac{b-a}{2}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|+\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]\mathrm{d}t$$
$$\le \frac{b-a}{2}\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\mathrm{d}t\right)^{1-1/q}\left[\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q\mathrm{d}t\right)^{1/q}\right. \tag{2}$$
$$\left.+\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q\mathrm{d}t\right)^{1/q}\right].$$

Let $0<\mu\le 1\le\eta$ and $0<s,t\le 1$. Then it was deduced in [1, p.4] that

$$\mu^{t^s}\le\mu^{st} \quad\text{and}\quad \eta^{t^s}\le\eta^{st+1-s}. \tag{3}$$

Considering the condition that $|f'|^q$ is decreasing and s-geometrically convex on $[a,b]$ and making use of the inequalities in (3) yield

$$\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q \le \left|f'\left(a^{(1-t)/2}b^{(1+t)/2}\right)\right|^q$$

$$\le |f'(a)|^{q(1-t)^s/2^s}|f'(b)|^{q(1+t)^s/2^s} \le |f'(a)f'(b)|^{q(1-s/2)}\left[\frac{|f'(b)|}{|f'(a)|}\right]^{stq/2}$$

and

$$\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q \le |f'(a)f'(b)|^{q(1-s/2)}\left[\frac{|f'(b)|}{|f'(a)|}\right]^{stq/2}.$$

Similarly or straightforwardly, we acquire

$$\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|dt=\frac{5}{36},$$

$$\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q \mathrm{d}t \le |f'(a)f'(b)|^{q(1-s/2)}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\frac{|f'(b)|}{|f'(a)|}\right]^{stq/2}\mathrm{d}t$$
$$=|f'(a)f'(b)|^{q(1-s/2)}h_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right), \tag{4}$$

$$\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q \mathrm{d}t \le |f'(a)f'(b)|^{q(1-s/2)}h_1\left(\frac{1}{\alpha(sq/2,sq/2)}\right). \tag{5}$$

Substituting the inequalities (4) and (5) into the inequality (2) and simplifying result in the inequality (1). Theorem 3 is thus proved.

Corollary 1. Under the conditions of Theorem 3, if q=1, then

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le\frac{b-a}{2}\left\{h_1\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)+h_1\left(\frac{1}{\alpha(s/2,s/2)}\right)\right\}|f'(a)f'(b)|^{1-s/2},$$

where $\alpha(u,v)$ *and* $h_1(\alpha)$ are defined as in Theorem 1.

Corollary 2. Under the conditions of Theorem 3, if s=1, then

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le\frac{b-a}{2}\left(\frac{5}{36}\right)^{1-1/q}\left\{\left[h_1\left(\alpha\left(\frac{q}{2},\frac{q}{2}\right)\right)\right]^{1/q}+\left[h_1\left(\frac{1}{\alpha(q/2,q/2)}\right)\right]^{1/q}\right\}|f'(a)f'(b)|^{1/2},$$

where $\alpha(u,v)$ *and* $h_1(\alpha)$ are defined as in Theorem 1.

By virtue of the same ideas and approaches as in the proof of Theorem 3, we can find out the following results.

Theorem 4 (Corrected version of Theorem 2). Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1([a,b])$, where $a,b \in I^\circ$ with $a<b$. If $|f'(x)|^q$ is s-geometrically convex and decreasing on $[a,b]$ for $s \in (0,1]$ and $q>1$ with $\frac{1}{q}+\frac{1}{p}=1$, then

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{2}\left[\frac{2\left(1+2^{p+1}\right)}{6^{p+1}(p+1)}\right]^{1/p}$$

$$\times\left\{\left[h_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{1/q}+\left[h_3\left(\frac{1}{\alpha(sq/2,sq/2)}\right)\right]^{1/q}\right\}\left|f'(a)f'(b)\right|^{1-s/2},$$

where the function $\alpha(u,v)$ is defined as in Theorem 1 and the function $h_3(\alpha)$ is defined as in Theorem 2.

Corollary 3. Under the conditions of Theorem 3, if s=1, then

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{2}\left[\frac{2\left(1+2^{p+1}\right)}{6^{p+1}(p+1)}\right]^{1/p}$$

$$\times\left\{\left[h_3\left(\alpha\left(\frac{q}{2},\frac{q}{2}\right)\right)\right]^{1/q}+\left[h_3\left(\frac{1}{\alpha(q/2,q/2)}\right)\right]^{1/q}\right\}\left|f'(a)f'(b)\right|^{1/2},$$

where the function $\alpha(u,v)$ is defined as in Theorem 1 and the function $h_3(\alpha)$ is defined as in Theorem 2.

3. Corrected versions of three propositions in the paper [2]

In this section, we will apply several integral inequalities of the Hermite-Hadamard type for s-geometrically convex functions to construct some inequalities for means.

For two positive numbers $a>0$ and $b>0$, define

$$A(a,b)=\frac{a+b}{2}, \quad G(a,b)=\sqrt{ab},$$

and

$$L_P(a,b)=\begin{cases}\left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & a\neq b, p\neq 0,-1;\\ a, & a=b.\end{cases}$$

These means are respectively called the arithmetic mean, the geometric mean, and the generalized logarithmic mean of two positive number $a>0$ and $b>0$.

Let $f(x)=\dfrac{x^s}{s}$ for $x>0$, $0<s<1$, and $q>0$. Then the function $|f'(x)|=x^{s-1}$ is geometrically convex on $x\in(0,1]$. Since the inequality

$$\left|f'\left(x^t y^{1-t}\right)\right|=\left(x^{s-1}\right)^t\left(y^{s-1}\right)^{1-t}\leq\left(x^{s-1}\right)^{t^s}\left(y^{s-1}\right)^{(1-t)^s}=\left[f'(x)\right]^{t^s}\left[f'(y)\right]^{(1-t)^s} \quad (6)$$

holds for all $x,y\in(0,1]$ and $t\in[0,1]$, so the function $\left|f'(x)\right|^q=x^{(s-1)q}$ is s-geometrically convex in $x\in(0,1]$.

Theorem 5. Let $0<a<b\leq 1$ and $0<s<1$. Then

$$\left|\frac{2A(a^s,b^s)+4[A(a,b)]^s}{6}-[L_s(a,b)]^s\right|$$
$$\leq\frac{(b-a)s}{2}[G(a,b)]^{s-1}[h_1(\beta(a,b))+h_1(\beta(b,a))],$$

where $\beta(a,b)=\left(\dfrac{b}{a}\right)^{(s-1)/2}$ and $h_1(\alpha)$ is defined as in Theorem 1.

Proof. Using Lemma 1, we obtain

$$\begin{aligned}
&\frac{1}{s}\left|\frac{A(a^s,b^s)+2[A(a,b)]^s}{6}-[L_s(a,b)]^s\right| \\
&\le \frac{b-a}{2}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|+\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]\mathrm{d}t \\
&= \frac{b-a}{2}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)^{s-1}+\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)^{s-1}\right]\mathrm{d}t \\
&\le \frac{b-a}{2}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\left(a^{(1-t)/2}b^{(1+t)/2}\right)^{s-1}+\left(a^{(1+t)/2}b^{(1-t)/2}\right)^{s-1}\right]\mathrm{d}t \\
&= \frac{b-a}{2}[G(a,b)]^{s-1}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\left(\frac{a}{b}\right)^{(s-1)t/2}+\left(\frac{b}{a}\right)^{(s-1)t/2}\right]\mathrm{d}t \\
&= \frac{b-a}{2}[G(a,b)]^{s-1}[h_1(\beta(a,b))+h_1(\beta(b,a))].
\end{aligned} \tag{7}$$

Theorem 5 is thus proved.

Corollary 4. Let $0<a<b\le 1$ and $0<s<1$. Then

$$\left|\frac{2A(a^s,b^s)+4[A(a,b)]^s}{6}-[L_s(a,b)]^s\right|\le\frac{5(b-a)s}{36}A(a^{s-1},b^{s-1}).$$

Proof. By the inequality (7) and the geometric-arithmetic mean inequality, we have

$$\begin{aligned}
&\frac{1}{s}\left|\frac{A(a^s,b^s)+2[A(a,b)]^s}{6}-[L_s(a,b)]^s\right| \\
&\le \frac{b-a}{2}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\left(a^{(1-t)/2}b^{(1+t)/2}\right)^{s-1}+\left(a^{(1+t)/2}b^{(1-t)/2}\right)^{s-1}\right]\mathrm{d}t \\
&\le \frac{b-a}{2}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left(\frac{1-t}{2}a^{s-1}+\frac{1+t}{2}b^{s-1}+\frac{1+t}{2}a^{s-1}+\frac{1-t}{2}b^{s-1}\right)\mathrm{d}t \\
&= (b-a)A(a^{s-1},b^{s-1})\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\mathrm{d}t \\
&= \frac{5(b-a)}{36}A(a^{s-1},b^{s-1}).
\end{aligned}$$

Corollary 4 is thus proved.

Under the conditions of Theorem 5, from the inequalities (6) and (7), it follows that

$$\frac{1}{s}\left|\frac{A(a^s,b^s)+2[A(a,b)]^s}{6}-[L_s(a,b)]^s\right|$$
$$\le \frac{b-a}{2}[G(a,b)]^{s-1}[h_1(\beta(a,b))+h_1(\beta(b,a))]$$
$$=\frac{b-a}{2}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\left(a^{(1-t)/2}b^{(1+t)/2}\right)^{s-1}+\left(a^{(1+t)/2}b^{(1-t)/2}\right)^{s-1}\right]\mathrm{d}t$$
$$\le \frac{b-a}{2}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\left(a^{(1-t)^s/2^s}b^{(1+t)^s/2^s}\right)^{s-1}+\left(a^{(1+t)^s/2^s}b^{(1-t)^s/2^s}\right)^{s-1}\right]\mathrm{d}t.$$

Therefore, we can correct [2, Propositions 3.1, 3.2, and 3.2] as follows.

Corollary 5 (Corrected version of [2, Propositions 3.1 and 3.3])**.** Let $0<a<b\le 1$, $0<s<1$, and $q\ge 1$. Then

$$\left|\frac{1}{6}\left[\frac{2A(a^s,b^s)+4[A(a,b)]^s}{s}\right]-\frac{[L_s(a,b)]^s}{s}\right|\le\frac{b-a}{2}\left(\frac{5}{36}\right)^{1-1/q}$$
$$\times G(a^{(s-1)(2-s)},b^{(s-1)(2-s)})\left\{\left[h_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{1/q}+\left[h_1\left(\frac{1}{\alpha(sq/2,sq/2)}\right)\right]^{1/q}\right\},$$

where $h_1(\alpha)$ is defined as in Theorem 1 and $\alpha(u,v)=\dfrac{b^{(s-1)v}}{a^{(s-1)u}}$ for $u,v>0$.

Corollary 6 (Corrected version of [2, Propositions 3.2])**.** Let $0<a<b\le 1$, $0<s<1$, and $q>1$ with $\frac{1}{q}+\frac{1}{p}=1$. Then

$$\left|\frac{1}{6}\left[\frac{2A(a^s,b^s)+4[A(a,b)]^s}{s}\right]-\frac{[L_s(a,b)]^s}{s}\right|\le\frac{b-a}{2}\left(\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right)^{1-1/q}$$
$$\times G(a^{(s-1)(2-s)},b^{(s-1)(2-s)})\left\{\left[h_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{1/q}+\left[h_3\left(\frac{1}{\alpha(sq/2,sq/2)}\right)\right]^{1/q}\right\},$$

where $h_3(\alpha)$ is defined as in Theorem 2 and $\alpha(u,v)=\dfrac{b^{(s-1)v}}{a^{(s-1)u}}$ for $u,v>0$.

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