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Commutators of fractional integral with variable kernel on variable exponent Herz-Morrey spaces

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Abstract: By using the boundedness results for the commutators of the fractional integral with variable kernel on variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, the boundedness results are established on variable exponent Herz–Morrey spaces $M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$.

Keywords: Fractional integral, variable kernel, commutators, variable exponents, Herz-Morrey spaces.

MSC: 42B20, 42B25.

1. Introduction

Muckenhoupt and Wheeden [1] have proved the boundedness of the fractional integral operators $T_{\Omega,\mu}$ with power weights from L^p to L^q . The boundedness of the fractional integral operators were studied by Calderón and Zygmund [2]. Ding, Chen and Fan [3] introduced the properties of $T_{\Omega,\mu}$ on Hardy spaces. The theory of the variable exponent function spaces has been rapidly developed after the work [4] where Kováčik and Rákosník proved fundamental properties of Lebesgue spaces with variable exponent. After that, many researchers work in this direction, see [5–11]. Izuki [7] defined the class of Herz-Morrey spaces with variable exponent and considered the boundedness of the fractional integral on these spaces. In [11] the author's studied the boundedness of the fractional integral with variable kernel on $M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$ spaces.

Let S^{n-1} ($n \geq 2$) be the unit sphere in $\mathbb{R}^n$ with normalized Lebesgue measure $d\sigma(x')$. A function $\Omega(x, z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be in $L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ ($r \geq 1$), if Ω satisfies the following two conditions:

(i) For any $x, z \in \mathbb{R}^n$ and any $\lambda > 0$, one has $\Omega(x, \lambda z) = \Omega(x, z)$;

(ii) $\|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})} := \sup_{x \in \mathbb{R}^n} \left(\int_{S^{n-1}} |\Omega(x, z')|^r d\sigma(z') \right)^{\frac{1}{r}} < \infty$.

For $0 \leq \mu < n$ the fractional integral operator with variable kernel $T_{\Omega,\mu}$ is defined by

$$T_{\Omega,\mu}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} f(y) dy.$$

The commutators of the fractional integral is defined by

$$[b^m, T_{\Omega,\mu}]f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} (b(x) - b(y))^m f(y) dy.$$

Throughout this paper, Let E be a Lebesgue measurable set in $\mathbb{R}^n$ with measure $|E| > 0$, χ_E means its characteristic function. Now, introduce the definition of the variable exponent Lebesgue spaces.

Definition 1. [5] Let $p(\cdot) : E \rightarrow [1, \infty)$ be a measurable function, the Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E) = \{f \text{ is measurable} : \int_E \left(\frac{|f(x)|}{\eta} \right)^{p(x)} dx < \infty \text{ for some constant } \eta > 0\}.$$

The space $L_{loc}^{p(\cdot)}(E)$ is defined by

$$L_{loc}^{p(\cdot)}(E) = \{f \text{ is measurable} : f \in L^{p(\cdot)}(K) \text{ for all compact } K \subset E\}.$$

The Lebesgue spaces $L^{p(\cdot)}(E)$ is a Banach spaces with the norm defined by

$$\|f\|_{L^{p(\cdot)}(E)} = \inf \left\{ \eta > 0 : \int_E \left(\frac{|f(x)|}{\eta} \right)^{p(x)} dx \leq 1 \right\}.$$

We denote $p_- = \text{ess inf } \{p(x) : x \in E\}$, $p_+ = \text{ess sup } \{p(x) : x \in E\}$. Then $\mathcal{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$, $\chi_k = \chi_{C_k}$, $k \in \mathbb{Z}$.

Definition 2. [7] Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 \leq \lambda < \infty$. The homogeneous Herz–Morrey spaces with variable exponent $M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$ is defined by

$$M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n) = \left\{ f \in L_{Loc}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} := \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{K=-\infty}^L 2^{k\alpha q} \|f\chi_k\|_{L^{p(\cdot)}}^q \right\}^{1/q}.$$

2. Properties of variable Lebesgue spaces

In this section, we state some properties of variable exponent Lebesgue spaces.

Proposition 3. Assume that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies the follows inequalities:

$$\begin{aligned} |p(x) - p(y)| &\leq \frac{-C}{\text{Log}(|x - y|)}, & |x - y| &\leq 1/2; \\ |p(x) - p(y)| &\leq \frac{C}{\text{Log}(e + |x|)}, & |y| &\geq |x|; \end{aligned}$$

then, we have $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$.

Lemma 4. [4] Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$, for all function f and g , there exists the fact

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

Lemma 5. [5] Assume that E is a Lebesgue measurable subset of $\mathbb{R}^n$ with positive measure and $p(\cdot) \in \mathcal{P}$, if $f : E \times E \rightarrow \mathbb{R}$ be a measurable function (with respect to product measure) such that for almost every $y \in E$, $f(\cdot, y) \in L^{p(\cdot)}(E)$. Then we have

$$\left\| \int_E f(\cdot, y) dy \right\|_{L^{p(\cdot)}(E)} \leq C \int_E \|f(\cdot, y)\|_{L^{p(\cdot)}(E)} dy.$$

Lemma 6. [12] If $0 < \mu < n$, and $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ ($r \geq 1$) satisfies the L^r –Dini condition. If there exists an $0 < \alpha < 1/2$ such that $|y| < \alpha_0 R$, we have

$$\left(\int_{R < |x| < 2R} \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right|^r dx \right)^{\frac{1}{r}} \leq CR^{\left(\frac{n}{r} - n + \mu\right)} \left(\frac{|y|}{R} + \int_{|y|/2R}^{|y|/R} \frac{\omega_r(\delta)}{\delta} d\delta \right).$$

Lemma 7. [13] If $x \in \mathbb{R}^n$ and defined $\tilde{q}(x)$ by $\frac{1}{\tilde{q}(x)} = \frac{1}{q} + \frac{1}{\tilde{q}(x)}$, for all measurable function f and g , we have

$$\|f(x)g(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|g(x)\|_{L^r(\mathbb{R}^n)} \|f(x)\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}.$$

Lemma 8. [13] Suppose that $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ and $0 < p^- \leq p^+ < \infty$,

- (1) For any cube (or ball) and $|Q| \leq 2^n$, all the $\chi \in Q$, then: $\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |Q|^{1/p(x)}$.
- (2) For any cube (or ball) and $|Q| \geq 1$, then $\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |Q|^{1/p_\infty}$, where $p_\infty = \lim_{x \rightarrow \infty} p(x)$.

Lemma 9. [8] If $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, then there exist constants $C, \delta, \delta_1 > 0$ such that for all balls B in $\mathbb{R}^n$ and all measurable subset $S \subset B$

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|S|}{|B|} \right)^\delta, \quad \frac{\|\chi_S\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|S|}{|B|} \right)^{\delta_1}.$$

Lemma 10. [8] If $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, there exists constant $C > 0$ such that for any balls B in $\mathbb{R}^n$, we have

$$\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 11. [14] Assume that $b^m \in BMO(\mathbb{R}^n)$, $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. Let $0 < \gamma \leq \frac{n}{(p_1)_+}$, $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\gamma}{n}$, then for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, we have

$$\|T_{\Omega, \gamma}^{b^m} f\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{BMO(\mathbb{R}^n)}^m \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

Lemma 12. [15] Assume that $b^m \in Lip_\beta(\mathbb{R}^n)$, $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. Let $0 < \gamma \leq \frac{n}{(p_1)_+}$, $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\gamma+m\beta}{n}$, then for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ we have

$$\|T_{\Omega, \gamma}^{b^m} f\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{Lip_\beta}^m \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

Lemma 13. [10] Let $b^m \in Lip_\beta(\mathbb{R}^n)$; m is a positive integer, and there exist constants $C > 0$, such that for any $k, j \in \mathbb{Z}$ with $k > j$, we have

- (1) $C^{-1} \|b\|_{Lip_\beta}^m \leq |B|^{-m\beta/n} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{Lip_\beta}^m$;
- (2) $\|(b - b_{B_j})^m \chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C |B_k|^{m\beta/n} \|b\|_{Lip_\beta}^m \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$

Lemma 14. [16] Let $b^m \in BMO(\mathbb{R}^n)$; m is a positive integer, and there exist constants $C > 0$, such that for any $k, j \in \mathbb{Z}$ with $k > j$, we have

- (1) $C^{-1} \|b\|_*^m \leq \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_*^m$;
- (2) $\|(b - b_{B_j})^m \chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C (k - j)^m \|b\|_*^m \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$

3. Main theorems and their proofs

In this section, we will prove the main results of this paper.

Theorem 15. Assume that $b^m \in BMO(\mathbb{R}^n)$, $0 < \mu < n$, $0 < \beta \leq 1$, $\lambda < \alpha < n\delta_1 + \beta$, $0 < q_1 \leq q_2 < \infty$. Let $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ ($r > p_2^+$), and the integral modulus of continuity $\omega_r(\delta)$ satisfying

$$\int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\beta}} d\delta < \infty. \quad (1)$$

If $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ such that $0 < \mu \leq \frac{n}{(p_1)_+}$, $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\mu}{n}$. Then for all $f \in \dot{MK}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)$, we have

$$\|T_{\Omega, \mu}^{b^m} f\|_{\dot{MK}_{q_2, p_2(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)} \leq C \|b\|_*^m \|f\|_{\dot{MK}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}.$$

Proof. If $f \in \dot{MK}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)$ arbitrarily, we use the following inequality

$$\left(\sum_{k=1}^{\infty} a_k \right)^q \leq \sum_{k=1}^{\infty} a_k^q \quad \text{such that } (a_1, a_2, \dots) \geq 0,$$

we have

$$\begin{aligned} \|T_{\Omega,\mu}^{b^m} f\|_{MK_{q_2,p_2(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1} &= \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{K=-\infty}^L 2^{K\alpha q_2} \|T_{\Omega,\mu}^{b^m}(f) \chi_K\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_2} \right\}^{q_1/q_2} \\ &\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{K=-\infty}^L 2^{K\alpha q_1} \|T_{\Omega,\mu}^{b^m}(f) \chi_K\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} \right\}. \end{aligned}$$

Let $f(x) = \sum_{k=-\infty}^{\infty} f(x) \chi_k = \sum_{k=-\infty}^{\infty} f_j(x) \chi_k$. Then we have

$$\begin{aligned} \|T_{\Omega,\mu}^{b^m} f\|_{MK_{q_2,p_2(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1} &\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=-\infty}^{\infty} \|T_{\Omega,\mu}^{b^m}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\ &\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=-\infty}^{k-2} \|T_{\Omega,\mu}^{b^m}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\ &\quad + \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k-1}^{\infty} \|T_{\Omega,\mu}^{b^m}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\ &= U_1 + U_2. \end{aligned}$$

First, we consider U_1 . By the vanishing condition of f_j , applying Lemma 5 and Minkowski inequality when $j \leq k-2$ we have

$$\begin{aligned} \|T_{\Omega,\mu}^{b^m}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq \int_{B_j} |f_j(y)| \left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\| |b(x) - b(y)|^m \chi_k \Big\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} dy \\ &\leq \int_{B_j} |f_j(y)| \left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\| |b(x) - b_B|^m \chi_k \Big\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} dy \\ &\quad + \int_{B_j} |b_B - b(y)|^m |f_j(y)| \left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\| \chi_k \Big\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} dy \\ &= U_{11} + U_{12}. \end{aligned}$$

For U_{11} , we define $\frac{1}{p_2(x)} = \frac{1}{r} + \frac{1}{\tilde{p}_2(x)}$, such that $\tilde{p}_2(x) > 1$, by Lemma 7 and Lemma 14, we have

$$\begin{aligned} &\left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\| |b(x) - b_B|^m \chi_k \Big\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq \left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\|_{L^r(\mathbb{R}^n)} \| |b(x) - b_B|^m \chi_k \|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)} \\ &\leq \left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\|_{L^r(\mathbb{R}^n)} \| |b(x) - b_{B_j}|^m \chi_{B_k} \|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)} \\ &\leq \left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\|_{L^r(\mathbb{R}^n)} \| |b(x) - b_{B_j}|^m \chi_{B_k} \|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)} \\ &\leq \left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\|_{L^r(\mathbb{R}^n)} (k-j)^m \|b\|_*^m \| \chi_{B_k} \|_{L^{\tilde{p}_2(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

According to Lemma 8 and the formula $\frac{1}{\tilde{p}_2(x)} = \frac{1}{p_2(x)} - \frac{1}{r}$, we have

$$\| \chi_{B_k} \|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)} \approx \| \chi_{B_k} \|_{L^{p_1(x)}(\mathbb{R}^n)} |B|^{\frac{1}{r} - \frac{\mu}{n}}.$$

Applying Lemma 6, noting that $2^{j-k} \leq 2^{(j-k)\beta}$ we get

$$\begin{aligned}
\left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\|_{L^r(\mathbb{R}^n)} &\leq CR^{\left(\frac{n}{r}-n+\mu\right)} \left(\frac{|y|}{2^{k-1}} + \int_{|y|/2^k}^{|y|/2^{k-1}} \frac{\omega_r(\delta)}{\delta} d\delta \right) \\
&\leq CR^{\left(\frac{n}{r}-n+\mu\right)} \left(2^{j-k} + 2^{(j-k)\beta} \int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\beta}} d\delta \right) \\
&\leq CR^{\left(\frac{n}{r}-n+\mu\right)} 2^{(j-k)\beta} \left(1 + \int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\beta}} d\delta \right) \\
&= C 2^{(k-1)\left(\frac{n}{r}-n+\mu\right)} 2^{(j-k)\beta}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
U_{11} &\leq C(k-j)^m \|b\|_*^m 2^{-kn+(j-k)\beta} \|\chi_{B_k}\|_{L^{p_1(x)}(\mathbb{R}^n)} \int_{B_j} |f_j(y)| dy \\
&\leq C(k-j)^m \|b\|_*^m 2^{-kn+(j-k)\beta} \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For U_{12} , similar to U_{11} , we have

$$\begin{aligned}
\left\| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right| \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq \left\| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right\|_{L^r(\mathbb{R}^n)} \|\chi_k\|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)} \\
&\leq C 2^{-kn+(j-k)\beta} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

By Lemma 14, we obtain

$$\begin{aligned}
U_{12} &\leq C 2^{-kn+(j-k)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|b_{B_j} - b(y)\|^m \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C \|b\|_*^m 2^{-kn+(j-k)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

So,

$$\|T_{\Omega, \mu}^{b^m}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C(k-j)^m \|b\|_*^m 2^{-kn+(j-k)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}.$$

Using Lemma 4, Lemma 9 and Lemma 10, we have

$$\begin{aligned}
\|T_{\Omega, \mu}^{b^m}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq C(k-j)^m \|b\|_*^m 2^{-kn+(k-j)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C(k-j)^m \|b\|_*^m 2^{(j-k)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} 2^{-kn} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C(k-j)^m \|b\|_*^m 2^{(j-k)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}} \\
&\leq C(k-j)^m \|b\|_*^m 2^{(j-k)\beta} 2^{(j-k)n\delta_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C(k-j)^m \|b\|_*^m 2^{(j-k)(\beta+n\delta_1)} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Then we have

$$\begin{aligned}
U_1 &\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=-\infty}^{k-2} (k-j)^m 2^{\alpha k} 2^{(j-k)(\beta+n\delta_1)} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=-\infty}^{k-2} (k-j)^m 2^{\alpha j} 2^{(j-k)(\beta+n\delta_1-\alpha)} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1}
\end{aligned}$$

When $1 < q_1 < \infty$, take $1/q_1 + 1/q'_1 = 1$. Noting that $\alpha < n\delta_1 + \beta$, by the Hölder inequality we obtain

$$\begin{aligned}
 U_1 &\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=-\infty}^{k-2} 2^{\alpha j q_1} 2^{(j-k)(\beta+n\delta_1-\alpha)q_1/2} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \right) \\
 &\quad \times \left(\sum_{j=-\infty}^{k-2} (k-j)^{mq'_1} 2^{(j-k)(\beta+n\delta_1-\alpha)q'_1/2} \right)^{q_1/q'_1} \\
 &\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \sum_{j=-\infty}^{k-2} 2^{\alpha j q_1} 2^{(j-k)(\beta+n\delta_1-\alpha)q_1/2} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
 &\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=j+2}^L 2^{(j-k)(\beta+n\delta_1-\alpha)q_1/2} \\
 &\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
 &\leq C \|b\|_*^{mq_1} \|f\|_{M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{q_1}. \tag{2}
 \end{aligned}$$

When $0 < q_1 \leq 1$, we have

$$\begin{aligned}
 U_1 &\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \sum_{j=-\infty}^{k-2} (k-j)^{mq_1} 2^{\alpha j q_1} 2^{(j-k)(\beta+n\delta_1-\alpha)q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
 &\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=j+2}^L (k-j)^{mq_1} 2^{(j-k)(\beta+n\delta_1-\alpha)q_1} \\
 &\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
 &\leq C \|b\|_*^{mq_1} \|f\|_{M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{q_1}. \tag{3}
 \end{aligned}$$

Finally, we estimate U_2 . By the boundedness of the $T_{\Omega, \mu}^{b^m}$ on $L^{p(\cdot)}(\mathbb{R}^n)$ (Lemma 11), we have

$$\begin{aligned}
 U_2 &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k-1}^{\infty} \|T_{\Omega, \mu}^{b^m}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
 &\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k-1}^{\infty} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
 &\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=k-1}^{\infty} 2^{\alpha j} 2^{(k-j)\alpha} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
 &\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=k-1}^{k+1} 2^{\alpha j} 2^{(k-j)\alpha} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
 &\quad + \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=k+2}^{\infty} 2^{\alpha j} 2^{(k-j)\alpha} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
 &= U_{21} + U_{22}.
 \end{aligned}$$

First, we consider U_{21} , then we have

$$U_{21} \leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=k-1}^{k+1} 2^{\alpha j} 2^{(k-j)\alpha} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1}$$

$$\begin{aligned}
&\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{\alpha k q_1} \|f_k\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq C \|b\|_*^{mq_1} \|f\|_{M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{q_1}.
\end{aligned} \tag{4}$$

For U_{22} , when $\lambda < \alpha$ we have

$$\begin{aligned}
U_{22} &\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=k+2}^{\infty} 2^{\alpha j} 2^{(k-j)\alpha} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} \sum_{k=-\infty}^L 2^{(K-L)\lambda q_1} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha-\lambda)} 2^{-j\lambda} 2^{j\alpha} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} \sum_{k=-\infty}^L 2^{(K-L)\lambda q_1} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha-\lambda)} 2^{-j\lambda} \left\{ \sum_{n=-\infty}^j 2^{n\alpha} \|f_n\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \right\}^{\frac{1}{q_1}} \right)^{q_1} \\
&\leq C \|b\|_*^{mq_1} \sup_{L \in \mathbb{Z}} \sum_{k=-\infty}^L 2^{(K-L)\lambda q_1} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha-\lambda)} \right)^{q_1} \|f\|_{M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{q_1} \\
&\leq C \|b\|_*^{mq_1} \|f\|_{M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{q_1}.
\end{aligned} \tag{5}$$

Thus, by (2)-(5), we finishes the proof of Theorem 15. $\square$

Theorem 16. Assume that $b^m \in Lip_\beta(\mathbb{R}^n)$, $0 < \mu < n$, $0 < \beta \leq 1$, $\lambda < \alpha < n\delta_1 + \beta$, $0 < q_1 \leq q_2 < \infty$. Let $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$ ($r > p_2^+$), and the integral modulus of continuity $\omega_r(\delta)$ satisfying 1. If $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ such that $0 < \mu + m\beta \leq \frac{n}{(p_1)_+}$, $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\mu + m\beta}{n}$. Then for all $f \in M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)$, we have

$$\|T_{\Omega, \mu}^{b^m} f\|_{M\dot{K}_{q_2, p_2(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)} \leq C \|b\|_{Lip_\beta}^m \|f\|_{M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}.$$

Proof. If $f \in M\dot{K}_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)$ arbitrarily, we use the following inequality

$$\left(\sum_{k=1}^{\infty} a_k \right)^q \leq \sum_{k=1}^{\infty} a_k^q \quad \text{such that } (a_1, a_2, \dots) \geq 0,$$

we have

$$\begin{aligned}
\|T_{\Omega, \mu}^{b^m} f\|_{M\dot{K}_{q_2, p_2(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{q_1} &= \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{K=-\infty}^L 2^{K\alpha q_1} \|T_{\Omega, \mu}^{b^m}(f) \chi_K\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_2} \right\}^{q_1/q_2} \\
&\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \left\{ \sum_{K=-\infty}^L 2^{K\alpha q_1} \|T_{\Omega, \mu}^{b^m}(f) \chi_K\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} \right\}.
\end{aligned}$$

Let $f(x) = \sum_{k=-\infty}^{\infty} f(x) \chi_k = \sum_{k=-\infty}^{\infty} f_j(x)$. Then we have

$$\begin{aligned}
\|T_{\Omega, \mu}^{b^m} f\|_{M\dot{K}_{q_2, p_2(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{q_1} &\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=-\infty}^{\infty} \|T_{\Omega, \mu}^{b^m}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=-\infty}^{k-2} \|T_{\Omega, \mu}^{b^m}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\quad + \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k-1}^{\infty} \|T_{\Omega, \mu}^{b^m}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&= A_1 + A_2.
\end{aligned}$$

First, we consider A_1 . By the vanishing condition of f_j and Lemma 5, the Minkowski inequality when $j \leq k-2$ we have

$$\begin{aligned} \|T_{\Omega,\mu}^{b^m}(f_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq \int_{B_j} |f_j(y)| \left\| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right| |b(x) - b(y)|^m \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} dy \\ &\leq \int_{B_j} |f_j(y)| \left\| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right| |b(x) - b_B|^m \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} dy \\ &\quad + \int_{B_j} |b_B - b(y)|^m |f_j(y)| \left\| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right| \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} dy \\ &= A_{11} + A_{12}. \end{aligned}$$

For A_{11} , we define $\frac{1}{p_2(x)} = \frac{1}{r} + \frac{1}{\tilde{p}_2(x)}$ such that $\tilde{p}_2(x) > 1$, by Lemma 7 and Lemma 13, we have

$$\begin{aligned} &\left\| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right| |b(x) - b_B|^m \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq \left\| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right| \right\|_{L^r(\mathbb{R}^n)} \| |b(x) - b_B|^m \chi_k \|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)} \\ &\leq \left\| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right| \right\|_{L^r(\mathbb{R}^n)} \| |b(x) - b_{B_j}|^m \chi_{B_k} \|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)} \\ &\leq \left\| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right| \right\|_{L^r(\mathbb{R}^n)} \| |b(x) - b_{B_j}|^m \chi_{B_k} \|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)} \\ &\leq \left\| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right| \right\|_{L^r(\mathbb{R}^n)} |B_k|^{m\beta/n} \|b\|_{Lip_\beta(\mathbb{R}^n)}^m \| \chi_{B_k} \|_{L^{\tilde{p}_2(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

According to Lemma 8 and the formula $\frac{1}{\tilde{p}_2(x)} = \frac{1}{p_2(x)} - \frac{1}{r}$, we have

$$\| \chi_{B_k} \|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)} \approx \| \chi_{B_k} \|_{L^{p_1(x)}(\mathbb{R}^n)} |B|^{\frac{-1}{r} - \frac{(\mu+m\beta)}{n}}.$$

By Lemma 7 know that

$$\left\| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right| \right\|_{L^r(\mathbb{R}^n)} \leq C 2^{(k-1)(\frac{n}{r}-n+\mu)} 2^{(j-k)\beta}.$$

Thus, we have

$$\begin{aligned} A_{11} &\leq C |B_k|^{m\beta/n} \|b\|_{Lip_\beta(\mathbb{R}^n)}^m 2^{(k-1)(\frac{n}{r}-n+\mu)} 2^{(j-k)\beta} \| \chi_{B_k} \|_{L^{p_1(\cdot)}(\mathbb{R}^n)} |B|^{\frac{-1}{r} - \frac{(\mu+m\beta)}{n}} \int_{B_j} |f_j(y)| dy \\ &\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^m 2^{-kn+(j-k)\beta} \| \chi_{B_k} \|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \int_{B_j} |f_j(y)| dy \\ &\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^m 2^{-kn+(j-k)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \| \chi_{B_j} \|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} \| \chi_{B_k} \|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

For A_{12} , we obtain that

$$\begin{aligned} \left\| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right| \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq \left\| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x, x)}{|x|^{n-\mu}} \right| \right\|_{L^r(\mathbb{R}^n)} \| \chi_k \|_{L^{\tilde{p}_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C |B_k|^{-m\beta/n} 2^{-kn+(j-k)\beta} \| \chi_{B_k} \|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \end{aligned}$$

Then, by Lemma 13, we get

$$A_{12} \leq C |B_k|^{-m\beta/n} 2^{-kn+(j-k)\beta} \| \chi_{B_k} \|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \| |b_{B_j} - b(y)|^m \chi_{B_j} \|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}$$

$$\begin{aligned}
&\leq C \|b\|_{Lip(\mathbb{R}^n)}^m \frac{|B_j|^{m\beta/n}}{|B_k|^{m\beta/n}} 2^{-kn+(j-k)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^m 2^{-kn+(j-k)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

So, we have that

$$\|T_{\Omega,\mu}^{b^m}(f_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C 2^{-kn+(j-k)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}.$$

Using Lemma 4, Lemma 9 and Lemma 10, we have

$$\begin{aligned}
\|T_{\Omega,\mu}^{b^m}(f_j)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^m 2^{-kn+(j-k)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^m 2^{(j-k)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} 2^{-kn} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^m 2^{(j-k)\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}} \\
&\leq \|b\|_{Lip_\beta(\mathbb{R}^n)}^m 2^{(j-k)\beta} 2^{(j-k)n\delta_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\leq \|b\|_{Lip_\beta(\mathbb{R}^n)}^m 2^{(j-k)(\beta+n\delta_1)} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

When $1 < q_1 < \infty$, take $1/q_1 + 1/q'_1 = 1$. Noting that $\alpha < n\delta_1 + \beta$, by the Hölder inequality we have

$$\begin{aligned}
A_1 &\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=-\infty}^{k-2} 2^{\alpha j q_1} 2^{(j-k)(\beta+n\delta_1-\alpha)q_1/2} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \right) \\
&\quad \times \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(\beta+n\delta_1-\alpha)q'_1/2} \right)^{q_1/q'_1} \\
&\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \sum_{j=-\infty}^{k-2} 2^{\alpha j q_1} 2^{(j-k)(\beta+n\delta_1-\alpha)q_1/2} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=j+2}^L 2^{(j-k)(\beta+n\delta_1-\alpha)q_1/2} \\
&\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1}. \tag{6}
\end{aligned}$$

When $0 < q_1 \leq 1$, we have

$$\begin{aligned}
A_1 &\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \sum_{j=-\infty}^{k-2} 2^{\alpha j q_1} 2^{(j-k)(\beta+n\delta_1-\alpha)q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=j+2}^L 2^{(j-k)(\beta+n\delta_1-\alpha)q_1} \\
&\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{j=-\infty}^{L-2} 2^{\alpha j q_1} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1}. \tag{7}
\end{aligned}$$

Finally, we estimate A_2 . By the boundedness of the $T_{\Omega,\mu}^{b^m}$ on $L^{p(\cdot)}(\mathbb{R}^n)$ (Lemma 12), we have

$$\begin{aligned}
A_2 &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k-1}^{\infty} \|T_{\Omega, \mu}^{b^m}(f_j) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L 2^{k\alpha q_1} \left(\sum_{j=k-1}^{\infty} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=k-1}^{\infty} 2^{\alpha j 2^{(k-j)\alpha}} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\leq C \|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=k-1}^{k+1} 2^{\alpha j 2^{(k-j)\alpha}} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
&\quad + \|b\|_{Lip_\beta(\mathbb{R}^n)}^{mq_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^L \left(\sum_{j=k+2}^{\infty} 2^{\alpha j 2^{(k-j)\alpha}} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1}.
\end{aligned}$$

The rest of the proof is the same as the proof of U_2 in Theorem 15, we omit the details there. Then, we can easily see that

$$A_2 \leq C \|b\|_{Lip(\mathbb{R}^n)}^{mq_1} \|f\|_{MK_{q_1, p_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{q_1}. \quad (8)$$

By (6)-(8) the proof of Theorem 16 is complete. $\square$

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