SHRINKAGE ESTIMATION OF $P(X < Y)$

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Abstract: Let X and Y be two random variables having cdf F and G respectively. Based on two independent random samples from F and G respectively, the problem is to estimate $R = P(X < Y)$, when we have a priori information R_0 about R . Various estimators for R are proposed. Performance of the proposed estimators is studied by simulation.

Keywords: Cdf, Estimators

Introduction:

Let X and Y be two independent random variables with cdf F and G . Problem is to estimate $R = P(X < Y)$, when we have apriori information about R in the form of a point R_0 . There are three basic ways to incorporate apriori information while constructing a point estimator. These are namely- Shrinkage technique, Preliminary test technique and Bayesian approach. These techniques have extensively used by researchers in numbers of situations. An extensive bibliography on the topic is due to Bancroft and Han (1977) and Han, Rao and Ravichandran (1988). Some of the references in this area worth to mention are due to Bancroft (1944), Kale and Bancroft (1967), Thompson (1968a, 1968b), Ahmed and Rohatgi (1996).

In the present work we adopt parametric as well as nonparametric approach in estimating R . It is motivated from Baklizi and El-Masri (2004), which is related to shrinkage estimation of R for two independent exponential distributions with common location parameter. The main objective of the paper is to give unified treatment to shrinkage estimation of R in the presence of prior information under mild assumptions. The procedure proposed here can go for number of situations and has been illustrated with the model considered by Baklizi and El-Masri (2004). The rest of the paper is organized as follows. Following

section gives parametric procedure, wherein shrinkage as well as pretest estimation of R is discussed. Use of bootstrap is recommended in both the cases. Pretest estimation for the problem considered by Baklizi and El-Masri (2004) is revisited. Performance of the proposed estimator has been discussed in the section 3. Section 4 is devoted to the nonparametric approach. Various estimators along with their mean square errors have been reported therein. We conclude the paper with major findings.

Parametric Approach

Shrinkage Estimator:

We note that

$$\begin{aligned} R &= P(X < Y) \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^y f_X(x; \underline{\theta}_1) dx \right] f_Y(y; \underline{\theta}_2) dy \\ &= h(\underline{\theta}_1; \underline{\theta}_2) \end{aligned}$$

An expression for R can have all the parameters or it can have only few of the all the parameters.

Let $X_1, X_2, \dots, X_m$ and $Y_1, Y_2, \dots, Y_n$ be two independent samples observed from $F(\cdot)$ and $G(\cdot)$ respectively. Assume that, $\hat{\theta}_1$ and $\hat{\theta}_2$ be the maximum likelihood estimators (mle) of θ_1 and θ_2 respectively. Then $\hat{R}_{ML} = h(\hat{\theta}_1; \hat{\theta}_2)$ will be

the mle of R . Therefore the parametric shrinkage estimator of R is

$$\hat{R}_{PSR} = \nu \hat{R}_{ML} + (1 - \nu) R_0$$

where $0 < \nu < 1$ is to be chosen suitably. One of the ways to find optimum value of ν is, to minimize $MSE(\hat{R}_{PSR})$

$$MSE(\hat{R}_{PSR}) = E(\nu \hat{R}_{ML} + (1 - \nu) R_0 - R)^2$$

It can be shown that

$$\nu = \frac{(R - R_0)E(\hat{R}_{ML} - R_0)}{E(\hat{R}_{ML}^2) - 2R_0E(\hat{R}_{ML}) + R_0^2}$$

However this value depends on R , $E(\hat{R}_{ML})$ and $E(\hat{R}_{ML}^2)$, which are unknown. In most of the cases moments of $\hat{R}_{ML}$ are difficult to compute. After having obtained these moments, we replace unknown parameters involved therein by their respective mles.

To overcome this problem, we propose to use Bootstrap technique to estimate $E(\hat{R}_{ML})$ and $E(\hat{R}_{ML}^2)$. These Bootstrap estimates can be simply plugged in ν so as to get $\hat{\nu}^*$. In the following we outline Preliminary test approach.

Preliminary Test Estimator:

In this technique, we first test the validity of the apriori information using suitable

Theorem (2.1): The MSE of the pretest estimator defined in (2.1) is give by

$$MSE(\hat{R}_{ASPT}) = (\beta^2 - 2\beta)\Phi(a, b, 0, 2, \sigma^2(\hat{R}_{ML})) + 2\beta(1 - \beta)(R_0 - R)\Phi(a, b, 0, 1, \sigma^2(\hat{R}_{ML})) + \beta^2(R_0 - R)^2\Phi(a, b, 0, 0, \sigma^2(\hat{R}_{ML})) + \sigma^2(\hat{R}_{ML})$$

where $\Phi(a, b, k, 0, \sigma^2) = \int_a^b z^k \phi(z; 0, \sigma^2) dz$ and $\sigma^2(\hat{R}_{ML})$ is the asymptotic variance of $\hat{R}_{ML}$.

Lemma (2.1): $\hat{R}_{ASPT}$ is consistent estimator of R provided

$$\sigma^2(\hat{R}_{ML}) \rightarrow 0 \text{ as } N = n_1 n_2 \rightarrow \infty.$$

Again choice of ' β ' is subjective. Traditionally p- Value of the test is used to estimate ' β '. (Please refer to Tse and Tso (1996).) We propose here Bootstrap estimate of ' β ' following Ratnaparakhi, Waikar and Schuurmann (2001) for two-stage testimator

statistical test and based on the result of the test an estimator is suggested. In other words preliminary test estimator is conditional estimator in contrast to the shrinkage estimator.

Consider testing $H_0: R = R_0$ vs. $H_1: R \neq R_0$

the parametric PTE is defined as

$$\hat{R}_{PTE} = \begin{cases} \beta \hat{R}_{ML} + (1 - \beta) R_0 & \text{if } H_0 \text{ is not rejected} \\ \hat{R}_{ML} & \text{if } H_0 \text{ is rejected} \end{cases}$$

where we can use one of the three tests namely-

1. Test based on exact distribution of $\hat{R}$,
2. Likelihood ratio test and
3. Asymptotic test $\left(\frac{\hat{R}_{ML} - E_0(\hat{R}_{ML})}{\sqrt{Var_0(\hat{R}_{ML})}} \right)$, as $\hat{R}_{ML}$

CAN for R with AVAR ($\hat{R}_{ML}$).

Following the third type of test,

$$\hat{R}_{ASPT} = \begin{cases} \beta \hat{R}_{ML} + (1 - \beta) R_0 & \text{if } |Z| \leq Z_{1-\alpha/2} \\ \hat{R}_{ML} & \text{Otherwise.} \end{cases} \quad (2.1)$$

of the normal mean. It is given in the following.

Bootstrap approach to estimate ' β '

Based on $\underline{X}$ and $\underline{Y}$ sample we adopt following approach to estimate ' β '.

1. Draw a bootstrap samples $\underline{X}^*$ and $\underline{Y}^*$ from the samples $\underline{X}$ and $\underline{Y}$ and compute Z^*

2.If $|Z^*| \leq Z_{1-\alpha/2}$, consider the corresponding bootstrap sample for estimating 'w' and compute $\beta^* = \frac{|Z^*|}{Z_{1-\alpha/2}}$ and if $|Z^*| > Z_{1-\alpha/2}$ ignore this bootstrap sample.

3. Repeat (1-2), B (=2000) number of times and estimate β_A^* by taking average of β^* for which $|Z^*| \leq Z_{1-\alpha/2}$. Also compute proportion of times $|Z^*| \leq Z_{1-\alpha/2}$ among the B bootstrap samples. This proportion is regarded as another estimate of β say, β_p^* .

Example:

Ballizi and El-Masri (2004), have considered shrinkage estimation of R based on random samples of size n_1 and n_2 from $E(\theta, \mu)$ and $E(\lambda, \mu)$ respectively.

In this case $R = \frac{\theta}{\theta + \lambda}$ and $\hat{R}_{ML} = \frac{\hat{\theta}}{\hat{\theta} + \hat{\lambda}}$,

where, $\hat{\theta} = \frac{n_1}{\sum (X_i - Z)}$ $\hat{\lambda} = \frac{n_2}{\sum (Y_j - Z)}$

and

$\hat{\mu} = Z = \min\{X_i, Y_j; i = 1, 2, \dots, n_1 \text{ \& } j = 1, 2, \dots, n_2\}$

The pdf of $\hat{R}$ is

$$h_{\hat{R}}(t) = \frac{\theta^{n_1} \lambda^{n_2}}{n_1 \theta + n_2 \lambda} * \frac{\Gamma(n_1 + n_2 - 1)}{\left(\frac{\theta(1-t)}{n_2} + \frac{\lambda t}{n_1}\right)^{n_1+n_2-1}} * \frac{1}{n_1^{n_2-1} n_2^{n_1-1}} * \left(\frac{t^{n_2-2} (1-t)^{n_1-1}}{\Gamma(n_1) \Gamma(n_2 - 1)} + \frac{t^{n_2-1} (1-t)^{n_2-2}}{\Gamma(n_2) \Gamma(n_1 - 1)} \right)$$

, $0 \leq t \leq 1$

Moments of $\hat{R}$ contains Gaussian hypergeometric function. BE (2004) approximated these moments and then plugged mles of θ and λ therein.

Define $U = \frac{n_2 \sum (X_i - Z)}{n_1 \sum (Y_j - Z) + 1}$. Then $\hat{R} = \frac{1}{U} = \Phi(U)$ say.

Using Lindley (1965), we have

$$E(\Phi(U)) \cong \Phi(E(U)) + \frac{Var(U) \Phi''(E(U))}{2} \text{ and}$$

$$Var(\Phi(U)) \cong (\Phi'(E(U)))^2 Var(U)$$

We only need to know $E(U)$ and $Var(U)$

$$E(U) = \left(\frac{n_1 \theta + \frac{n_2 (n_1 + n_2 - 2) \lambda}{n_2 - 1} + \frac{n_2^2 \lambda}{(n_2 - 2) \theta}}{n_1 \theta + n_2 \lambda} \right)$$

$$Var(U) = n_2^2 (n_1 + n_2 - 2) \lambda^2 \left(\frac{(n_1 - 1) n_1 (n_2 - 3) \theta^2 + 2 n_1 (n_2 - 2) (n_2 - 1) n_2 \theta \lambda + (n_2 - 1)^2 n_2^2 \lambda^2}{n_1 (n_2 - 3) (n_2 - 2)^2 (n_2 - 1)^2 \theta^2 (n_1 \theta + n_2 \lambda)^2} \right)$$

Using $E(U)$ and $Var(U)$, under null hypothesis, the desired test can be constructed.

Define $S = \theta T_1 (\theta T_1 + \lambda T_2)^{-1}$ It can be shown that S has mixed Beta distribution with mixing parameter

$\gamma = (1 + n_2 (1 - R) / (n_1 R))^{-1}$. Under null hypothesis, we have

$S_0 = (1 + (1 - R_0) T_2 / R_0 T_1)^{-1}$ has mixed beta distribution with parameter,

$\gamma_0 = (1 + n_2 (1 - R_0) / (n_1 R_0))^{-1}$.

Define $Z = (S_0 - E(S_0)) / \sqrt{V(S_0)}$ we obtain the desired pretest estimator (2.1).

Performance Of The Pretest Estimator

We consider the model studied by Baklizi and El-Masri (2004). A simulation study is conducted to investigate the performance of estimator proposed here. We choose the similar set of parameters as considered by Baklizi and El-Masri (2004). These are as follows:

n_1 = Sample size for the first population 10, 20.

n_2 = Sample size for the second population 10, 20.

$R = .1(.025) .9, R_0 = 0.25, 0.50, 0.75$.

For each combination of n_1, n_2, R and R_0 , we generate 5000 samples from both the populations with $\theta = 1, \lambda = (1/R - 1)$ and $\mu = 0$. For each sample we compute MLE and the proposed estimator and compute MSEs of the both. Efficiency of the proposed estimator is the ratio of MSE of MLE to the MSE of the PTE.

Results are shown in Figure 1-3.

For each combination of n_1, n_2, R and R_0 , we generate 2000 B bootstrap samples from

both the populations with $\theta = 1$, $\lambda = (1/R - 1)$ and $\mu = 0$. For each sample we compute proportion of times $|Z^*| \leq Z_{1-\alpha/2}$ among the B bootstrap samples. This proportion is regarded as another estimate of β say, β_p^* . Compute MLE and the

proposed estimator and compute MSEs of the both. Efficiency of the proposed estimator is the ratio of MSE of MLE to the MSE of the PTE.

Efficiencies of ML to PTE estimator when β is estimated (Proportion of $|Z| \leq 1.96$

R=.25 R=.50 R=.75

Ro	n ₁ =10,n ₂ =10	n ₁ =10,n ₂ =20	n ₁ =20,n ₂ =20	n ₁ =10,n ₂ =10	n ₁ =10,n ₂ =20	n ₁ =20,n ₂ =20	n ₁ =10,n ₂ =10	n ₁ =10,n ₂ =20	n ₁ =20,n ₂ =20
0.1	0.8155623	0.7936373	0.5727857	0.335647	0.2792389	0.189309	0.1671512	0.06757	0.06494
0.2	1.203453	1.247997	1.103778	0.502182	0.4422252	0.3060647	0.2546692	0.09938	0.09770
0.3	1.174008	1.162549	1.08615	0.773710	0.6901933	0.5326474	0.3692421	0.14351	0.15136
0.4	0.7713979	0.6777891	0.5318005	1.090049	1.096237	0.9353358	0.5497544	0.20650	0.26605
0.5	0.4391117	0.3523001	0.2691758	1.269377	1.280395	1.270035	0.871919	0.36620	0.53027
0.6	0.2824964	0.2304058	0.1530846	1.085666	1.007019	0.9336012	1.213383	1.1067443	1.08751
0.7	0.1903591	0.1500728	0.09736974	0.747451	0.6250686	0.5338378	1.215710	1.17814	1.61097
0.8	0.1294796	0.1048997	0.06630765	0.504632	0.3979473	0.3178993	0.896672	1.125550	1.25312
0.9	0.1062477	0.0763007	0.0467161	0.361725	0.2629554	0.1958819	0.5909238	0.640879	0.76213

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