SOME RESULTS ON DIFFERENTIAL POLYNOMIALS

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shankar62pawar@gmail.com**Preliminaries:**

Let $f(z)$ be a transcendental meromorphic function in the finite complex plane. Let $P(f)$ denote the homogeneous differential polynomial in f . As in Hayman [5], such functions will be called differential polynomials in f . Thus a differential polynomial P in f is the sum of a finite number of terms of the form $a f^{(1)} \dots f^{(m)}$ where $f^{(1)}, f^{(2)}, \dots$ are successive derivatives of f and $l_0, l_1, \dots, l_m$ are non-negative integers. If $l_0 + l_1 + \dots + l_m = n$ (a fixed positive integer) in every term of P , then P is called homogeneous differential polynomial in f of degree n . In general, if $\max(l_0 + l_1 + \dots + l_m) = n$ where the maximum is taken over the term of P , then P is said to be a differential polynomial in f of degree at most n .

Definition 1: If f_1, f_2 are meromorphic functions, we denote by $S(r; f_1, f_2)$ a function of r such that

$$S(r; f_1, f_2) = O\left(\sum_{i=1}^2 T(r, f_i)\right) \text{ as } r \rightarrow \infty \text{ through all values if } f_i \text{ s are of finite orders and outside a set of finite linear measure.}$$

Here we prove the theorems by using following lemmas.

Lemma 1. [9] If P is a homogeneous differential polynomial in f of degree

$$n \geq 1, \text{ then } m\left(r, \frac{P}{f^n}\right) = S(r, f).$$

Lemma 2. : [1] Let P be a homogeneous differential polynomial in f of degree n and suppose that P does not involve f . That is, P is a homogeneous differential polynomial of degree n in $f^{(1)}, f^{(2)}, \dots$ with coefficients of the form $a(z)$. If P is not a constant and $b_1, b_2, \dots, b_q$ are distinct elements of $\mathbb{C}$ (where q is any positive integer), then

$$n \sum_{i=1}^q m(r, b_i, f) + N\left(r, \frac{1}{P}\right) \leq T(r, P) + S(r, f)$$

Lemma 3. [10] Let f_1, f_2 be two non-constant meromorphic functions such that $a_1 f_1 + a_2 f_2 \equiv 1$, where a_1, a_2 are constants. Then for $i = 1, 2$

$$T(r, f_i) < \bar{N}\left(r, \frac{1}{f_1}\right) + \bar{N}\left(r, \frac{1}{f_2}\right) + \bar{N}(r, f_i) + S(r, f_1, f_2) \quad \dots\dots\dots(1)$$

Lemma 4 : [1] Let f be a meromorphic function satisfying

$$\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) = S(r, f)$$

and let P be a homogeneous differential polynomial in f . Suppose that P is not a constant. Then the order of P is equal to the order of f and

$$\bar{N}(r, P) + \bar{N}\left(r, \frac{1}{P}\right) = S(r, P)$$

so that $\bar{N}\left(r, \frac{1}{P-a}\right) \neq S(r, P)$ and $\Theta(a, P) = 0$ for all $a \neq 0, \neq \infty$ and there exists no evB for P for

distinct zeros in $\bar{C} - \{0, \infty\}$.

A.P. Sing and Dukane [11] have proved the following result.

Theorem A. Let $f(z)$ be a meromorphic function and $\pi_n(f)$ be a homogeneous differential Polynomial of degree n .

Let $\frac{T(r, \pi_n(f))}{T(r, f)} \rightarrow \alpha$ as $r \rightarrow \infty$ where $\alpha \geq n$, then

$$\Theta(\infty, f) \leq 1 + \frac{1}{m} - \frac{\alpha}{pmn},$$

where m is the highest derivative occurring in $\pi_n(f)$ and p is the number of terms in $\pi_n(f)$.

Here we shall prove the following improvement of the above theorem. The result here is independent of the number of terms in P .

Theorem 1: Let $f(z)$ be a meromorphic function and $P(f)$ be a homogeneous differential Polynomial of degree n .

Let $\frac{T(r, P)}{T(r, f)} \rightarrow \alpha$ as $r \rightarrow \infty$ where $\alpha \geq n$, then

$$\Theta(\infty, f) \leq 1 + \frac{1}{m} - \frac{\alpha}{mn}, \quad \dots\dots\dots(2)$$

where m is the highest derivative occurring in $P(f)$.

Proof : Let

$\frac{T(r, P)}{T(r, f)} \rightarrow \alpha$ as $r \rightarrow \infty$ where $\alpha \geq n$

$$\begin{aligned} \text{Now } m(r, P) &\leq m\left(r, \frac{P}{f^n}\right) + m(r, f^n) \\ &= n m(r, f) + S(r, f) \quad \text{by Lemma 1} \quad \dots\dots(3) \end{aligned}$$

At a pole of f of order P which is not a pole of any of the coefficient $a(z)$ of P , P has a pole of order at most $pn + mn$.

So,

$$N(r, P) \leq nN(r, f) + mn\bar{N}(r, f) + S(r, f). \quad \dots\dots\dots(4)$$

From (3) and (4), we get.

$$T(r, P) \leq nT(r, f) + mn\bar{N}(r, f) + S(r, f).$$

Since $\frac{T(r, P)}{T(r, f)} \rightarrow \alpha$ it follows that

$$\alpha T(r, f) \leq nT(r, f) + mn\bar{N}(r, f) + S(r, f). \quad \dots\dots(5)$$

Dividing (5), by $T(r, f)$ and taking limit superior, we get

$$\alpha - n \leq \limsup_{r \rightarrow \infty} \frac{nm\bar{N}(r, f)}{T(r, f)} + \limsup_{r \rightarrow \infty} \frac{S(r, f)}{T(r, f)}.$$

$$\text{Thus } \alpha - n \leq nm(1 - \Theta(\infty, f))$$

Consequently,

$$nm\Theta(\infty, f) \leq nm + n - \alpha \text{ and so}$$

$$\Theta(\infty, f) \leq 1 + \frac{1}{m} - \frac{\alpha}{nm}$$

Remark - If $m = 1$, $n=1$ then $\Theta(\infty, f) \leq 2 - \alpha$ this is an interesting generalization of theorem 3 of S.K. Singh and V. N. Kulkarni [12].

Bhoosnurmath [1] has proved the following result.

Theorem B. Let f be a meromorphic function of finite order. If P is a homogeneous differential polynomial in f of degree n and if P does not involve f , then

$$n \sum_{b \in \mathbb{C}} \delta(b, f) \leq \delta(O, P) \limsup_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)}$$

and

$$n \sum_{b \in \mathbb{C}} \delta(b, f) \leq \Delta(O, P) \liminf_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)}$$

provided that P does not reduce to a constant.

In view of this, we can obtain the following theorem.

Theorem 2. Let f be a meromorphic function of finite order. If P is a homogenous differential polynomial in f of degree n and if P does not involve f , and

$$\liminf_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)} = \limsup_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)} = \alpha, \text{ then}$$

$$n \sum_{a \in C} \delta(a, f) \leq \alpha \delta(O, P) \quad \dots(6)$$

$$\text{and } n \sum_{a \in C} \delta(a, f) \leq \alpha \Delta(O, P) \quad \dots(7)$$

provided that P does not reduce to a constant.

Corollary 2.2.1 : Let f be a meromorphic function of finite order with

$$\lim_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)} = \alpha, \sum_{a \in C} \delta(a, f) = 1$$

If P is a homogeneous differential polynomial in f of degree n and not involving f , then

$$n \leq \alpha \delta(O, P) \text{ and } n \leq \alpha \Delta(O, P) \quad \dots\dots(8)$$

$$\text{If } \alpha = n, \delta(O, P) = \Delta(O, P) = 1 \quad \dots\dots(9)$$

provided that P does not reduce to a constant. In particular (10) and (11) hold if f is an entire function

$$\lim_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)} \rightarrow \alpha \text{ and } \sum_{a \in C} \delta(a, f) = 1.$$

Theorem 3: Let $P[f]$ be a homogeneous differential polynomial such that each term of P involves f , then the order of $P[f]$ and order of f are equal.

Proof. We have

$$P[f] = \sum a f^{l_0} (f^{(1)})^{l_1} \dots (f^{(m)})^{l_m}$$

$$T(r, P[f]) \leq T(r, a) + l_0 T(r, f) + l_1 T(r, f') + \dots + l_m T(r, f^{(m)}).$$

$$T(r, P[f]) \leq T(r, f) \left(l_0 + l_1 \frac{T(r, f')}{T(r, f)} + l_2 \frac{T(r, f'')}{T(r, f)} + \dots + \frac{T(r, f^{(m)})}{T(r, f)} \right) + S(r, f) \quad \dots(10)$$

we have

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)} &\leq n[(m+1) - m\Theta(\infty, f)] \\ \limsup_{r \rightarrow \infty} \frac{T(r, f^{(m)})}{T(r, f)} &\leq m+1, \text{ for all } m \geq 1 \end{aligned} \quad \dots(11)$$

Substituting (11) in (10), we get

$$\log T(r, P[f]) \leq \log T(r, f) + \log C$$

$$\begin{aligned} \rho_p &= \limsup_{r \rightarrow \infty} \frac{\log T(r, P(f))}{\log r} \leq \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} = \rho_f \\ \therefore \rho_p &\leq \rho_f \end{aligned} \quad \dots(12)$$

Since a zero or a pole of f , which is not a pole of any coefficient $a(z)$ of P , is a pole of $\frac{P}{f^n}$ of degree mn

at most, we have

$$\begin{aligned} N\left(r, \frac{P}{f^n}\right) &\leq mn \left(\bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f) \right) + S(r, f) \\ nT(r, f) &= T(r, f^n) \end{aligned}$$

$$\begin{aligned}
&\leq T\left(r, \frac{f^n}{P}\right) + T(r, P) \\
&\leq T\left(r, \frac{P}{f^n}\right) + T(r, P) + O(1) \\
&\leq T(r, P) + mn \left(\bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f) \right) + S(r, f) \\
&\leq T(r, P) + mn \left(\bar{N}\left(r, \frac{1}{P}\right) + \bar{N}(r, P) \right)
\end{aligned}$$

Since each term of P involves f term, it follows that

$$\begin{aligned}
&\bar{N}(r, f) \leq N(r, P) \text{ and } \bar{N}\left(r, \frac{1}{f}\right) < \bar{N}\left(r, \frac{1}{P}\right) \\
&nT(r, f) \leq T(r, P) + mn \left(\bar{N}\left(r, \frac{1}{P}\right) + \bar{N}(r, P) \right) + S(r, f) \\
&\leq T(r, P) + mn(2T(r, P)) + S(r, f) \\
&\leq (1 + 2mn)T(r, P) + S(r, f) \\
&\rho_f = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} \leq \limsup_{r \rightarrow \infty} \frac{\log T(r, P[f])}{\log r} = \rho_P \quad \dots\dots(13)
\end{aligned}$$

From (12) and (13) we conclude that $\rho_P = \rho_f$

Theorem 4 : Let f be a meromorphic function satisfying

$$\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) = S(r, f).$$

Let P(f) be a homogeneous differential polynomial which does not reduce to a constant. If G is a meromorphic function such that,

$$\bar{N}\left(r, \frac{1}{G}\right) = S(r, G),$$

then the identity, $P + G \equiv 1$ is impossible.

This theorem was proved by H.S. Gopalkrishna and S.S. Bhoosnumath [1] but here we improve this theorem removing the condition $\Theta(0, G) > 0$. It is also interesting to note that here we use a different technique.

Proof. Suppose $P + G \equiv 1$ holds.

$$\begin{aligned}
\text{Now} \quad &\bar{N}(r, P) \leq \bar{N}(r, f) + S(r, f) \\
&= S(r, f), \text{ by hypothesis} \\
\text{Therefore} \quad &\bar{N}(r, P) = S(r, P). \quad \dots\dots(14) \\
\text{Because} \quad &S(r, f) = S(r, P). \\
&\bar{N}(r, P) = S(r, f) = S(r, P).
\end{aligned}$$

Also

$$\begin{aligned}
&\bar{N}\left(r, \frac{1}{P}\right) \leq (1 + mn) \bar{N}\left(r, \frac{1}{f}\right) + mn \bar{N}(r, f) + S(r, f) \\
&= S(r, f), \text{ by hypothesis}
\end{aligned}$$

Therefore

$$\bar{N}\left(r, \frac{1}{P}\right) = S(r, f) = S(r, P) \quad \dots\dots(15)$$

Then by Lemma 3, we have

$$T(r, P) < \bar{N}\left(r, \frac{1}{P}\right) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}(r, P) + S(r, P, G)$$

$$T(r, P) < \bar{N}\left(r, \frac{1}{G}\right) + S(r, G) + S(r, P, G), \quad \dots(16)$$

by Lemma 4.

Since $P + G = 1$ holds, it follows that

$$T(r, P) \sim T(r, G) \text{ and } S(r, G) = S(r, P). \quad \dots(17)$$

By (16) and (17), we have

$$T(r, G) < \bar{N}\left(r, \frac{1}{G}\right) + S(r, G) \text{ Or } T(r, G) < S(r, G) \text{ by hypothesis.}$$

This is a contradiction.

Thus, $P + G \equiv 1$ is impossible.

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