Asymptotic Estimates of Fundamental Solutions of Third order Nonlinear Difference Equation

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ABSTRACT: We find asymptotic estimates of fundamental solutions of third order Difference equation

$$y(n+3) - (2 + e^{-n-2})y(n+2) - \left(1 + \frac{1}{n^2+1}\right)y(n+1) + 2y(n) = 0$$

KEYWORDS: Difference equation; Oscillation and Non oscillation; asymptotic properties; polynomial.

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1. INTRODUCTION

The asymptotic behavior of third order difference equation play a central role in many branches of pure and applied mathematics such as continued fractions, special functions, orthogonal polynomials and combinatorics. In this paper we will utilize the special characteristics of third order equation to obtain a deeper understanding of the asymptotic of their solution. The difference equation of the form

$$y(n+3) - (2 + e^{-n-2})y(n+2) - \left(1 + \frac{1}{n^2+1}\right)y(n+1) + 2y(n) = 0 \quad (1)$$

The characteristic equation is given by $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$ with roots $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$. Notice that $p_1(n) = -e^{-n-2}, p_2(n) = -\frac{1}{n^2+1}, p_3(n) = 0$ all satisfy below condition (2). Hence from the corollary (1) applies to produce solutions $y_1(n), y_2(n)$ and $y_3(n)$ defined as follows
 $y_1(n) = [1 + o(1)]2^n, y_2(n) = 1 + o(1), y_3(n) = [1 + o(1)](-1)^n$.

Corollary (1) is due to Evgrafov. It says that for each characteristic root of polynomial (2), at least one solution behaves in formula $y_i(n) = [1 + o(1)]\lambda_i^n$, provided that the rate of convergence of the coefficients is not too slow.

Corollary: 1

Suppose that the polynomial

$$p(\lambda) = \lambda^k + a_1\lambda^{k-1} + \dots + a_k \quad (2)$$

Have distinct roots $\lambda_1, \lambda_2, \dots, \lambda_k$
and that

$$\sum_{n=1}^{\infty} |p_i(n)| < \infty \quad \text{for } 1 \leq i \leq k. \quad (3)$$

Then equation

$$y(n+k) + (a_1 + p_1(n))y(n+k-1) + \dots + (a_k + p_k(n))y(n) = 0 \quad (A)$$

has k solutions

$$y_1(n), y_2(n), \dots, y_k(n)$$

with

$$y_i(n) = [1 + o(1)]\lambda_i^n \quad (4)$$

Proof:

First we put into the form of a k-dimensional system

$$(n+1) = [A + B(n)]z(n) \quad (5)$$

$y(n+k) + (a_1 + p_1(n))y(n+k-1) + \dots + (a_k + p_k(n))y(n) = 0$
where

$$A = \begin{pmatrix} 0 & 1 & \cdot & \cdot & 0 \\ 0 & 9 & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_k & -a_{k-1} & \cdot & \cdot & -a_1 \end{pmatrix}$$

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$$B(n) = \begin{pmatrix} 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_k(n) & -p_{k-1}(n) & \cdot & \cdot & -p_1(n) \end{pmatrix}$$

$$z(n) = (y(n), y(n+1), \dots, y(n+k-1))^T$$

Notice that polynomial (2) is the characteristic polynomial of the matrix A. Furthermore, for each eigen value λ_i there corresponds the eigen vectors $\xi = (1, \lambda_i^1, \lambda_i^2, \dots, \lambda_i^{k-1})^T$. In addition, the matrix $B(n)$ satisfies condition

$$\sum_{n=n_0}^{\infty} \frac{1}{|\lambda_i(n)|} \|B(n)\| < \infty \quad (6)$$

Hence one may apply theorem (1.1) to conclude that there are k solutions

$z_1(n), z_2(n), \dots, z_k(n)$ Of (5) such that, for $1 \leq i \leq k$.

$$z_i(n) = \begin{pmatrix} y_i(n) \\ y_i(n+1) \\ y_i(n+2) \\ \cdot \\ y_i(n+k-1) \end{pmatrix} = (1 + o(1)) \lambda_i^n \begin{pmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \\ \cdot \\ \lambda_i^{k-1} \end{pmatrix}$$

Hence $y_i(n) = [1 + o(1)] \lambda_i^n$

Theorem 1.1

Suppose that the matrix A has k linearly independent eigen values $\xi_1, \xi_2, \dots, \xi_k$ and k corresponding eigen vectors $\lambda_1, \lambda_2, \dots, \lambda_k$. If condition $\sum_{n=n_0}^{\infty} \frac{1}{|\lambda_i(n)|} \|B(n)\| < \infty$ holds for $B(n)$, then system of equation

$$y(n+1) = [A + B(n)]y(n) \quad (7)$$

has solutions

$$y_i(n), 1 \leq i \leq k, \text{ such that } y_i(n) = [\xi_i + o(1)] \lambda_i^n \quad (8)$$

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Proof:

In order to be able to apply theorem 1.2 we need to diagonalize the matrix A. This may be accomplished by letting

$$y = Tz \quad (9)$$

$$\text{In equation (6) where } T = (\xi_1, \xi_2, \dots, \xi_k) \quad (10)$$

That is the i-th column of T is ξ_i . Then we obtain

$$Tz(n+1) = [A + B(n)]Tz(n)$$

(or)

$$z(n+1) = [D + B(n)]z(n) \quad (11)$$

Where $D = T^{-1}AT = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_k)$ and $\overline{B(n)} = T^{-1}B(n)T$. It is easy to see that $\overline{B(n)}$ satisfies conditions (A). Now formula (9) follows by applying theorem 1.1.

Theorem 1.2

Suppose that condition

Condition

$$L = \begin{cases} \prod_{r=0}^n \left| \frac{\lambda_i(r)}{\lambda_j(r)} \right| \rightarrow +\infty \\ \text{and } \prod_{r=n_1}^{n_2} \left| \frac{\lambda_i(r)}{\lambda_j(r)} \right| \geq \mu > 0, \text{ for all } 0 \leq n_1 \leq n_2, \\ \text{or } \prod_{r=n_1}^{n_2} \left| \frac{\lambda_i(r)}{\lambda_j(r)} \right| \leq k, \text{ for all } 0 \leq n_1 \leq n_2 \end{cases}$$

Holds and for each $i, 1 \leq i \leq k, \sum_{n=n_0}^{\infty} \frac{1}{|\lambda_i(n)|} \|B(n)\| < \infty$. (A)

Then the equation $y(n+1) = (D(n) + B(n))y(n)$ has a fundamental set of k solutions $y_i(n)$ such that

$$y_i(n) = (e_i + o(1)) \prod_{r=n_0}^{n-1} \lambda_i(r)$$

Where e_i is the standard unit vector in R^+ where its components are all zero, except that the i-th component is 1.

Theorem 1.3

Suppose that the polynomial (2) has a k-fold root of 1 and that

$$\sum_{n=1}^{\infty} n^{k-1} |p_i(n)| < \infty \text{ for } 1 \leq i \leq k \quad (12)$$

Then

$$y(n+k) + (a_1 + p_1(n))y(n+k-1) + \dots + (a_k + p_k(n))y(n) = 0$$

Have k solutions $y_1(n), y_2(n), \dots, y_k(n)$ with

$$y_i(n) = n^{i-1} (1 + o(1)), n \rightarrow \infty \quad (13)$$

Solution:

We remark here that the actual result of Coffman is stronger than the statement of theorem 1.3. Indeed, he proved that

$$\Delta^m y_i(n) = \begin{cases} \binom{n}{i-m} + o\left(\binom{n}{i-m}\right) & \text{for } 1 \leq m \leq i \\ o\left(\binom{n}{i-m}\right) & \text{for } i \leq m \leq k-1 \end{cases}$$

The curious reader might wonder coffman's theorem applies if the polynomial (2) has a k-fold root not equal to 1. Luckily by a very simple trick, one is able to do exactly that. Assume that the characteristic equation (2) has a k-fold root $\mu \neq 1$. Then polynomial (2) may be written as

$$(\lambda - \mu)^4 = 0 \quad (14)$$

Letting $y(n) = \mu^n x(n)$ in equation (A), we obtain
 $\mu^{n+k} x(n+k) + \mu^{n+k-1} (a_1 + p_1(n)) + \dots + \mu^n (a_k + p_k(n)) x(n) = 0$

Or

$$x(n+k) + \frac{1}{\mu} (a_1 + p_1(n)) x(n+k-1) + \dots + \frac{1}{\mu^k} (a_k + p_k(n)) x(n) = 0 \quad (15)$$

The characteristic equation (15) is given by

$$\lambda^k + \frac{a_1}{\mu} \lambda^{k-1} + \frac{a_2}{\mu^2} \lambda^{k-2} + \dots + \frac{a_k}{\mu^k} = 0$$

Which has k-fold root $\lambda=1$. If $p_i(n)$, $1 \leq i \leq k$, satisfies condition (A), then so does $\left(\frac{1}{\mu^i}\right) p_i(n)$. Hence theorem (1.3) applies to (15) to yield solutions $x_1(n), x_2(n), \dots, x_k(n)$ with $x_i(n) = n^{i-1} (1 + o(1))$, $n \rightarrow \infty$

Consequently, there are solutions $y_1(n), y_2(n), \dots, y_k(n)$ of (A) such that $y_i(n) = n^{i-1} (1 + o(1)) \mu^n$

We now summarize the above observation in the following corollary.

Corollary 2

Suppose that the polynomial (2) has a k-fold root μ and that condition (12) holds. Then

$$y(n+k) + (a_1 + p_1(n)) y(n+k-1) + \dots + (a_k + p_k(n)) y(n) = 0$$

Have k solutions $y_1(n), y_2(n), \dots, y_k(n)$ such that $y_i(n) = n^{i-1} (1 + o(1)) \mu^n$

(16)

Example: 1

Investigate the asymptotic behavior of solutions of the difference equation

$$y(n+3) - (6 + e^{-n-2}) y(n+2) + \left(12 - \frac{1}{(n+1)^4}\right) y(n+1) - 8y(n) = 0$$

Solution:

The characteristic equation is given by $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$ with roots $\lambda_1 = \lambda_2 = \lambda_3 = 2$. Also $p_1(n) = -e^{-n-2}$, $p_2(n) = -1/(n+1)^4$, and $p_3(n) = 0$

All satisfy condition (12). Hence, by corollary 2 there are three solutions

$$y_1(n) = (1 + o(1)) 2^n, \quad y_2(n) = n(1 + o(1)) 2^n, \text{ and } y_3(n) = n^2(1 + o(1)) 2^n.$$

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