Teaching Choice Function Using of Axioms of Sets and Well Ordered Countably Infinite Sets

Küme Aksiyomları Kullanılarak Seçme Fonksiyonunun ve İyi Sıralı Sayılabilir Sonsuz Kümelerin Öğretimi

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Abstract: Set theory has an important place in the historical development of mathematics. This theory, which helps to reveal almost all mathematical problems and develop solution methods with Zermelo-Fraenkel set theory, has a special importance in terms of building the base of set theory. Despite the application of mathematics, students have difficulty understanding set axioms and well-ordered countably infinite sets. Due to the inevitability of this accumulation of knowledge, firstly the historical development of set theory has been included in this article, the $\mathbb{Q}/\mathbb{Z}$ quotient group has been created, and then a choice function whose existence has been demonstrated by the choice axiom has been found. In other words, a choice function definition application has been made for an infinite set. Rather than the real method using a very abstract approach, appropriate approaches have been used. In addition, totally ordered set method was obtained. Then countably infinite sets are well-ordered. Finally, it is stated that each set can be obtained well-ordered using Zermelo-Fraenkel axiom system or Zorn's Lemma. These methods were taught to high school teachers. With the help of this method, students were able to identify infinite sets more easily and easily operate on these sets. The method of teaching, learning and evaluation of finding totally ordered and well-ordered countably sets and choice axiom has been examined in this study as an application of group and set theory. In this way, the method increased the students interest in algebraic concepts and contributed to the development of their mathematical knowledge and skills.

Structured Abstract: In this work, we first give the historical development process of the axiom of choice, because our object of research sticks much better to our mind and reminds us of the Zermelo-Fraenkel axioms, emphasized by technique for set theory. Methodological techniques are developed with special teaching teachings used. Basic information about quotient group $\mathbb{Q}/\mathbb{Z}$ is given. In mathematics, a technique for set theory is applied in mathematics. The class has the choice function of the quotient group $\mathbb{Q}/\mathbb{Z}$ using the preferred axiom. Applying the axiom of choice and well-ordered infinite sets to learning groups is applied. It is an active learning group with ten teachers, five female mathematics teachers and five male mathematics teachers. The other ten teachers are the traditional learning group, with six female mathematics teachers and four male mathematics teachers. The teaching method is carefully chosen with the mind of the teacher. They

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show success when using other concrete examples and tend to struggle with abstract concepts without any content. Therefore, abstract ideas are taught using practical examples. In addition, this practice aims to understand how abstract concepts can be used from judgments about practical scenarios. The assessment was conducted in consultation with the moderator, who has equipped in teaching pure mathematics to these students. This was given through a process of peer observation and discussion. Students attendance and scores were also used to evaluate the success of the set theory section of the choice axiom. Within this pare, the axiom of choice to teach the choice function of Set theory and this application on well-ordered infinite sets to teach the choice function of Set theory will be discussed. After defining the teaching environment in which students work, the connection with set theory is presented, along with examples of how a particular subject is taught. The students were asked to define the concept of one-to-one correspondence, totally ordered sets, countably infinite sets by applying the brainstorming technique. Students expressed their ideas freely and said definitions they saw as correct. The brainstorming technique was occasionally used in later stages of teaching. Then they listened to the lecture presentation on the teaching of the well-ordered infinite set. They told their students at their school what they were actively learning in the classroom setting. If we collect what the teacher in the Active Learning Group earns in the classroom. When explicit responses were requested, teachers cited several reasons for their high attendance at the sessions. In Active Learning Group sessions, many talked about personal motivation (seventy percent of students) and wanted to do well through this practice (sixty percent of students). Some have stated that they participated because they enjoyed this practice and found it interesting (eighty percent of students). The minority feared that they did not want to be left behind and would struggle to keep up (twenty percent of students). The situation is not very welcome in the Traditional Learning Group. Only two out of ten mathematics teachers are happy with this. Thus, about twenty percent of students in the Traditional Learning Group can achieve success. In this application, they take every set of infinite numbers with this application. First, with this study they learn well-arranged sets and with this problem they can grasp the basics of the axiom of choice. In this study, special attention is given to the teaching of this subject, and they can examine the differences and similarities between finite and infinite sets, types of infinite sets, the function of choice in sets, and the proof of the Union of finite or infinite sets. This shows the success of the method developed in the article. The theory of infinite sets provides great ease to the study. With the help of this method, it will enable mathematics teachers who will gain good gains in future generalizations. Above all, it conceptualizes mathematics beyond memorization. Secondary school students who understand the importance of cluster theory can class all the concepts they will encounter throughout their lives in a concrete way. They can add some kind of theory to the concepts they classify.

Keywords: Set Theory, Choice Function, Discovery Learning, Infinite Set, Countably Set.

Öz: Kümeler kuramı matematiğin tarihsel gelişiminde önemli bir yere sahiptir. Zermelo-Fraenkel kümeler kuramı ile günümüzde hemen hemen tüm matematiksel problemlerin ortaya atılması ve çözüm yöntemlerinin geliştirilmesine yardımcı olan bu kuram, kümeler kuramının tabanının inşa etmesi açısından da ayrı bir öneme sahiptir. Matematik uygulamasına yer verilmesine rağmen, öğrenciler küme aksiyomları ve iyi sıralı sayılabilir sonsuz kümeleri kavramakta güçlük çekerler. Bu bilgi birikiminin kaçınılmaz olmasından dolayı ilk olarak kümeler kuramının tarihsel gelişimine yer verilen bu makalede Q / Z bölüm grubu oluşturulmuş, daha sonra ise varlığını seçim aksiyomu ile gösterilmiş bir seçim fonksiyonu bulunmuştur. Başka bir deyişle, sonsuz bir küme için bir seçim fonksiyonu tanımlama uygulaması yapılmıştır. Çok soyut bir yaklaşım kullanan gerçek yöntemden ziyade, uygun yaklaşımlardan yararlanılmıştır. Buna ek olarak, tam sıralı kümeler metodu elde edilmiştir. Sonra sayılabilir sonsuz kümeler iyi sıralanmıştır. Son olarak, her kümenin Zermelo-Fraenkel aksiyom sistemi veya Zorn Lemma kullanılarak iyi bir şekilde elde edilebileceği belirtilmiştir. Bu yöntemler lise öğretmenlerine öğretilmiştir. Bu yöntem yardımıyla öğrenciler sonsuz kümeleri daha kolay belirleyebilmiş ve bu kümeler üzerinde kolaylıkla işlem yapabilmişlerdir. Tam sıralı ve iyi sıralı sayılabilir kümeleri ve seçim aksiyomunu bulma öğretme, öğrenme ve değerlendirme yöntemi, bu çalışmada grup ve küme teorisinin bir uygulaması olarak incelenmiştir. Bu şekilde yöntemin öğrencilerin cebirsel kavramlara olan ilgisini artırmış ve matematiksel bilgi ve becerilerinin geliştirilmesinde katkı sağlamıştır.

Anahtar Kelimeler: Kümeler Kuramı, Seçme Fonksiyonu, Keşfederek Öğrenme, Sonsuz Küme, Sayılabilir Küme.

Introduction

If we are informed historical development process of axiom of choice, our research object sticks in our mind much better. A great amount of mathematician studied in the sense of set theory. But Cantor's research worked provides original development on set theory. Cantor relied on intuition, rather than any set of axioms, in deciding which objects were to be sets. However common sense turned out not to be good enough lighthouse to keep set theory from being wrecked on the shoals of the paradoxes (Burton, 2011).

The first prospering axiomatic treatment of set theory was published by the German mathematician Ernst Zermelo in 1908. Zermelo based his argument on a powerful new, and suspect, device, the axiom of choice. The axiom of choice (a name given by Zermelo) asserted that from any given collection of disjoint nonempty sets, it would be possible to choose exactly one element from each set and thereby from a new set. Incommodiously, tis axiom provided for a simultaneous but independent selection from each of an infinite number of sets. The idea of making infinitely many choices was not entirely new, having been an important part of many mathematical arguments around 1900. Zermelo did not say what sets are but simply postulated them together with their basic properties. The conciseness of his system of axioms is surprising. Only seven axioms, involving just two undefined technical terms (set and membership) sufficed to build up the set theory required for practically all mathematics (Burton, 2011).

The original axioms of Zermelo, as amended by Fraenkel, von Neumann, and others, are now called the Zermelo-Fraenkel axioms for set theory. Zermelo, to avoid paradoxes that would render his system useless, refused to admit into the club of decent sets those collections that were too big. He observed that mathematicians would not normally think of using such sets as the set of all sets or the set of all sets which are not elements of themselves. He held that the sets needed in practice were always built up by means of given operations from certain simple sets (like the sets of natural numbers or real numbers). Thus, Zermelo formulated a principle by which he guaranteed the existence of certain subsets. For more historical narrative referable to (Burton, 2011).

Firstly we remind Zermelo-Fraenkel axioms in (Zermelo,1930):

The Zermelo-Fraenkel axioms are the basis for Zermelo-Fraenkel set theory. In the following (Jech, 1997), $\exists$ stands for exists, $\forall$ means for all, $\in$ stands for 'is an element of', $\emptyset$ for the empty set, $\Rightarrow$ for implies, $\wedge$ for AND, $\vee$ for OR, and $\equiv$ for 'is equivalent to'.

- 1) Axiom of the Empty Set: There exists a set that has no elements.

$$\exists X \forall y (y \notin X)$$

- 2) Axiom of Extensionality: If X and Y have the same elements, then $X = Y$.

$$\forall u (u \in X \equiv u \in Y) \Rightarrow X = Y$$

- 3) Axiom of the Unordered Pair: For any a and b there exists a set $\{a, b\}$ that contains exactly a and b . (also called Axiom of Pairing)

$$\forall a \forall b \exists c \forall x (x \in c \equiv (x = a \vee x = b))$$

- 4) Axiom of Subsets: If φ is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X \mid \varphi(u, p)\}$ that contains all those $u \in X$ that have the property φ . (also called Axiom of Separation or Axiom of Comprehension)

$$\forall X \forall p \exists Y \forall u (u \in Y \equiv (u \in X \wedge \varphi(u, p)))$$

- 5) Axiom of the Sum Set: For any X there exists a set $Y = \bigcup X$, the union of all elements of X . (also called Axiom of Union)

$$\forall X \exists Y \forall u (u \in Y \equiv \exists z (z \in X \wedge u \in z))$$

- 6) Axiom of the Power Set: For any X there exists a set $Y = P(X)$, the set of all subsets of X .

$$\forall X \exists Y \forall u (u \in Y \equiv u \subseteq X)$$

- 7) Axiom of Infinity: There exists an infinite set.

$$\exists S [\emptyset \in S \wedge (\forall x \in S) [x \cup \{x\} \in S]]$$

- 8) Axiom of Replacement: If F is a function, then for any X there exists a set $Y = F[X] = \{F(x) \mid x \in X\}$.

$$\forall x \forall y \forall z [\varphi(x, y, p) \wedge \varphi(x, z, p) \Rightarrow y = z] \Rightarrow \forall X \forall Y \forall y [y \in Y \equiv (\exists x \in X) \varphi(x, y, p)]$$

- 9) Axiom of Foundation: Every nonempty set has an $\in$ -minimal element. (also called Axiom of Regularity)

$$\forall S [S \neq \emptyset \Rightarrow (\exists x \in S) S \cap x = \emptyset]$$

- 10) Axiom of Choice: Every family of nonempty sets has a choice function.

$$\forall x \in a \exists A(x, y) \Rightarrow \exists y \forall x \in a A(x, y(x))$$

The system of axioms 1-9 is called Zermelo-Fraenkel set theory, denoted ‘ZF’. The system of axioms 1-9 minus the axiom of replacement (i.e., axioms 1-7 plus 9) is called Zermelo set theory, denoted ‘Z’. The set of axioms 1-10 with the axiom of choice is usually denoted ‘ZFC’. Unfortunately, there seems to be some disagreement in the literature about just what axioms constitute ‘Zermelo set theory’. Mendelson does not include the axioms of choice or foundation in Zermelo set theory, but does include the axiom of replacement in (Mendelson, 1997). Enderton includes the axioms of choice and foundation, but does not include the axiom of replacement in (Enderton, 1977). It includes an Axiom of the empty set, which can be gotten from 6 and 3, via $\exists X (X = X)$ and $\emptyset = \{u \mid u \neq u\}$. Abian proved consistency and independence of four of the Zermelo-Fraenkel axioms in (Abian, 1969).

Constituting a basis for set theory, Cantor primarily studied sets occurring in mathematics, such as sets of integers and the real numbers and their subsets. If we want to study infinite sets in general, we have to clarify, when we consider a set as given, what kind of sets exist. In the initial ‘romantic’ period of set theory, this problem did not arise. Every ‘conceivable’ set was thought to exist, every collection for which it was possible to say in some way what its elements were considered a set. It soon turned out, however, that this viewpoint is untenable, since, as we will see, the set of all elements can not possibly exist. This turn of events eventually led to the development of axiomatic set theory. The usual axiomatic approach postulates only the existence of a single set, and other sets can be obtained from this set with the aid of the so-called conditional set existence axioms (Burton, 2011).

In what follows, we consider a few sets commonly used in mathematics, such as the set of integers, of the rationals, and of the reals. Beyond these, we only consider sets whose existence can be derived from the conventions enumerated below. These conventions in effect amount to the usual Zermelo-Fraenkel axiom system of set theory. This axiom system will be formulated in full precision only in the (Hajnal et al. 1999). There will be sketch how the axiomatic development of set theory can be carried out on the basis of these axioms, how the nonnegative integers, the rational and real numbers can be defined in this development.

Cantor’s great synoptic work of 1895, tried to define what he meant by a set. By a set we are to understand any collection into a whole M of definite and distinguishable objects of our intuition

or our thought. These objects are called the elements of M . Although ‘collection into a whole’ is at best a paraphrase of the notion of set, the terms definite and distinguishable had a clear meaning to Cantor. The intended meaning of the former was that given a set M , one should be able to decide whether any particular element would belong to M : the attribute distinguishable is interpreted as meaning that any two elements of the same set are different. The implication is that a set is determined solely by what is in it, by its elements. Cantor conceived of the notion of set in as general a way as possible. There was no restriction whatever on the nature of the considered objects nor on the way there were collected into a whole. Because his definition was not precise enough to prohibit him from considering such things as the ‘set of all sets’, it ultimately led to some famous paradoxes concerning the infinite. These paradoxes, which threatened the very foundations of logic and mathematics, necessitated the refinement of Cantor’s naive concept of ‘set’. The attempted improvements in the definition were so unsuccessful in identifying the notion that today we find it convenient to take set and element as undefined terms. Cantor also interested in the properties of infinite sets (Burton, 2011).

Cantor define one to one correspondence between the two sets: Two sets M and M' are equivalent, written $M \sim M'$, if there exists a one to one correspondence between their elements. It is clear that two finite sets are equivalent provided that they have the same number of elements. But Cantor’s definition of equivalence does not use the notion of finiteness in any essential way. It depends only on the idea of one to one correspondence, which can be applied to all sets, finite or not. Up to now the terms finite set and infinite set have been used in an informal way but they can be given a precise meaning through the notion of equivalence. Because everyday experience involves encounters with finite sets only, the usual custom is first to define a finite set in the positive sense, and then to take an infinite set as one that is not finite (Burton, 2011).

A set M is finite if either it is empty or there exists a natural number n such that $M \sim \{1, 2, \dots, n\}$; otherwise, M is infinite. A set M is infinite if it is equivalent to a proper subset of itself; in the contrary case, M is finite. Cantor spent considerable effort defending himself against the opposition of many mathematicians who regarded the infinite more as a description of unbounded growth expressed by some symbol like ∞ , than of an attained quantity. According to the traditional conception, the infinite was something increasing above all bounds, but always remaining finite (Burton, 2011).

Kronecker in his violent opposition of Cantor’s work succeeded only in curbing its early development for two decades. These are ‘countable’ was later used by Cantor to describe the simplest kind of infinite, one that has the power of the natural numbers. A set A is said to be countable if there is a one to one correspondence between it and the set N of natural numbers. Infinite sets which are not countable are called uncountable. Refer to (Burton 2011) for more details.

In Turkey, graduate study increases the significance on pure mathematics in consequence of technological developments. The mathematics degree at the University is based on real, applicable and current uses of mathematics. The degree programme has several themes running throughout including abstract algebra and number theory. There is a large range of using technology and the development of employability skills. There is not a focus of getting students ready for academia, although some do move on to postgraduate study. Upper secondary education is significant on mathematics. Current use of mathematics is indispensable part of our lives. The degree programme has several themes running throughout including abstract algebra. There is a large range of using technology and the development of employability skills.

I preferred a technic application for set theory in mathematics lessons. This application is present our twenty secondary mathematics teachers. In this application ten teachers are active learning group which is five female mathematics teachers and five male mathematics teachers. The other ten teachers are traditional learning group which is six female mathematics teachers and four

male mathematics teachers. Teaching method was carefully selected with teacher mind. They show success when using concrete other examples and tend to struggle with abstract concepts without any content. Therefore, the abstract ideas are taught through the use of practical examples. In addition to this, this application aims to increase the understanding of how abstract concepts can be used to from judgements about practical scenarios.

Within this pare, this application on axiom of choice to teach the choice function part of set theory and well ordered infinite sets to teach the choice function of set theory will be discussed. After describing the teaching environment that the students work within, the connection with set theory are presented, along with examples of how specific topic is taught.

Methodology Techniques

Most of the questions were about the specific teaching techniques used. Responses to the following questions have been used:

- Did using this application help you understand the concepts?
- Did you find the application of choice function?
- Would you have preferred to study group theory without axioms of sets?
- Do you find it easier to understand abstract ideas when you see a practical example?
- Did being in a support group help you learn the application for group theory?
- Do you like the partially printed notes?

These questions had three possible responses (yes, no or sometimes) and were followed by questions asking why they had responded the way they had. There was also an open question about why they chose Abstract Algebra as an elective along with questions asking whether they attended most of the lectures and why this was the case.

In addition to this, evaluation has taken place by consultation with the moderator, who has a back ground in teaching pure mathematics to these students. This was done through a process of peer observation and discussions. The attendance of the students and the marks they obtained have also been used when evaluating the success of the set theory part of axiom of choice.

Rudiment about Quotient Group $\mathbb{Q}/\sim$

We consider then an equivalence relation $\sim$ on rational number group $\mathbb{Q}$; we seek a group $\mathbb{Q}/\sim$ and a group homomorphism $\pi: \mathbb{Q} \rightarrow \mathbb{Q}/\sim$ satisfying the appropriate universal property.

It is natural to try to construct the group $\mathbb{Q}/\sim$ by defining an operation $+$ on the set $\mathbb{Q}/\sim$. The situation is tightly constrained by the requirement that the quotient map $\pi: \mathbb{Q} \rightarrow \mathbb{Q}/\sim$ be a group homomorphism: for if $[a] = \pi(a)$, $[b] = \pi(b)$ are elements of $\mathbb{Q}/\sim$ (that is, equivalence classes with respect to $\sim$), then the homomorphism condition forces

$$[a] + [b] = \pi(a) + \pi(b) = \pi(a + b) = [a + b]$$

For the operation to be well-defined ‘ in the first factor ’, it is necessary that if

$$[a] = [\beta], \text{ then } [a + \varphi] = [\beta + \varphi]$$

Regardless of what φ is; that is,

$$(\forall \vartheta \in \mathbb{Q}): \alpha \sim \beta \Rightarrow \alpha + \vartheta \sim \beta + \vartheta \quad (*)$$

Similarly, for the operation to be well-defined in the second factor we need

$$(\forall \vartheta \in \mathbb{Q}): \alpha \sim \beta \Rightarrow \vartheta + \alpha \sim \vartheta + \beta .$$

We constitute that this operation is well-defined. Consequently, the operation

$$[\alpha] + [\beta] := [\alpha + \beta]$$

Defines a group structure on $\mathbb{Q}/\sim$. So we say that $\sim$ is compatible with the group structure of $\mathbb{Q}$ by means of above condition hold. Since the operation $+$ on the quotient $\mathbb{Q}/\sim$ is uniquely determined by the operation on $\mathbb{Q}$, we yield to usual abuse of language and omit it. If $\sim$ is compatible, we call $\mathbb{Q}/\sim$ the quotient group of $\mathbb{Q}$ by $\sim$. Since above condition hold, they lead to a complete description of all compatible relations on a group $\mathbb{Q}$.

In fact, each of these two conditions leads to a description of the relations satisfying it, and we analyze them separately. Let $\sim$ be an equivalence relation on a group $\mathbb{Q}$, satisfying above conditions. Then the equivalence class of 0 is a subgroup $\mathbb{Z}$ of $\mathbb{Q}$ and

$$a \sim b \Leftrightarrow -a + \beta \in \mathbb{Z} \Leftrightarrow a + \mathbb{Z} = \beta + \mathbb{Z}.$$

This shows that the equivalence classes of an equivalence relation satisfying (*) are in fact all in the form $\alpha + \mathbb{Z}$, as α ranges in $\mathbb{Q}$. This important subset determined by a subgroup $\mathbb{Z}$ deserve a name: left cosets. The left cosets of a subgroup $\mathbb{Z}$ in a group $\mathbb{Q}$ are sets $\alpha + \mathbb{Z}$, for $\alpha \in \mathbb{Q}$. The right-cosets of $\mathbb{Z}$ are sets $\mathbb{Z} + \alpha$, $\alpha \in \mathbb{Q}$. Since every subgroup of abelian group is normal subgroup, $\mathbb{Z}$ is a normal subgroup of a group $\mathbb{Q}$. The quotient group of $\mathbb{Q}$ modulo $\mathbb{Z}$, denoted $\mathbb{Q}/\mathbb{Z}$, is the group $\mathbb{Q}/\sim$ obtained from the relation $\sim$ defined above. In terms of (left-)cosets, the sum in $\mathbb{Q}/\mathbb{Z}$ is defined by $(\alpha + \mathbb{Z}) + (\beta + \mathbb{Z}) = \alpha + \beta + \mathbb{Z}$. The identity element $0_{\mathbb{Q}/\mathbb{Z}}$ of the quotient group $\mathbb{Q}/\mathbb{Z}$ is the coset of identity, $0 + \mathbb{Z} = \mathbb{Z}$. For more information, we refer to (Aluffi, 2009) about quotient groups.

Finding Choice Function of Quotient Group $\mathbb{Q}/\mathbb{Z}$ Using Axiom of Choice in the Classroom

Every element of $\mathbb{Q}/\mathbb{Z}$ is a set in the form of

$$q + \mathbb{Z} = \{q + n \mid n \in \mathbb{Z}\}$$

For every rational number q . Here we must like minded the set of

$$\mathbb{Q}/\mathbb{Z} = \{q + \mathbb{Z} \mid q \in \mathbb{Q}\}$$

Is a family of set with the classroom. Now we want to choose an element of the set $q + \mathbb{Z}$. Some students can expect that the set of $q + \mathbb{Z}$ determine the element of q . Solely, it is not correct consideration.

In this place, the teacher must give an example which for any rational numbers q and q' such that $q + \mathbb{Z} = q' + \mathbb{Z}$. For example

$$\frac{3}{4} + \mathbb{Z} = \frac{11}{4} + \mathbb{Z}.$$

In that the equation

$$\frac{11}{4} + \mathbb{Z} = \frac{3}{4} + 2 + \mathbb{Z} = \frac{3}{4} + \mathbb{Z}$$

is hold as $2 \in \mathbb{Z}$. Our main problem is how do we find a choice function of the set $\mathbb{Q}/\mathbb{Z}$. For this, we can choose an element for every set $q + \mathbb{Z}$ such that q is a rational number. Let α be any element of $\mathbb{Q}/\mathbb{Z}$. Then there exists an element q of $\mathbb{Q}$ such that $\alpha = q + \mathbb{Z}$.

The teacher must consult this question: ‘Well, can we do this?’ or ‘how does we do this?’ They don’t forget that students have no knowledge of axiom of choice in the level of postgraduate students. Because of this, the teacher must use numerical axis for understanding choosing of the element q . We must choose an element q of $[0,1) \cap \mathbb{Q}$ applying two steps process.

In the first step, we take an element r of $\mathbb{Q}$ such that $\alpha = r + \mathbb{Z}$.

In the second step we add on 1 enough to r or we subtract on 1 enough to r .

Consequently, r is been chosen without any rule. Actually, q is equal to $r - \llbracket r \rrbracket$, here $\llbracket \cdot \rrbracket$ is a greatest integer function. Students understand clearly this element of q is a unique with the above proof. So q is a unique element of set $\alpha \cap [0,1) \cap \mathbb{Q}$. For every element α , we can choose $r \in \alpha$.

In the axiom of sets, these choosing structures create a set. Since $|\alpha \cap [0,1) \cap \mathbb{Q}| = 1$ for every element α of $\mathbb{Q}/\mathbb{Z}$. Finally, we find a choice function f of the set $\mathbb{Q}/\mathbb{Z}$ such that for every $\alpha \in \mathbb{Q}/\mathbb{Z}$, $f(\alpha) = q \in \alpha$. In this way, we show to students that they can find a choice function on infinite sets.

Application of Axiom of Choice and Well ordered Infinite Sets to Learning Groups

The effects of the teaching methods were assessed by comparing the performance of prospective teachers who participated in a formal, traditional course. Two groups of prospective secondary school mathematics teachers studying in a Turkish state university participated in the study. Ten prospective mathematics teachers participated in a formal course for teaching choice function of axioms of sets and well ordered infinite sets and ten prospective mathematics teachers participated in traditional teaching course.

In determining the groups totaling twenty mathematics teachers, minimum requirements Identification test for understanding axioms of choice and well ordered infinite set (ACMRIT), which includes the prerequisite concepts required to comprehend axiom of choice such as choice function has been used and the well ordered countably infinite set such as set, finite and infinite set, countably infinite set, well ordered relation has been used. In order for mathematics teachers in Active Learning Group to explain these concepts more clearly, (Cohen 2008; Jech 1971; Hungerford 1974; Kuratowski et al 1968; Jech 1997; Erdetón,1977) source books were proposed.

According to the results of ACMRIT, mathematics teachers were ordered from top down using their achievement scores and first mathematics teacher was assigned to first group, second teacher to second group, third mathematics teacher to first group, fourth mathematics teacher to second group and so on, and two groups were formed. Groups then were randomly assigned to one of the control and experimental groups. Distribution of groups is shown in Table 1.

Table 1: Distribution of Sample According to Groups and Gender

Gender	Active Learning Group	Traditional Learning Group	Total
Female	5	6	11
Male	5	4	9
Total	10	10	20

This study is a randomized pretest-posttest experimental design. Pretest-posttest research method with control group was employed in this study. Before teaching the subject to groups using two different methods, axiom of choice and well ordered countably infinite sets achievement test (ACWOISAT) was administered to both groups as the pretest. After the pretest, both groups were taught by the same instructor. In the control group, traditional, formal instructional methods were used, with time to time question and answer and whole class discussion sessions. After the teaching period, ACWOISAT was administered to both groups as the posttest.

Active Learning Based Course in Well Ordered Countably Infinite Sets

It is not easy problem to well order of the infinite set. For example the set of every subset of the set $\mathbb{N}$ of natural numbers, i.e. $P(\mathbb{N})$ power set. If we want to totally order $P(\mathbb{N})$, the question come to mind elements of $P(\mathbb{N})$ has to compare as far as which relation is concerned. Mathematically, in the set $P(\mathbb{N}) = \{A \mid A \subseteq \mathbb{N}\}$, we establish a relation $A \leq B \Leftrightarrow ?$ for every $A, B \in P(\mathbb{N})$. Can we get a result from here? Argumentative. However, there is a very important method used to alleviate this problem. It is also associated each set of natural number with a real number between 0 and 1. That is, to encode every natural number with a real number between $[0,1]$. Thus, using the attached real numbers, the natural numbers will be totally ordered. Say $A \in P(\mathbb{N})$ is a prime number set. To find the real number attached to this set, start counting from 0 and if the natural numbers are prime numbers, let's code 0 if there is no prime number.

Table 2: Real Number Encoding Prime Numbers

Non-prime:	0	1		4	6		8	9	10	12		14	15	16		18	20				
	↓	↓		↓	↓		↓	↓	↓	↓		↓	↓	↓		↓	↓				
	0,0	0	1	1	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0...
			↑	↑		↑	↑				↑	↑				↑	↑				
Prime:				2	3		5	7				11	13				17	19			

With this method, $\emptyset \in P(\mathbb{N})$ is set to 0, $\mathbb{N} \in P(\mathbb{N})$ is set to 0,1111 ... and even number set is set to 0,101010 Say $n(A)$ to the number corresponding to a natural number set A . If $A \subseteq \mathbb{N}$ (or $A \in P(\mathbb{N})$) and $i \in \mathbb{N}$, we get

$$A(i) = \begin{cases} 0, & \text{if } i \notin A \\ 1, & \text{if } i \in A \end{cases} \quad (1)$$

piecewise function (1).

$$n(A) = \sum_{i=0}^{\infty} \frac{A(i)}{10^i} \quad (2)$$

Thus the number corresponding to a natural number set A as is equality (2). We can compare the natural number sets A and B according to the size of the corresponding numbers. In the following, we define a relation (3) on family of natural number sets.

$$A \leq B \Leftrightarrow n(A) \leq n(B) \quad (3)$$

So the natural number sets are totally ordered. A set that is equivalent to a set of natural numbers, put another way, a set that is one to one with a set of natural numbers, that is to say, is a countable infinite set. Every infinite subset of countably infinite set $\mathbb{N}$ of natural number is a countably infinite set. Any infinite subset of an infinite set that can be counted in general is infinite.

Totally ordered sets with the smallest element are called well ordered sets. We know that the set $\mathbb{N}$ of natural numbers is well ordered. Our aim is to a well order a countably infinite set. Let X be countably infinite set. Then there exists a one to one and surjective function $f: X \rightarrow \mathbb{N}$. Using this

one to one correspondence with $\mathbb{N}$ is a well ordered set, can we sort X ? For this, we can see the $f(x) \in \mathbb{N}$ element as the number of $x \in X$. For every $x, y \in X$,

$$x \leq y \Leftrightarrow f(x) \leq f(y) \quad (4)$$

So we move the order of $\mathbb{N}$ to X exactly. As a result, we can list the set with a one to one correspondence with a well ordered set. So far, we have achieved that every infinite subset of $\mathbb{N}$ can be well ordered. We can expand the set $\mathbb{N}$ a little more. $\mathbb{N}$ is a countably infinite set and the set $\mathbb{N} \times \mathbb{N}$ is countably infinite. Then there exists a one to one and surjective function $g: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. Non-negative rational numbers set $\mathbb{Q}^{\geq 0}$ and rational numbers set $\mathbb{Q}$ are countably infinite sets. Here, with the help of one to one correspondence $\mathbb{N} \rightarrow \mathbb{N} \times (\mathbb{N} \setminus \{0\})$ the first to be seen as $\mathbb{N} \times (\mathbb{N} \setminus \{0\})$ is a countably infinite set. Then for every element $(a, b) \in \mathbb{N} \times (\mathbb{N} \setminus \{0\})$, $h(a, b) = \frac{a}{b}$ defined function $h: \mathbb{N} \times (\mathbb{N} \setminus \{0\}) \rightarrow \mathbb{Q}^{\geq 0}$ is a surjective function. Since $\mathbb{N} \times (\mathbb{N} \setminus \{0\})$ is countably infinite, $h(\mathbb{N} \times (\mathbb{N} \setminus \{0\})) = \mathbb{Q}^{\geq 0}$ is a countably infinite set. By similar method, it can be shown that negative rational number set $\mathbb{Q}^{< 0}$ is countably infinite. Since union of a finite number of countably infinite sets is a countably infinite set, $\mathbb{Q} = \mathbb{Q}^{\geq 0} \cup \mathbb{Q}^{< 0}$ is a countably infinite set. Let's go back to the well ordered set method we gave for countably infinite sets. Power set $\mathbb{N} \times \mathbb{N}$ and the set $\mathbb{Q}$ of rational numbers are well ordered countably infinite sets. A little bit further, let us put forward the idea that each set can be well ordered. For this, we need to accept the theory (ZF) in Zermelo-Fraenkel set theory (ZFC) and Zorn lemma. So we need Zorn lemma or Choice axiom to sort out the totally ordered set $P(\mathbb{N})$ which we first sampled.

Expectations of Above Application

Most of the questions were about the specific teaching techniques used. Responses to the following questions have been used:

- Did using this application help you understand the concepts?
- Did you find the application of choice function?
- Would you have preferred to study group theory without axioms of sets?
- Do you find it easier to understand abstract ideas when you see a practical example?
- Did being in a support group help you learn the application for group theory?
- Do you like the partially printed notes?

These questions had three possible responses (yes, no or sometimes) and were followed by questions asking why they had responded the way they had. There was also an open question about why they chose Abstract Algebra as an elective along with questions asking whether they attended most of the lectures and why this was the case.

In addition to this, evaluation has taken place by consultation with the moderator, who has a background in teaching pure mathematics to these students. This was done through a process of peer observation and discussions. The attendance of the students and the marks they obtained have also been used when evaluating the success of the set theory part of axiom of choice.

Evaluation

By application of brainstorming technique, students were asked to define the concept of one to one correspondence, totally ordered set, countably infinite sets. Students freely expressed their ideas and stated the definitions, which they regarded as true.

Brainstorming technique was employed from time to time during the later stages of instruction. Then they listened to the lecture presentation on teaching well ordered infinite set. They told their students in their schools what they actively learned in the classroom environment.

If we gather what the teacher in the Active Learning Group have acquired in the classroom. When asked for open responses, the teachers stated several different reasons for their high attendance at the sessions. In Active Learning Group sessions, several mentioned personal motivation (seventy percent of the students) and wanting to do well by means of this application (sixty percent of the students). Some mentioned that they attend because they enjoy this application and find it interesting (eighty percent of the students).

Minority said they did not want to fall behind and feared that they would struggle to catch up (twenty percent of the students). In Traditional Learning Group, the situation isn't so welcoming. Only two of ten mathematics teachers are satisfied with this situation. So in Traditional Learning Group about twenty percent of the students success can be achieved.

In this application they get every countably infinite sets via this application. Firstly they learn well ordered sets via this study and they can comprehend the basis of axiom of choice via this problem. Secondly special attention is given to the teaching of this subject in this study and they can study the differences and similarities between finite and infinite sets, types of infinite sets, choice function on sets and the proof of countability of union of finite countable or infinite sets.

This situation shows the success of the method developed in the article. It provides great convenience to the study of infinite sets theory. With the help of this method, it will enable the mathematics teachers who will gain good gains to future generalizations. Above all, it conceptualizes mathematics beyond memorization. Secondary school students who understand the importance of set theory can classify all the concepts they will encounter throughout their life in a concrete way. They can also add some kind of theory to the concepts they classifies.

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