Turkish Studies *Information Technologies and* *Applied Sciences*

Volume 14 Issue 2, 2019, p. 167-177

DOI: 10.29228/TurkishStudies.22692

ISSN: 2667-5633

Skopje/MACEDONIA-Ankara/TURKEY

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Research Article / Araştırma Makalesi

Article Info / Makale Bilgisi

✍ *Received/Geliş:* 02.04.2019

✓ *Accepted/Kabul:* 10.06.2019

✍ *Report Dates/Rapor Tarihleri:* Referee 1 (25.04.2019)- Referee 2 (30.04.2019)

This article was checked by iThenticate.

A RECENT GENERALIZATION OF RAD-SUPPLEMENTING MODULES

Figen ERYILMAZ - Esra ÖZTÜRK SÖZEN***

ABSTRACT

In this study, generalized δ -supplementing and ample generalized δ -supplementing modules are defined as a new generalization of Rad-supplementing modules and modules with the property $(\delta-E)$ (briefly δ -supplementing) which were studied in (Özdemir, 2016) and (Sözen et. al., 2017) respectively and some basic properties of them are investigated. While preparing this paper besides the article (Özdemir, 2016), the articles (Çalışıcı et al., 2012 and Türkmen, 2013) related with modules having a supplement and Rad-supplement in every cofinite extension are used. Generalized δ -supplementing modules are of course a generalization of injective modules such as Zöschinger's modules with the property (E) . It is shown that δ -supplementing and δ -radical modules are both generalized δ -supplementing. In general, generalized δ -supplementing modules need not be δ -supplementing. We give an example which supports this reality. It is proved that every direct summand of a generalized δ -supplementing module is generalized δ -supplementing. Moreover, it is proved that generalized δ -supplemented modules are preserved under extensions below some special conditions. Following this, it is given that an immediate consequence such that every module with composition series is generalized δ -supplementing. Moreover, it is pointed that the cases

* Dr. Öğr. Üyesi, Ondokuz Mayıs Üniversitesi, E-posta: fyuzbasi@omu.edu.tr

** Dr. Öğr. Üyesi, Sinop Üniversitesi, E-posta: esozen@sinop.edu.tr

being generalized δ -supplementing and injectivity are coincide for modules over δ - V -rings. It is wondered whether the factor module of a generalized δ -supplementing module is also generalized δ -supplementing and so it is found a positive answer under a special condition. It is also proved that for a module M , the necessary and sufficient condition of being ample generalized δ -supplementing is that every submodule of M is generalized δ -supplementing. As a result, it is obtained that every ample generalized δ -supplementing module is a generalized δ -supplemented module.

Keywords: δ -supplement, generalized δ -supplement, supplementing module.

2010 Mathematics Subject Classification: 16D10, 16L30, 16D90

STRUCTURED ABSTRACT

In this study, generalized δ -supplementing and ample generalized δ -supplementing modules are defined as a new generalization of Rad-supplementing modules and modules with the property $(\delta-E)$ (briefly δ -supplementing) which were studied in (Özdemir, 2016) and (Sözen et. al., 2017) respectively and some basic properties of them are investigated. While preparing this paper besides the article (Özdemir, 2016), the articles (Çalışıcı et al., 2012 and Türkmen, 2013) related with modules having a supplement and Rad-supplement in every cofinite extension are used. Throughout the paper, R will be an associative ring with identity and all modules are unital left R -modules unless otherwise specified. Generalized δ -supplementing modules are of course a generalization of injective modules such as Zöschinger's modules with the property (E) .

By (Zhou, 2000), a submodule L of M is called δ -small in M denoted by $L \ll_{\delta} M$, if for any submodule N of M with $\frac{M}{N}$ singular, $M = N + L$ implies that $M = N$. The sum of δ -small submodules of a module M is denoted by $\delta(M)$. Let K, N be submodules of M . Then N is called a δ -supplement of K in M , if $M = N + K$ and $N \cap K \ll_{\delta} N$. A module M is called δ -supplemented if every submodule of M has a δ -supplement in M (Koşan, 2007) and (Wang, 2007). On the other hand, the submodule N is said to have ample δ -supplements in M if every submodule L of M with $M = N + L$ contains a δ -supplement of N in M . The module M is called amply δ -supplemented if every submodule of M has ample δ -supplements in M (Tribak, 2012).

Let N and K be any submodules of M with $M = N + K$. If $N \cap K \subseteq \delta(N)$ then N is called a generalized δ -supplement of K in M . Following (Talebi et.al., 2009), M is called a generalized δ -supplemented module (briefly δ -GS-module) if every submodule N of M has a generalized δ -supplement K in M . Also, in (Talebi et.al., 2009) M is called a generalized amply δ -supplemented module (briefly δ -GAS-module) if whenever $M = N + K$ for submodules N, K of M then N contains a generalized δ -supplement of K in M .

Modules that have supplements (ample supplements) in every module in which it is contained as a submodule are called modules with the property (E)((EE)) in (Zöschinger, 1974). Such modules are also called supplementing modules in (Clark et. al., 2006). In a recent paper, (Çalışıcı and Türkmen, 2012), modules that have a supplement in every cofinite extension have been studied and these modules are called modules with the property (CE)((CEE)). After this paper, generalization of these modules has been studied in (Türkmen, 2013). Another type of these modules were studied in (Türkmen et. al., 2018) and (Sözen et. al., 2017). In addition, (ample) Rad-supplementing were defined in (Özdemir, 2016). By considering these concepts, we define and study (ample) generalized δ -supplementing modules. A module M is called generalized δ -supplementing if it has a generalized δ -supplement in each module in which it is contained as a submodule. Similarly, a module M is called ample generalized δ -supplementing if it has an ample generalized δ -supplement in each module in which it is contained as a submodule where a submodule $U \subseteq M$ has ample generalized δ -supplements in M for every $L \subseteq M$ with $M = U + L$, there is a generalized δ -supplement L_1 of U with $L_1 \subseteq L$. Some properties of these modules are investigated. It is shown that δ -supplementing and δ -radical modules are both generalized δ -supplementing. In general, generalized δ -supplementing modules need not be δ -supplementing. We give an example which supports this reality. It is proved that every direct summand of a generalized δ -supplementing module is generalized δ -supplementing. Moreover, it is proved that generalized δ -supplemented modules are preserved under extensions below some special conditions. Following this, it is given that an immediate consequence such that every module with composition series is generalized δ -supplementing. Moreover, it is pointed that the cases being generalized δ -supplementing and injectivity are coincide for modules over δ -V-rings. It is wondered whether the factor module of a generalized δ -supplementing module is also generalized δ -supplementing and so it is found a positive answer under a special condition. It is also proved that for a module M , the necessary and sufficient condition of being ample generalized δ -supplementing is that every submodule of M is generalized δ -supplementing. As a result, it is obtained that every ample generalized δ -supplementing module is a generalized δ -supplemented module.

Keywords: δ – supplement, generalized δ – supplement, supplementing module.

2010 Mathematics Subject Classification: 16D10, 16L30, 16D90

RAD-TÜMLEYENLİ MODÜLLERİN YENİ BİR GENELLEŞTİRİLMESİ

ÖZ

Bu çalışmada sırasıyla (Özdemir, 2016) ve (Sözen ve ark., 2017) kaynaklarında çalışılan Rad-tümleyenli ve $(\delta - E)$ özelliğine sahip (kısaca δ – tümleyenli) modüllerin yeni bir genelleştirmesi olarak genelleştirilmiş δ – tümleyenli ve bol genelleştirilmiş δ – tümleyenli modüller tanımlandı ve bazı temel özellikleri incelendi. Bu çalışma hazırlanırken (Özdemir, 2016) makalesinin yanısıra her dual sonlu genişlemesinde tümleyene ve Rad-tümleyene sahip modüller (Çalışıcı ve ark., 2012 ve Türkmen, 2013) makalelerinden yararlanıldı. Genelleştirilmiş δ – tümleyenli modüller tıpkı Zöschinger’ in (E) özelliğine sahip modülleri gibi injektif modüllerin bir genelleştirilmesidir. δ – tümleyenli ve δ – radikal modüllerin genelleştirilmiş δ – tümleyenli modül olduğu gösterildi. Genel olarak, genelleştirilmiş δ – tümleyenli modüller δ – tümleyenli değildir. Bu iddiayı doğrulayan bir örneğe yer verildi. Genelleştirilmiş δ – tümleyenli bir modülün her bir direkt toplam teriminin de genelleştirilmiş δ – tümleyenli olduğu ispatlandı. Dahası, genelleştirilmiş δ – tümleyenli modüllerin genişlemeler altında bazı özel şartlar altında korunduğu ispatlandı. Buna dayanarak, kompozisyon serisine sahip her modülün genelleştirilmiş δ – tümleyenli olduğu verildi. Ayrıca, $\delta - V$ – halka üzerine kurulan modüllerin injektifliği ile genelleştirilmiş δ – tümleyenli kavramının çakıştığı görüldü. Genelleştirilmiş δ – tümleyenli modüllerin bölüm modüllerinin de genelleştirilmiş δ – tümleyenli olup olmadığı sorgulandı ve özel bir şart altında bunun doğruluğu bulundu. Bir M modülünün bol genelleştirilmiş δ – tümleyenli olması için gerekli ve yeterli koşulun M nin her alt modülünün genelleştirilmiş δ – tümleyenli olması gerektiği ispatlandı. Bunun sonucu olarak da her bol genelleştirilmiş δ – tümleyenli modülün genelleştirilmiş δ – tümleyenli modül olduğu elde edildi.

Anahtar Kelimeler: δ – tümleyen, genelleştirilmiş δ – tümleyen, tümleyenli modül.

1. Introduction

Throughout the paper, R will be an associative ring with identity and all modules are unital left R – modules unless otherwise specified. Let M be an R – module. By $N \subseteq M$ we mean that N

is a submodule of M or M is an extension of N . Recall that a submodule $N \subseteq M$ is called *small*, denoted by $N \ll M$, if $N + L \neq M$ for all proper submodules L of M , and that $L \subseteq M$, is said to be *essential* in M , denoted by $L \leq M$, if $L \cap K \neq 0$ for each nonzero submodule $K \subseteq M$. A module M is said to be *singular* if $M \cong \frac{N}{L}$ for some module N and a submodule $L \subseteq N$ with $L \leq N$. $Rad(M)$ will indicate Jacobson radical of M . M is called *supplemented*, if every submodule N of M has a *supplement* in M , i.e. a submodule K minimal with respect to $N + K = M$. K is a supplement of N in M if and only if $N + K = M$ and $N \cap K \ll K$ (Wisbauer, 1991).

By (Zhou, 2000), a submodule L of M is called δ -small in M denoted by $L \ll_{\delta} M$, if for any submodule N of M with $\frac{M}{N}$ singular, $M = N + L$ implies that $M = N$. The sum of δ -small submodules of a module M is denoted by $\delta(M)$. It is easy to see that every small submodule of a module M is δ -small in M , so $Rad(M) \subseteq \delta(M)$ and $Rad(M) = \delta(M)$ if M is singular. Also any non-singular semisimple submodule of M is δ -small in M and δ -small submodules of a singular module are small submodules. For more detailed discussion on δ -small submodules, we refer to (Zhou, 2000).

Let K, N be submodules of M . Then N is called a δ -supplement of K in M , if $M = N + K$ and $N \cap K \ll_{\delta} N$. A module M is called δ -supplemented if every submodule of M has a δ -supplement in M (Koşan, 2007) and (Wang, 2007). On the other hand, the submodule N is said to have *ample δ -supplements* in M if every submodule L of M with $M = N + L$ contains a δ -supplement of N in M . The module M is called *amply δ -supplemented* if every submodule of M has ample δ -supplements in M (Tribak, 2012).

Let N and K be any submodules of M with $M = N + K$. If $N \cap K \subseteq \delta(N)$ then N is called a generalized δ -supplement of K in M . Following (Talebi et.al., 2009), M is called a *generalized δ -supplemented* module (briefly δ -GS-module) if every submodule N of M has a generalized δ -supplement K in M . Also, in (Talebi et.al., 2009) M is called a *generalized amply δ -supplemented* module (briefly δ -GAS-module) if whenever $M = N + K$ for submodules N, K of M then N contains a generalized δ -supplement of K in M .

Let M be an R -module and N, K be any submodules of M with $M = N + K$. If $N \cap K \subseteq Rad(K)$ then K is called a generalized supplement of N in M . Following (Wang et. al., 2006), M is called *generalized supplemented* module (briefly GS-module) if every submodule N of M has a generalized supplement K in M . For characterizations of generalized supplemented modules we refer to (Wang et. al., 2006) and (Xue, 1996).

Modules that have supplements (ample supplements) in every module in which it is contained as a submodule are called modules with the property (E)((EE)) in (Zöschinger, 1974). Such modules are also called supplementing modules in (Clark et. al., 2006). In a recent paper, (Çalışıcı and Türkmen, 2012), modules that have a supplement in every cofinite extension have been studied and these modules are called *modules with the property (CE)((CEE))*. After this paper, generalization of these modules has been studied in (Türkmen, 2013). Another type of these modules were studied in (Türkmen et. al., 2018) and (Sözen et. al., 2017). In addition, (ample) Rad-supplementing were defined in (Özdemir, 2016). By considering these concepts, we define and study (ample) generalized δ -supplementing modules. A module M is called *generalized δ -supplementing* if it has a

generalized δ -supplement in each module in which it is contained as a submodule. Similarly, a module M is called *ample generalized δ -supplementing* if it has an ample generalized δ -supplement in each module in which it is contained as a submodule where a submodule $U \subseteq M$ has ample generalized δ -supplements in M for every $L \subseteq M$ with $M = U + L$, there is a generalized δ -supplement L_1 of U with $L_1 \subseteq L$. Some properties of these modules are investigated.

2. Generalized δ -Supplementing Modules

Recall that a module M is called δ -radical if $\delta(M) = M$ and the sum of all δ -radical submodules of the module M is denoted by $P_\delta(M)$, that is, $P_\delta(M) = \sum \{U \subseteq M : \delta(U) = U\}$ (Talebi et.al., 2009).

Proposition 2.1. δ -supplementing modules and δ -radical modules are generalized δ -supplementing.

Proof. Let M be a module and N be any extension of M . If M is δ -supplementing, then it has a δ -supplement, and so a generalized δ -supplement in N . Hence M is generalized δ -supplementing. In the second case, if $\delta(M) = M$, then N is a generalized δ -supplement of M in N .

It is clear that, for any module M , $P_\delta(M)$ is the largest δ -radical submodule of M . So we have the following corollary.

Corollary 1. For a module M , $P_\delta(M)$ is generalized δ -supplementing.

Now we show a kind of a module which is generalized δ -supplementing but need not be δ -supplementing.

Example 1. Let R be the ring of polynomials over a field F in countably many commuting indeterminates $x_1, x_2, \dots$ modulo the ideal generated by $\{x_1^2, x_2^2 - x_1, x_3^2 - x_2, \dots\}$. As observed in (Example 4.4, Zhou, 2000) R is a local ring with the unique maximal ideal $Jac(R) = J = J^2 = \langle \{x_1, x_2, \dots\} \rangle / \langle \{x_1^2, x_2^2 - x_1, x_3^2 - x_2, \dots\} \rangle$. Let $M = J^{(\mathbb{N})}$. Since $\delta(M) = M$, M is δ -radical and so generalized δ -supplementing. On the other hand, M does not have a δ -supplement in $R^{(\mathbb{N})}$ by (Example 2.8, Tribak, 2015). So M is not δ -supplementing.

Proposition 2.2. Every direct summand of a generalized δ -supplementing module is generalized δ -supplementing.

Proof. Let M be a generalized δ -supplementing module, U be a direct summand of M and N be any extension of U . Then $M = A \oplus U$ for some submodule $A \subseteq M$. By the hypothesis, M has a generalized δ -supplement in the module $A \oplus N$ containing M , that is, there exists a submodule V of $A \oplus N$ such that $(A \oplus U) + V = A \oplus N$ and $(A \oplus U) \cap V \leq \delta(V)$. Let $g : A \oplus N \rightarrow N$ be a projection onto N . Then we get

$$U + g(V) = g(A \oplus U) + g(V) = g((A \oplus U) + V) = g(A \oplus N) = N$$

and

$$U \cap g(V) = g((A \oplus U) \cap V) \subseteq g(\delta(V)) \subseteq \delta(g(V))$$

by (Lemma 1.5, Zhou, 2000). So, $g(V)$ is a generalized δ -supplement of U in N . Therefore U is a generalized δ -supplementing module.

Proposition 2.3. Let M be a module. Then the following are equivalent.

- (i) M has a generalized δ -supplement in every essential extension.
- (ii) M has a generalized δ -supplement in its injective envelope $E(M)$.

Proof. (i) $\Rightarrow$ (ii) It is clear.

(ii) $\Rightarrow$ (i) Let $M \subseteq N$ with $M \cap N = 0$. Let $f: M \rightarrow N$ and $g: M \rightarrow E(M)$ be inclusion maps. Since $E(M)$ is injective, there exists a homomorphism $h: N \rightarrow E(M)$ such that $h \circ f = g$. Equivalently, we have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow g & \square & h \\ E(M) & & \end{array}$$

Now, we show that h is a monomorphism. Let $a \in \text{Ker}(h)$. If $a \in M$, then $0 = g(a) = (hf)(a) = h(a)$ and so $a \in \text{Ker}(g) = 0$. It means that $M \cap \text{Ker}(h) = 0$. Since $M \cap N = 0$, $\text{Ker}(h) = 0$, and hence h is a monomorphism. By the hypothesis, M has a generalized δ -supplement in $E(M)$, say K , that is, $E(M) = M + K$ and $M \cap K \subseteq \delta(K)$. Since $M \subseteq h(N)$, we obtain that

$$h(N) = h(N) \cap E(M) = h(N) \cap (M + K) = M + (K \cap h(N)).$$

Now, taking any $n \in N$, we have $h(n) = m + h(n_1) = h(m + n_1)$ where $m \in M$ and $h(n_1) \in h(N) \cap K$. So, $n = m + n_1 \in M + h^{-1}(K)$ since h is monic and thus $N = M + h^{-1}(K)$. Moreover, $M \cap h^{-1}(K) = h^{-1}(M \cap K) \subseteq h^{-1}(\delta(K)) \subseteq \delta(h^{-1}(K))$ since $h^{-1}(M) = M$. Hence $h^{-1}(K)$ is a generalized δ -supplement of M in N .

Proposition 2.4. Let B a module and A be a submodule of B . If A and $\frac{B}{A}$ are generalized δ -supplementing, then so is B .

Proof. Let N be any extension of B . By the hypothesis, there is a generalized δ -supplement $\frac{V}{A}$ of $\frac{B}{A}$ in $\frac{N}{A}$ and a generalized δ -supplement W of A in V . We claim that W is a generalized δ -supplement of B in N . We have epimorphisms $f: W \rightarrow \frac{V}{A}$ and $g: \frac{V}{A} \rightarrow \frac{N}{B}$ such

that $\text{Ker}(f) = W \cap A \subseteq \delta(W)$ and $\text{Ker}(g) = \left(\frac{V}{A}\right) \cap \left(\frac{B}{A}\right) \subseteq \delta\left(\frac{V}{A}\right)$. Then $g \circ f : W \rightarrow \frac{N}{B}$ is an epimorphism such that $W \cap B = \text{Ker}(g \circ f) \subseteq \delta(W)$ by Lemma 2.11 in (Yüzbaşı et. al., 2013). It is clear that $N = V + B = (W + A) + B = W + B$. Hence B is generalized δ -supplementing.

Corollary 2. Let B a module and $A = \delta(A)$. If $\frac{B}{A}$ is generalized δ -supplementing, then so is B .

Corollary 3. If M_1 and M_2 are generalized δ -supplementing modules, then so is $M_1 \oplus M_2$.

Proof. Consider the short exact sequence $0 \rightarrow M_1 \rightarrow M_1 \oplus M_2 \rightarrow M_2 \rightarrow 0$. Thus the result follows from the above proposition.

Lemma 2.1. Every simple submodule S of a module M is either a direct summand of M or small in M (Özdemir, 2016).

Proposition 2.5. Every simple module is (generalized) δ -supplementing.

Proof. Let S be a simple module and N be any extension of S . Then by the above lemma, $S \sqsubseteq N$ or $N = S \oplus S_1$ for a submodule $S_1 \subseteq N$. In the first case, N is a (generalized) δ -supplement of S in N and in the second case, S_1 is a (generalized) δ -supplement of S in N . So, in each case S has a (generalized) δ -supplement in N , that is, S is (generalized) δ -supplementing.

Recall from (Wisbauer, 1991) that a normal series $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots \subseteq M_k = M$ is called a *composition series* of M if all factors $\frac{M_i}{M_{i-1}}$ are simple modules.

Theorem 2.1. Every module with composition series is (generalized) δ -supplementing.

Proof. Let $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots \subseteq M_n = M$ be a composition series of a module M . The proof is by induction on $n \in \mathbb{N}$. If $n = 1$, then $M = M_1$ is simple and so M is (generalized) δ -supplementing by Proposition 2.5. Suppose that this is true for each $k \leq n-1$. Then M_{n-1} is (generalized) δ -supplementing. Since $\frac{M_n}{M_{n-1}}$ is also (generalized) δ -supplementing as a simple module, we obtain by Proposition 2.4 that $M = M_n$ is (generalized) δ -supplementing.

In (Ungor et. al., 2014) a ring R is said to be *left δ -V-ring* if for any left R -module M , $\delta(M) = 0$.

Proposition 2.6. Let R be a δ -V-ring. Then an R -module M is generalized δ -supplementing if and only if M is injective.

Proof. ($\Rightarrow$) Suppose that M is a generalized δ -supplementing module. Let $M \subseteq N$ be any extension of M . Then there is a generalized δ -supplement V of M in N , that is, $N = M + V$ and $M \cap V \subseteq \delta(V)$. Since $\delta(V) = 0$ by (Ungor et. al., 2014), we have $N = V \oplus M$.

($\Leftarrow$) If M is injective and $M \subseteq N$ be any extension of M , then $N = M \oplus K$ for some submodule $K \subseteq N$. It is clear that K is a generalized δ -supplement of M in N . And so, M is generalized δ -supplementing.

Lemma 2.2. (see Zöschinger, 1974, Lemma 1.3 (a)) Let $A \subseteq B \subseteq C$ be modules with $\frac{C}{A}$ injective and N be a module containing $\frac{B}{A}$. Then there exists a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \rightarrow & B & \rightarrow & \frac{B}{A} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A & \rightarrow & P & \rightarrow & N \rightarrow 0 \end{array}$$

Proposition 2.7. Let $A \subseteq B \subseteq C$ be modules with $\frac{C}{A}$ injective. If B is generalized δ -supplementing so is $\frac{B}{A}$.

Proof. Let N be any extension of $\frac{B}{A}$. By Lemma 2.2, we have the following commutative diagram with exact rows since $\frac{C}{A}$ is injective:

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \rightarrow & B & \xrightarrow{\sigma} & \frac{B}{A} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow f \\ 0 & \rightarrow & A & \rightarrow & P & \xrightarrow{g} & N \rightarrow 0 \end{array}$$

Since h is monic and B is generalized δ -supplementing, $B \cong \text{Im}(h)$ has a generalized δ -supplement in P , say V , that is, $P = \text{Im}(h) + V$ and $\text{Im}(h) \cap V \subseteq \delta(V)$. We claim that $g(V)$ is a generalized δ -supplement of $\frac{B}{A}$ in N . Therefore

$$N = g(P) = g(h(B)) + g(V) = (f\sigma)(B) + g(V) = \left(\frac{B}{A}\right) + g(V)$$

and

$$\left(\frac{B}{A}\right) \cap g(V) = f(\sigma(B)) \cap g(V) = g[h(B) \cap V] \subseteq g(\delta(V)) \subseteq \delta(g(V)).$$

Proposition 2.8. A module M is ample generalized δ -supplementing if and only if every submodule of M is generalized δ -supplementing.

Proof. ($\Leftarrow$) Let M be a module and N be any extension of M . Suppose that for a submodule $X \subseteq N$, $N = X + M$. By the hypothesis the submodule $X \cap M$ of M has a generalized δ -supplement V in X containing $X \cap M$, that is $X = (X \cap M) + V$ and $(X \cap M) \cap V \subseteq \delta(V)$. Then

$$N = M + X = M + (X \cap M) + V = M + V \text{ and } M \cap V = M \cap (V \cap X) \subseteq \delta(V).$$

Hence V is a generalized δ -supplement of M in N such that $V \subseteq X$.

($\Rightarrow$) Let U be a submodule of M and N be any submodule containing U . We can draw the pushout for the inclusion homomorphisms

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & F \\ \uparrow i_2 & & \uparrow \beta \\ U & \xrightarrow{i_1} & N \end{array}$$

where $i_1 : U \rightarrow N$ and $i_2 : U \rightarrow M$.

In the diagram, α and β are also monomorphisms by the properties of pushout [see, Rotman, 2009, Exercise 5.10]. Let $M_1 = \text{Im}(\alpha)$ and $N_1 = \text{Im}(\beta)$. Then $F = M_1 + N_1$ by the properties of pushout. So by the hypothesis, $M_1 \cong M$ has a generalized δ -supplement V in F such that $V \subseteq N_1$, $M_1 + V = F$ and $M_1 \cap V \subseteq \delta(V)$. Then we have $N_1 = N_1 \cap F = N_1 \cap (M_1 + V) = (M_1 \cap N_1) + V$ and $(M_1 \cap N_1) \cap V = M_1 \cap V \subseteq \delta(V)$. Therefore, V is a generalized δ -supplement of $M_1 \cap N_1$ in N_1 . Since $\beta : N \rightarrow F$ is a monomorphism with $N_1 = \text{Im}(\beta)$, we have an isomorphism $\tilde{\beta} : N \rightarrow N_1$ defined as $\tilde{\beta}(x) = \beta(x)$ for all $x \in N$. By this isomorphism, since V is a generalized δ -supplement of $M_1 \cap N_1$ in N_1 , we obtain $\tilde{\beta}^{-1}(V)$ is a generalized δ -supplement of $\tilde{\beta}^{-1}(M_1 \cap N_1)$ in $\tilde{\beta}^{-1}(N_1)$. It can be easily shown that $\tilde{\beta}^{-1}(V) = \beta^{-1}(V)$, $\tilde{\beta}^{-1}(N_1) = N$ and $\tilde{\beta}^{-1}(M_1 \cap N_1) = U$. So $\beta^{-1}(V)$ is a generalized δ -supplement of U in N .

Corollary 4. Every ample generalized δ -supplementing module is both generalized δ -supplementing and generalized δ -supplemented.

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