Lagrangian Mechanical Systems Using Para-Quaternionic *Kähler* Manifolds

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Abstract:-

In this paper, we obtain Euler-Lagrange equations for mechanical systems using a canonical local basis $\{F, G, H\}$ of V that they defined on a generalized-quaternionic *kähler* manifold (M, g, V) .

1 – Introduction:-

Today; many branches of science are in our lives. One the branches is mathematics that has multiple applications. In particular, differential geometry and mathematical physics have a lot of different applications. One of them are on geodesics. Geodesics are known the shortest route between two points. Time-dependent equations of geodesics can be easily found with the help of the Euler-Lagrange equations. We can say that differential geometry provides a good working area for studying Lagrangians of classical mechanics and field theory. The dynamic equation for moving bodies is obtained the Lagrangian mechanic. These dynamic equation is illustrated as follows:

Lagrange Dynamics Equation [1,2,3]: let M be an n -dimensional manifold and TM its tangent bundle with canonical projection $TM: TM \rightarrow M$ is called the phase space of velocities of the base manifold M .

Let $L: TM \rightarrow \mathbb{R}$ be differentiable function on TM called the Lagrangian function. We consider the closed 2-form on TM given by $\Phi_L = dd_J L$ (if $J^2 = -I, J$ is a complex structure and if $J^2 = I, J$ is a paracomplex structures, $T_r(J) = 0$) Consider the equation:

$$i_X \Phi_L = dE_L \quad \rightarrow \quad (1)$$

Then X is a vector field, we shall see that (1) under a certain condition on X is the intrinsical

expression of the Euler-Lagrange equations of motion. This equation is named as Lagrange dynamical equation. We shall see that for motion in potential, $E_L = V(L) - L$ is an energy function and $V = J(X)$ a Liouville vector field. Here dE_L denotes the differential of E . The triple (TM, Φ_L, X) is known as Lagrangian system on the tangent bundle TM . If it is continued the operations on (1) for any coordinate system $(q^i(t), p_i(t))$, infinite dimension Lagrange's equation is obtained the form below:

$$\frac{dq^i}{dt} = \dot{q}^i, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) = \frac{\partial L}{\partial q^i}, \quad i = 1, \dots, n \quad \rightarrow \quad (2)$$

There are many studies about Lagrangian dynamics, mechanics, formalisms, systems and equations (see detail [4]). There are real, complex, paracomplex and other analogues. It is well-known that Lagrangian analogues are very important tools. They have a simple method to describe the model for mechanical systems. The models about mechanical systems are given as follows.

Some examples of the Lagrangian is applied to model the problems include harmonic oscillator, charge Q in electromagnetic fields, Kepler problem of the earth in orbit around the sun, pendulum, molecular and fluid dynamics. LC networks, Atwood's machine, symmetric to etc. Let's remember some work done. Vries shown that the Lagrangian motion equations have a very simple interpretation in relativistic quantum mechanics [5]. Paracomplex analogues of the Euler-Lagrange equations was obtained in the framework of Para-Kählerian manifold and the geometric results on a paracomplex mechanical systems were found by Tekkoyun [6]. Electronic origins, molecular dynamics simulations, computational nanomechanics, multiscale modeling of materials fields were contributed by Liu [7]. Bi-paracomplex analogue of Lagrangian systems was shown on Lagrangian distributions by Tekkoyun and sari [8]. Tekkoyun and Yayli presented generalized-quaternionic Kählerian analogue of Lagrangian and Hamiltonian mechanical systems. Eventually, the geometric-physical results related to generalized-quaternionic Kählerian mechanical systems are provided [9].

Nowadays, there are many studies about Euler-Lagrangian dynamics, mechanics, formalisms, systems and equations [2, 4, 10, 11, 12] and there in. There are real, complex, paracomplex and other analogues. As known it is possible to produce different analogous in different spaces. Quaternions were invented by Sir William Rowan Hamiltonian as an extension to the complex numbers. Hamiltonian's defining relation is most succinctly written as:

$$i^2 = j^2 = k^2 = -1, ijk = -1.$$

Generalized quaternions are defined as

$$i^2 = -a, j^2 = -b, k^2 = -ab, ijk = -ab.$$

If it is compared to the calculus of vectors, quaternions have slipped into the realm of obscurity.

They do however still find use in the computation of rotations. Lots of physical laws in classical, relativistic, and quantum mechanics can be written pleasantly by means of quaternions. Some physicists hope they will find deeper understanding of the universe by restating basic principles in terms of quaternion algebra [13, 14, 15, 16, 17, 18, 19].

In the present paper, we present equations related to Lagrangian mechanical systems on generalized-quaternionic Kähler manifold.

2 – Preliminaries:-

In this study, all the manifolds and geometric objects are C^∞ and the Einstein summation convention is in use. Also, $A, F(TM), \chi(TM)$ and $\Lambda^1(TM)$ denote the set of paracomplex numbers, the set of (para)-complex functions on TM, the set of (para)-complex vector fields on TM and the set of (para)-complex 1-forms on TM, respectively. The definitions and geometric structures on the differential manifold M given in [20] may be extended to TM as follows:

2 – 1 Theorem:-

Let f be differentiable ϕ, ψ are 1-form, then [21]

- $d(f\phi) = df \wedge \phi + f d\phi$
- $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$

2 – 2 Definition (Kronecker's delta):-

Kornecker's delta denote by δ and defined as follows [22]:

$$\delta_i^j = \begin{cases} 1 & , \text{ if } i = j \\ 0 & , \text{ if } i \neq j \end{cases}$$

3 – Conformal Geometry:-

In mathematics, conformal map is a function which preserves angles. In the most common case the function is between domains in the complex plane. Conformal maps can be defined between domains in higher dimensional Euclidean spaces, and more generally on a Riemann or semi-Riemann manifold. Conformal geometry is the study of the set of angle-preserving (conformal) transformations on space. In two real dimensions, conformal geometry is precisely the geometry of Riemann surfaces. In more than two dimensions, conformal geometry may refer either to the study of conformal transformations of “flat” spaces (such as Euclidean spaces or spheres), or more commonly, to the study of conformal manifolds which are Riemann or pseudo-Riemann manifolds with a class of metrics defined up to scale. A conformal manifold is a differentiable manifold equipped with an equivalence class of (pseudo) Riemann metric tensors, in which two metrics $\acute{g}$ and g are equivalent if and only if:

$$\acute{g} = \lambda^2 g \quad \rightarrow \quad (3)$$

Where $\lambda > 0$ is a smooth positive function. An equivalence class of such metrics is known as a conformal metric or conformal class [23].

4 – Conformal Structure:-

The linear distance between two points can be found easily by Riemann metric, which is very useful and is defined inner product. Many scientists have used the Riemann metric. Einstein was one of the first studies in this field. Einstein discovered which the Riemannian geometry and successfully used it to describe General Relativity in the 1910 that is actually a classical theory for gravitation. However, the universe is really completely not like Riemannian geometry. Each path between two points is not always linear. Also, orbits of move objects may change during movement. So, each two points in space may not be linear geodesic and need not to be. Therefore, new metric is needed for non-linear distances like spherical surface. Then, a method is required for converting non-linear distance to linear. Weyl introduced a metric with a conformal transformation in 1918.

Definition 4.1: Let M an n -dimensional smooth manifold. A conformal structure on M is an equivalence class G of Riemann metrics on M . A manifold with a conformal structure is called a conformal manifold

- (i) Two Riemann metrics g and $\acute{g}$ on M are said to be equivalent if and only if

$$\acute{g} = e^\lambda g \quad \rightarrow \quad (4)$$

Where λ is a smooth function on M . the equation given by (4) is called a conformal structure

- (ii) A Weyl structure on M is a map $F: G \rightarrow \Lambda^1 M$ satisfying

$$F(e^\lambda g) = F(g) - d\lambda \quad \rightarrow \quad (5)$$

Where G is a conformal structure. Note that a Riemann metric g and a one-form φ determine a Weyl structure, namely $F: G \rightarrow \Lambda^1 M$ where G is the equivalence class of g and

$$F(e^\lambda g) = \varphi - d\lambda.$$

Theorem 4.1: A connection on the metric bundle φ of a conformal manifold M naturally induces a map $F: G \rightarrow \Lambda^1 M$ and (5), and conversely. Parallel translation of points in φ by the connection is the same as their translation by F [24].

5 – Generalized-Quaternionic Kähler Manifolds:-

A generalized almost quaternion structure on the manifold M is a sub bundle of the bundle of endomorphism's of the tangent bundle M , whose standard fiber is the algebra of quaternions. A generalized almost quaternion structure on a pseudo-Riemannian manifold is called a generalized quaternion-Hermitian if the following conditions hold:

i) The endomorphism's F, G and H of $T_x M$ satisfy

$$F^2 = -aI, G^2 = -bI, H^2 = -abI, FG = H, GH = bF, HF = aG, \rightarrow (6)$$

ii) The compatibility equations are given by for $X, Y \in T_x M$,

$$g(FX, FY) = ag(X, Y), g(GX, GY) = bg(X, Y), g(HX, HY) = abg(X, Y) \rightarrow (7)$$

Where I denotes the identity tensor of type $(1, 1)$ in M . In particular, 2-form Q defined by $Q(X, Y) = (X, FY) = (X, GY) = (X, HY)$ on M is called the Kähler form Q on M is closed, i.e. $dQ = 0$, the manifold M is called a generalized-quaternionic Kähler manifold [25].

If $a = b = 1$, M is quaternion manifold. If $a = 1, b = -1$, M is Para-quaternion manifold. The bundle V is a set that locally admits basis $\{F, G, H\}$ satisfying (6) and (7) in any coordinate neighborhood $U \subset M$ such that $M = \cup U$ [14]. Then V is called a generalized-quaternionic structure in M . The pair (M, V) denotes a generalized-quaternionic manifold with V . The structure V with such a Riemannian metric g is called a generalized-quaternionic metric structure. The triple (M, g, V) denotes a generalized-quaternionic metric manifold. Let $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}, i = \overline{1, n}$ be a real coordinate system on a neighborhood U of M , and let $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{n+i}}, \frac{\partial}{\partial x_{2n+i}}, \frac{\partial}{\partial x_{3n+i}}\}$ and $\{dx_i, dx_{n+i}, dx_{2n+i}, dx_{3n+i}\}$ be natural bases over R of the tangent space $T(M)$ and the cotangent space $T^*(M)$ of M , respectively. Taking into consideration (6), then we can obtain the expressions as follows:

$$\begin{aligned} F\left(\frac{\partial}{\partial x_i}\right) &= a\frac{\partial}{\partial x_{n+i}}, F\left(\frac{\partial}{\partial x_{n+i}}\right) = -a\frac{\partial}{\partial x_i}, F\left(\frac{\partial}{\partial x_{2n+i}}\right) = a\frac{\partial}{\partial x_{3n+i}}, F\left(\frac{\partial}{\partial x_{3n+i}}\right) = -a\frac{\partial}{\partial x_{2n+i}} \\ G\left(\frac{\partial}{\partial x_i}\right) &= -b\frac{\partial}{\partial x_{2n+i}}, G\left(\frac{\partial}{\partial x_{n+i}}\right) = b\frac{\partial}{\partial x_{3n+i}}, G\left(\frac{\partial}{\partial x_{2n+i}}\right) = -b\frac{\partial}{\partial x_i}, G\left(\frac{\partial}{\partial x_{3n+i}}\right) = b\frac{\partial}{\partial x_{n+i}} \rightarrow (8) \\ H\left(\frac{\partial}{\partial x_i}\right) &= -ab\frac{\partial}{\partial x_{3n+i}}, H\left(\frac{\partial}{\partial x_{n+i}}\right) = -ab\frac{\partial}{\partial x_{2n+i}}, H\left(\frac{\partial}{\partial x_{2n+i}}\right) = -ab\frac{\partial}{\partial x_{n+i}}, H\left(\frac{\partial}{\partial x_{3n+i}}\right) = -ab\frac{\partial}{\partial x_i} \end{aligned}$$

6 – Generalized-Quaternionic Conformal Kähler Manifolds:-

Definition 6.1: Let $(M, g, \nabla, J \pm)$ be an almost para/pseudo-Hermitian Weyl manifold. If $\nabla(J \pm) = 0$, then one says that is a (para)-Kähler Weyl manifold. Note that necessarily $J \pm$ is integrable in this setting.

Theorem 6.1: If $(M, g, \nabla, J \pm)$ is a (para)-Kähler Weyl manifold with dimension $n \geq 6$ and with $H^1(M; R) = 0$ then the underlying Weyl structure on M is trivial.

Theorem 6.2: If $(M, g, \nabla, J \pm)$ is a curvature (para)-Kähler Weyl manifold with dimension $n \geq 6$ and with $H^1(M; R) = 0$ then the underlying Weyl structure on M is trivial.

Theorem 6.3: Let $n \geq 6$. If $(M, g, \nabla, J \pm)$ is a Kähler – Weyl structure, then the associated Weyl structure is trivial, i.e. there is a conformally equivalent metric $\tilde{g} = e^{2f}g$ so that $(M, \tilde{g}, J \pm)$ is Kähler and so that $\nabla = \nabla^{\tilde{g}}$ [26,27,28].

After this part W will be used instead of J . A manifold with a weyl structure is known as weyl manifold. λ second structure was chosen the minus sign. Because the condition of the structure required to provide. $W_{\pm}^2 = \pm Id$ [29]. If we rewrite (8) equation with conformal structure, we obtain the following equations:

$$\begin{aligned} W_F \left(\frac{\partial}{\partial x_i} \right) &= ae^{\lambda} \frac{\partial}{\partial x_{n+i}}, W_F \left(\frac{\partial}{\partial x_{n+i}} \right) = -ae^{-\lambda} \frac{\partial}{\partial x_i} \\ W_F \left(\frac{\partial}{\partial x_{2n+i}} \right) &= ae^{\lambda} \frac{\partial}{\partial x_{3n+i}}, W_F \left(\frac{\partial}{\partial x_{3n+i}} \right) = -ae^{-\lambda} \frac{\partial}{\partial x_{2n+i}} \\ W_G \left(\frac{\partial}{\partial x_i} \right) &= -be^{\lambda} \frac{\partial}{\partial x_{2n+i}}, W_G \left(\frac{\partial}{\partial x_{n+i}} \right) = be^{\lambda} \frac{\partial}{\partial x_{3n+i}} \rightarrow (9) \\ W_G \left(\frac{\partial}{\partial x_{2n+i}} \right) &= -be^{-\lambda} \frac{\partial}{\partial x_i}, W_G \left(\frac{\partial}{\partial x_{3n+i}} \right) = be^{-\lambda} \frac{\partial}{\partial x_{n+i}} \\ W_H \left(\frac{\partial}{\partial x_i} \right) &= -abe^{\lambda} \frac{\partial}{\partial x_{3n+i}}, W_H \left(\frac{\partial}{\partial x_{n+i}} \right) = -abe^{\lambda} \frac{\partial}{\partial x_{2n+i}} \\ W_H \left(\frac{\partial}{\partial x_{2n+i}} \right) &= -abe^{-\lambda} \frac{\partial}{\partial x_{n+i}}, W_H \left(\frac{\partial}{\partial x_{3n+i}} \right) = -abe^{-\lambda} \frac{\partial}{\partial x_i} \end{aligned}$$

We continue our studies thinking of the $(M, g, \nabla, W_{\pm})$ instead of the almost para/pseudo- *Kähler* – *weyl* manifolds $(M, g, \nabla, J_{\pm})$.

7 – Conformal Euler-Lagrangian Mechanical System:-

Here, we obtain Euler-Lagrange equations for quantum and classical mechanics by means of a canonical local basis $\{F, G, H\}$ of V that they defined on a generalized-quaternionic *Kähler* manifold (M, g, V) .

First:

$$\begin{aligned} a \frac{\partial}{\partial t} \left(e^{-\lambda} \frac{\partial L}{\partial x_i} \right) + \frac{\partial L}{\partial x_{n+i}} &= 0, a \frac{\partial}{\partial t} \left(e^{\lambda} \frac{\partial L}{\partial x_{n+i}} \right) - \frac{\partial L}{\partial x_i} = 0 \\ a \frac{\partial}{\partial t} \left(e^{-\lambda} \frac{\partial L}{\partial x_{2n+i}} \right) + \frac{\partial L}{\partial x_{3n+i}} &= 0, a \frac{\partial}{\partial t} \left(e^{\lambda} \frac{\partial L}{\partial x_{3n+i}} \right) - \frac{\partial L}{\partial x_{2n+i}} = 0 \end{aligned}$$

Second:

$$\begin{aligned} b \frac{\partial}{\partial t} \left(e^{-\lambda} \frac{\partial L}{\partial x_i} \right) + \frac{\partial L}{\partial x_{2n+i}} &= 0, b \frac{\partial}{\partial t} \left(e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \right) - \frac{\partial L}{\partial x_{3n+i}} = 0 \\ b \frac{\partial}{\partial t} \left(e^{\lambda} \frac{\partial L}{\partial x_{2n+i}} \right) + \frac{\partial L}{\partial x_i} &= 0, b \frac{\partial}{\partial t} \left(e^{\lambda} \frac{\partial L}{\partial x_{3n+i}} \right) - \frac{\partial L}{\partial x_{n+i}} = 0 \end{aligned}$$

Third: Let H take a local basis element on the generalized-quaternionic *Kähler* manifold (M, g, V) , and $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$ be its coordinate functions. Let semispray be the vector field Z determined by

$$Z = Z^i \frac{\partial}{\partial x_i} + Z^{n+i} \frac{\partial}{\partial x_{n+i}} + Z^{2n+i} \frac{\partial}{\partial x_{2n+i}} + Z^{3n+i} \frac{\partial}{\partial x_{3n+i}} \rightarrow (10)$$

Where $Z^i = \dot{x}_i, Z^{n+i} = \dot{x}_{n+i}, Z^{2n+i} = \dot{x}_{2n+i}, Z^{3n+i} = \dot{x}_{3n+i}$

This equation (5) can be written concise manner

$$Z = \sum_{a=0}^3 Z^{an+i} \frac{\partial}{\partial x_{an+i}} \rightarrow (11)$$

And the dot indicates the derivative with respect to time t .

The vector field defined by:

$$V_H(L) = H(Z) = -abZ^i e^\lambda \frac{\partial L}{\partial x_{3n+i}} - abZ^{n+i} e^\lambda \frac{\partial L}{\partial x_{2n+i}} - abZ^{2n+i} e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} - abZ^{3n+i} e^{-\lambda} \frac{\partial L}{\partial x_i}.$$

is named a conformal Liouville vector field on the generalized-quaternionic Kähler manifold (M, g, V) . For H the closed generalized-quaternionic Kähler form is the closed 2-form given by $\Phi_L^H = -dd_H L$ such that

$$d_H L = -abe^\lambda \frac{\partial L}{\partial x_{3n+i}} dx_i - abe^\lambda \frac{\partial L}{\partial x_{2n+i}} dx_{n+i} - abe^{-\lambda} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i} - abe^{-\lambda} \frac{\partial L}{\partial x_i} dx_{3n+i} : \mathcal{F}(M) \rightarrow \Lambda^1 M \rightarrow (12)$$

Then we have

$$\begin{aligned} \Phi_L^H = & abe^\lambda \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{3n+i}} dx_j \wedge dx_i + abe^\lambda \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_i + abe^\lambda \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{2n+i}} dx_j \wedge dx_{n+i} + \\ & abe^\lambda \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \wedge dx_{n+i} + abe^{-\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{n+i}} dx_j \wedge dx_{2n+i} + abe^{-\lambda} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_{2n+i} + \\ & abe^{-\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_i} dx_j \wedge dx_{3n+i} + abe^{-\lambda} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{3n+i} + abe^\lambda \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{3n+i}} dx_{n+j} \wedge dx_i + \\ & abe^\lambda \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_i + abe^\lambda \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{2n+i}} dx_{n+j} \wedge dx_{n+i} + abe^\lambda \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_{n+i} + \\ & abe^{-\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{n+i}} dx_{n+j} \wedge dx_{2n+i} + abe^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_{2n+i} + abe^{-\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_i} dx_{n+j} \wedge dx_{3n+i} \\ & + abe^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_{3n+i} + abe^\lambda \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{3n+i}} dx_{2n+j} \wedge dx_i + abe^\lambda \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} \wedge dx_i + \\ & abe^\lambda \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{2n+i}} dx_{2n+j} \wedge dx_{n+i} + abe^\lambda \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_{n+i} + \\ & abe^{-\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+j} \wedge dx_{2n+i} + abe^{-\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_{2n+i} + \\ & abe^{-\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_i} dx_{2n+j} \wedge dx_{3n+i} + abe^{-\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \wedge dx_{3n+i} + \\ & abe^\lambda \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{3n+i}} dx_{3n+j} \wedge dx_i + abe^\lambda \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_i + \\ & abe^\lambda \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{2n+i}} dx_{3n+j} \wedge dx_{n+i} + abe^\lambda \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_{n+i} + \\ & abe^{-\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{n+i}} dx_{3n+j} \wedge dx_{2n+i} + abe^{-\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_{2n+i} + \\ & abe^{-\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_i} dx_{3n+j} \wedge dx_{3n+i} + abe^{-\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \wedge dx_{3n+i} \rightarrow (13) \end{aligned}$$

Then we calculate

$$\begin{aligned} i_Z \Phi_L^H = & abZ^i e^\lambda \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{3n+i}} \delta_i^j dx_i - abZ^i e^\lambda \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{3n+i}} dx_j + \\ & abZ^i e^\lambda \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} \delta_i^j dx_i - abZ^i e^\lambda \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j + \end{aligned}$$

$$\begin{aligned}
 & abZ^i e^{\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{2n+i}} \delta_i^j dx_{n+i} - abZ^{n+i} e^{\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{2n+i}} dx_j + \\
 & abZ^i e^{\lambda} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta_i^j dx_{n+i} - abZ^{n+i} e^{\lambda} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j + \\
 & abZ^i e^{-\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{n+i}} \delta_i^j dx_{2n+i} - abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{n+i}} dx_j + \\
 & abZ^i e^{-\lambda} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta_i^j dx_{2n+i} - abZ^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j + \\
 & abZ^i e^{-\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_i} \delta_i^j dx_{3n+i} - abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_i} dx_j + \\
 & abZ^i e^{-\lambda} \frac{\partial^2 L}{\partial x_j \partial x_i} \delta_i^j dx_{3n+i} - abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j + \\
 & abZ^{n+i} e^{\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{3n+i}} \delta_{n+i}^{n+j} dx_i - abZ^i e^{\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{3n+i}} dx_{n+j} + \\
 & abZ^{n+i} e^{\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} \delta_{n+i}^{n+j} dx_i - abZ^i e^{\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} + \\
 & abZ^{n+i} e^{\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{2n+i}} \delta_{n+i}^{n+j} dx_{n+i} - abZ^{n+i} e^{\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{2n+i}} dx_{n+j} + \\
 & abZ^{n+i} e^{\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} \delta_{n+i}^{n+j} dx_{n+i} - abZ^{n+i} e^{\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} + \\
 & abZ^{n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{n+i}} \delta_{n+i}^{n+j} dx_{2n+i} - abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{n+i}} dx_{n+j} + \\
 & abZ^{n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} \delta_{n+i}^{n+j} dx_{2n+i} - abZ^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} + \\
 & abZ^{n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_i} \delta_{n+i}^{n+j} dx_{3n+i} - abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_i} dx_{n+j} + \\
 & abZ^{n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} \delta_{n+i}^{n+j} dx_{3n+i} - abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} + \\
 & abZ^{2n+i} e^{\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{3n+i}} \delta_{2n+i}^{2n+j} dx_i - abZ^i e^{\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{3n+i}} dx_{2n+j} + \\
 & abZ^{2n+i} e^{\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} \delta_{2n+i}^{2n+j} dx_i - abZ^i e^{\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} + \\
 & abZ^{2n+i} e^{\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{2n+i}} \delta_{2n+i}^{2n+j} dx_{n+i} - abZ^{n+i} e^{\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{2n+i}} dx_{2n+j} + \\
 & abZ^{2n+i} e^{\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} \delta_{2n+i}^{2n+j} dx_{n+i} - abZ^{n+i} e^{\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} + \\
 & abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{n+i}} \delta_{2n+i}^{2n+j} dx_{2n+i} - abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+j} + \\
 & abZ^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} \delta_{2n+i}^{2n+j} dx_{2n+i} - abZ^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} + \\
 & abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_i} \delta_{2n+i}^{2n+j} dx_{3n+i} - abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_i} dx_{2n+j} +
 \end{aligned}$$

$$\begin{aligned}
 & abZ^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} \delta_{2n+i}^{2n+j} dx_{3n+i} - abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} + \\
 & abZ^{3n+i} e^{\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{3n+i}} \delta_{3n+i}^{3n+j} dx_i - abZ^i e^{\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{3n+i}} dx_{3n+j} + \\
 & abZ^{3n+i} e^{\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} \delta_{3n+i}^{3n+j} dx_i - abZ^i e^{\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} + \\
 & abZ^{3n+i} e^{\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{2n+i}} \delta_{3n+i}^{3n+j} dx_{n+i} - abZ^{n+i} e^{\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{2n+i}} dx_{3n+j} + \\
 & abZ^{3n+i} e^{\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} \delta_{3n+i}^{3n+j} dx_{n+i} - abZ^{n+i} e^{\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} + \\
 & abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{n+i}} \delta_{3n+i}^{3n+j} dx_{2n+i} - abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{n+i}} dx_{3n+j} + \\
 & abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} \delta_{3n+i}^{3n+j} dx_{2n+i} - abZ^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} + \\
 & abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_i} \delta_{3n+i}^{3n+j} dx_{3n+i} - abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_i} dx_{3n+j} + \\
 & abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} \delta_{3n+i}^{3n+j} dx_{3n+i} - abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \rightarrow \quad (14)
 \end{aligned}$$

Energy function is:

$$\begin{aligned}
 E_L^H = V_H(L) - L = & -abZ^i e^{\lambda} \frac{\partial L}{\partial x_{3n+i}} - abZ^{n+i} e^{\lambda} \frac{\partial L}{\partial x_{2n+i}} - abZ^{2n+i} e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} - \\
 & abZ^{3n+i} e^{-\lambda} \frac{\partial L}{\partial x_i} - L \rightarrow \quad (15)
 \end{aligned}$$

And hence

$$\begin{aligned}
 dE_L^H = & -abZ^i e^{\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{3n+i}} dx_j - abZ^i e^{\lambda} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j - abZ^{n+i} e^{\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{2n+i}} dx_j - \\
 & abZ^{n+i} e^{\lambda} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j - abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{n+i}} dx_j - abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{n+i}} dx_j - \\
 & abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_i} dx_j - abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j - abZ^i e^{\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{3n+i}} dx_{n+j} - \\
 & abZ^i e^{\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} - abZ^{n+i} e^{\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{2n+i}} dx_{n+j} - abZ^{n+i} e^{\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} - \\
 & abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{n+i}} dx_{n+j} - abZ^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} - abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_i} dx_{n+j} - \\
 & abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} - abZ^i e^{\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{3n+i}} dx_{2n+j} - abZ^i e^{\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} - \\
 & abZ^{n+i} e^{\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{2n+i}} dx_{2n+j} - abZ^{n+i} e^{\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} - abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+j} - \\
 & abZ^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} - abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_i} dx_{2n+j} - abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} -
 \end{aligned}$$

$$\begin{aligned}
 & abZ^i e^\lambda \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{3n+i}} dx_{3n+j} - abZ^i e^\lambda \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} - abZ^{n+i} e^\lambda \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{2n+i}} dx_{3n+j} - \\
 & abZ^{n+i} e^\lambda \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} - abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{n+i}} dx_{3n+j} - abZ^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} - \\
 & abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_i} dx_{3n+j} - abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} - \frac{\partial L}{\partial x_j} dx_j - \frac{\partial L}{\partial x_{n+j}} dx_{n+j} - \\
 & \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} - \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} \rightarrow \quad (16)
 \end{aligned}$$

By means of Eq(1), we calculate the following expressions:

$$\begin{aligned}
 & abZ^i e^\lambda \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{3n+i}} \delta_i^j dx_i + abZ^i e^\lambda \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} \delta_i^j dx_i + abZ^i e^\lambda \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{2n+i}} \delta_i^j dx_{n+i} + \\
 & abZ^i e^\lambda \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta_i^j dx_{n+i} + abZ^i e^{-\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{n+i}} \delta_i^j dx_{2n+i} + abZ^i e^{-\lambda} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta_i^j dx_{2n+i} + \\
 & abZ^i e^{-\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_i} \delta_i^j dx_{3n+i} + abZ^i e^{-\lambda} \frac{\partial^2 L}{\partial x_j \partial x_i} \delta_i^j dx_{3n+i} + abZ^{n+i} e^\lambda \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{3n+i}} \delta_{n+i}^{n+j} dx_i + \\
 & abZ^{n+i} e^\lambda \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} \delta_{n+i}^{n+j} dx_i + abZ^{n+i} e^\lambda \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{2n+i}} \delta_{n+i}^{n+j} dx_{n+i} + abZ^{n+i} e^\lambda \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} \delta_{n+i}^{n+j} dx_{n+i} \\
 & + abZ^{n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{n+i}} \delta_{n+i}^{n+j} dx_{2n+i} + abZ^{n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} \delta_{n+i}^{n+j} dx_{2n+i} \\
 & + abZ^{n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_i} \delta_{n+i}^{n+j} dx_{3n+i} + abZ^{n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} \delta_{n+i}^{n+j} dx_{3n+i} \\
 & + abZ^{2n+i} e^\lambda \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{3n+i}} \delta_{2n+i}^{2n+j} dx_i + abZ^{2n+i} e^\lambda \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} \delta_{2n+i}^{2n+j} dx_i \\
 & + abZ^{2n+i} e^\lambda \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{2n+i}} \delta_{2n+i}^{2n+j} dx_{n+i} + abZ^{2n+i} e^\lambda \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} \delta_{2n+i}^{2n+j} dx_{n+i} \\
 & + abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{n+i}} \delta_{2n+i}^{2n+j} dx_{2n+i} + abZ^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} \delta_{2n+i}^{2n+j} dx_{2n+i} \\
 & + abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_i} \delta_{2n+i}^{2n+j} dx_{3n+i} + abZ^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} \delta_{2n+i}^{2n+j} dx_{3n+i} \\
 & + abZ^{3n+i} e^\lambda \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{3n+i}} \delta_{3n+i}^{3n+j} dx_i + abZ^{3n+i} e^\lambda \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} \delta_{3n+i}^{3n+j} dx_i \\
 & + abZ^{3n+i} e^\lambda \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{2n+i}} \delta_{3n+i}^{3n+j} dx_{n+i} + abZ^{3n+i} e^\lambda \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} \delta_{3n+i}^{3n+j} dx_{n+i} \\
 & + abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{n+i}} \delta_{3n+i}^{3n+j} dx_{2n+i} + abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} \delta_{3n+i}^{3n+j} dx_{2n+i} \\
 & + abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_i} \delta_{3n+i}^{3n+j} dx_{3n+i} + abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} \delta_{3n+i}^{3n+j} dx_{3n+i} + \frac{\partial L}{\partial x_j} dx_j \\
 & + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0 \rightarrow \quad (17)
 \end{aligned}$$

And thus:

$$\begin{aligned}
 & abZ^i e^\lambda \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{3n+i}} dx_j + abZ^{n+i} e^\lambda \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{3n+i}} dx_j + abZ^{2n+i} e^\lambda \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{3n+i}} dx_j + \\
 & abZ^{3n+i} e^\lambda \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{3n+i}} dx_j + abZ^i e^\lambda \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j + abZ^{n+i} e^\lambda \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_j + \\
 & abZ^{2n+i} e^\lambda \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_j + abZ^{3n+i} e^\lambda \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_j + \frac{\partial L}{\partial x_j} dx_j + \\
 & abZ^i e^\lambda \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{2n+i}} dx_{n+j} + abZ^{n+i} e^\lambda \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{2n+i}} dx_{n+j} + abZ^{2n+i} e^\lambda \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{2n+i}} dx_{n+j} + \\
 & abZ^{3n+i} e^\lambda \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{2n+i}} dx_{n+j} + abZ^i e^\lambda \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{n+j} + abZ^{n+i} e^\lambda \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} + \\
 & abZ^{2n+i} e^\lambda \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{n+j} + abZ^{3n+i} e^\lambda \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{n+j} + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \\
 & abZ^i e^{-\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{n+i}} dx_{2n+j} + abZ^{n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+j} + abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+j} + \\
 & abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+j} + abZ^i e^{-\lambda} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{2n+j} + abZ^{n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{2n+j} + \\
 & abZ^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} + abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{2n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \\
 & abZ^i e^{-\lambda} \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_i} dx_{3n+j} + abZ^{n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_i} dx_{3n+j} + abZ^{2n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_i} dx_{3n+j} + \\
 & abZ^{3n+i} e^{-\lambda} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_i} dx_{3n+j} + abZ^i e^{-\lambda} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_{3n+j} + abZ^{n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{3n+j} + \\
 & abZ^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{3n+j} + abZ^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0 \rightarrow (18)
 \end{aligned}$$

OR

$$\begin{aligned}
 & abe^\lambda [Z^i e^\lambda \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{3n+i}} + Z^{n+i} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{3n+i}} + Z^{2n+i} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{3n+i}} + Z^{3n+i} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{3n+i}} + Z^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} + \\
 & Z^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} + Z^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} + Z^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}}] dx_j + \frac{\partial L}{\partial x_j} dx_j + \\
 & abe^\lambda [Z^i \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{2n+i}} + Z^{n+i} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{2n+i}} + Z^{2n+i} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{2n+i}} + Z^{3n+i} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{2n+i}} + Z^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} + \\
 & Z^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} + Z^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} + Z^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}}] dx_{n+j} + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \\
 & abe^{-\lambda} [Z^i \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_{n+i}} + Z^{n+i} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_{n+i}} + Z^{2n+i} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_{n+i}} + Z^{3n+i} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_{n+i}} + Z^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} + \\
 & Z^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} + Z^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} + Z^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}}] dx_{2n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \\
 & abe^{-\lambda} [Z^i \frac{\partial \lambda}{\partial x_j} \frac{\partial L}{\partial x_i} + Z^{n+i} \frac{\partial \lambda}{\partial x_{n+j}} \frac{\partial L}{\partial x_i} + Z^{2n+i} \frac{\partial \lambda}{\partial x_{2n+j}} \frac{\partial L}{\partial x_i} + Z^{3n+i} \frac{\partial \lambda}{\partial x_{3n+j}} \frac{\partial L}{\partial x_i} + Z^i \frac{\partial^2 L}{\partial x_j \partial x_i} + Z^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} + \\
 & Z^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} + Z^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i}] dx_{3n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0
 \end{aligned}$$

In this equation can be concise manner

$$\begin{aligned}
 &abe^{\lambda} \sum_{a=0}^3 Z^{an+i} \left[\frac{\partial \lambda}{\partial x_{an+j}} \frac{\partial L}{\partial x_{3n+i}} + \frac{\partial^2 L}{\partial x_{an+j} \partial x_{3n+i}} \right] dx_j + \frac{\partial L}{\partial x_j} dx_j + \\
 &abe^{\lambda} \sum_{a=0}^3 Z^{an+i} \left[\frac{\partial \lambda}{\partial x_{an+j}} \frac{\partial L}{\partial x_{2n+i}} + \frac{\partial^2 L}{\partial x_{an+j} \partial x_{2n+i}} \right] dx_{n+j} + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \\
 &abe^{-\lambda} \sum_{a=0}^3 Z^{an+i} \left[\frac{\partial \lambda}{\partial x_{an+j}} \frac{\partial L}{\partial x_{n+i}} + \frac{\partial^2 L}{\partial x_{an+j} \partial x_{n+i}} \right] dx_{2n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \\
 &abe^{-\lambda} \sum_{a=0}^3 Z^{an+i} \left[\frac{\partial \lambda}{\partial x_{an+j}} \frac{\partial L}{\partial x_i} + \frac{\partial^2 L}{\partial x_{an+j} \partial x_i} \right] dx_{3n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0 \quad \rightarrow (19)
 \end{aligned}$$

Then we have the equations:

$$\begin{aligned}
 &ab \frac{\partial}{\partial t} \left(e^{-\lambda} \frac{\partial L}{\partial x_i} \right) + \frac{\partial L}{\partial x_{3n+i}} = 0, \quad ab \frac{\partial}{\partial t} \left(e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \right) + \frac{\partial L}{\partial x_{2n+i}} = 0 \\
 &ab \frac{\partial}{\partial t} \left(e^{\lambda} \frac{\partial L}{\partial x_{2n+i}} \right) + \frac{\partial L}{\partial x_{n+i}} = 0, \quad ab \frac{\partial}{\partial t} \left(e^{\lambda} \frac{\partial L}{\partial x_{3n+i}} \right) + \frac{\partial L}{\partial x_i} = 0 \quad \rightarrow (20)
 \end{aligned}$$

Such that the equations calculated in equation(20) are named Euler-Lagrange equations constructed on a generalized-quaternionic *Kähler* manifold (M, g, V) by means of Φ_L^H and thus the triple (M, Φ_L^H, X) is called a mechanical system on generalized-quaternionic *Kähler* manifold (M, g, V) .

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