PRINCIPLES OF RELATIVITY

Seungmin Lee

Chadwick International School

danielsml0718@gmail.com

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ABSTRACT

In this paper, I further simplify the founding principles of special relativity and its applications to the field of electrodynamics. Albert Einstein's special theory of relativity was officially published in the year 1905, well over a century ago. Yet, even to this day, its discussion continues, progressively becoming accessible to the general public. From the derivation of the mass-energy equivalence principle to the correction of maxwell's equations, special relativity forms the very basics of modern physics, and attempts to further simplify the field is imperative in perpetuating further applications of its concepts.

Keywords: Special Relativity, Lorentz transformations, Electrodynamics

1. THE FOUNDING PRINCIPLES OF RELATIVITY

Galilean Relativity

Before the introduction of special relativity, Galilean & Newtonian relativity were used to describe motion in physics [2]. First with the discussion of Galilean relativity, it is stated that the fundamental laws of physics applied the same in all inertial frames of reference - an observer inside a train moving in constant velocity would

not be able to perceive the motion of the train.

Galilean relativity allowed reference frames of an event to be described through cartesian planes with time and position: events can therefore be written as $E(t,x,y,z)$. If two reference frames $S(t,x,y,z)$ and $S'(t',x',y',z')$ were to move with velocity v relative to each other, it could be represented by the origin of one plane moving away with velocity v from the origin of the other.

This is known as the Galilean transformation.

To find $t', x', y',$ and z' in respect to $t, x, y,$ and z :

$$\begin{aligned}t' &= t \\x' &= x - vt \sin \theta \\y' &= y - vt \cos \theta \\z' &= z\end{aligned}$$

Galilean relativity kept time absolute [2] - time cannot be perceived differently by different frames of reference. Newtonian relativity remained similar to Galilei's just with the addition of the concept of "frames of absolute rest" - hinted at the law of inertia.

Ether and the speed of light

The existence of ether, the medium through which light supposedly traveled, was first proposed by Descartes in 1644. Light, like any other existing waves, was first conjectured to propagate through a certain medium to travel; light traveled from the Sun to Earth through ether, and when Earth moved, it was to cause a movement in the ether medium - an *ether wind*. To observe the existence of this wind, the famous Michelson-Morley experiment by physicists Albert A. Michelson and Edward W. Morley was conducted in 1887. The experiment set up was as follows:

If ether wind had existed, both paths of light must arrive at different times. If ether wind were to blow towards the right with velocity v , calculation of time using Newtonian physics would have been as follows:

$$t_{red} = \frac{\frac{d}{2}}{v+c} + \frac{d}{2c} + \frac{d}{c}$$

$$t_{blue} = \frac{d}{v+c} + \frac{\frac{d}{2}}{c-v} + \frac{d}{2c}$$

Where d is the distance from mirror B to the source of light or from mirror A to the light detector and c is the speed of light. Though mathematical calculations approve the existence of ether winds, the Michelson-Morley experiment proved ether wrong - both strands of light arrived at the same time.

The special theory of relativity

A more cohesive explanation of the nature of light came in 1905 by theoretical physicist Albert Einstein with his paper on the theory of special relativity. Einstein postulated:

1. The speed of light c is constant in any frame of reference
2. Physics applies in the same, simplest form for all inertial frames of reference

Einstein further explored changes to an object in mass, length, and time as it approached the speed of light. Einstein's thought experiment went as follows:

A source of light positioned exactly in the middle of a spacecraft moving with constant velocity v to the right emits 2 laser beams aiming front and back.

To an observer inside the spacecraft, light will travel equal distances to the walls of the spacecraft, perceiving the arrival simultaneously. However, from a stationary observer standing outside of the spacecraft will perceive light hitting the back wall first for the speed of light is constant for both light beams but the back wall will travel distance vt_0 closer (front wall will travel the same distance further). Einstein concluded that simultaneous events may be perceived differently depending on the state of motion of the observer.

Einstein's study on the special case of relativity got its "specialness" for its sole consideration of inertial-frames of reference. Laws of special relativity held false in non-inertial frames - acceleration and direction cannot be explained thoroughly. Einstein developed the general theory of relativity a decade after his release of special relativity to account for this limited aspect.

2. TIME DILATION AND LENGTH CONTRACTION

Derivation

Einstein's conclusion naturally led to the discussion of how certain physical measurements vary as objects reach speeds closer to c . A quick derivation of an example,

Where t_d is the dilated time observed by the observer, t_0 is the proper time (independent of v), c is the speed of light, and v is the speed of the spacecraft. Light travels vertically inside the spacecraft from one projector, yet the observer would perceive light traveling diagonally for the spaceship was traveling "with the light". For light speed has to stay constant in any inertial frame of reference, the diagonal speed of light remains as c . Applying the Pythagorean theorem, the following equation can be derived:

$$(ct_d)^2 = (ct_0)^2 + (vt_d)^2$$

$$t_d = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

As another leading example, the derivation of length contraction leads the discussion. Einstein, with special relativity, concluded that length is also relative along with time - moving frames perceive length differently from frames at rest. Consider a spacecraft traveling at speed v with a laser clock on board sending a deflectable laser beam:

With the spaceship at rest, proper time t_0 , the time takes for the beam of light to travel the length of the spacecraft, deflect, and return will equal to $2L_0/c$. With the spacecraft traveling at velocity v , an observer outside of the spacecraft will witness the beam of light traveling an extra distance of vt as the beam heads forward, and a shorter distance of vt as it travels back as the beam deflector moves position with the spacecraft. Hence, to the observer, the time it takes for light to complete its journey would be:

$$t = \frac{L}{c-v} + \frac{L}{c+v}$$

With the derived equation for time dilation and the definition of proper time t_0 , evaluating the value for L will give:

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

From the equation derived, it could be concluded that length contracts and time dilates by the Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

as speed increases. This factor is negligible for small values of v for c being such a numerically large number (300,000,000 m/s).

The muon paradox

The effects of length contraction and time dilation have been proven with empirical evidence - the famous muon decay experiment. Muons, elementary particles that continuously enter Earth's

surface, have been observed to completely decay in 2.2 microseconds when at rest. However, it has also been observed that muons travel roughly 700 meters before decaying. Velocity resulting in $\frac{700}{2.2 \times 10^{-6}} \approx 3.210^8$ m/s, faster than the speed of light, contradicting Einstein's postulates. In reality, muons are able to travel much more than expected strictly as a result of time dilation and length contraction. As expected, in the muon's frame of reference, time until decay will equal to $2.2\mu s$. However, in the experimenter's perspective, $2.2\mu s$ for the muon will be dilated by a factor of γ . Similarly, 700 meters in the experimenter's frame of reference will be contracted in the moving muon's perspective by a factor of γ . Correcting calculations:

$$L_{muon} = 700 \sqrt{1 - \frac{V_{muon}^2}{c^2}}$$

$$V_{muon} = \frac{700 \sqrt{1 - \frac{V_{muon}^2}{c^2}}}{2.2 \times 10^{-6}}$$

$$V_{muon} \approx 2.18 \times 10^8 \text{ m/s}$$

A muon's velocity gives less than the speed of light. Completing the calculation of values:

$$t_{experimenter} = \frac{2.2 \times 10^{-6}}{\sqrt{1 - \frac{V_{muon}^2}{c^2}}} \approx 3 \times 10^{-6} s$$

$$L_{muon} \approx 480.2 \text{ m}$$

3. LORENTZ TRANSFORMATIONS

The flaw of Galilean transformations is observable when shifting reference points while keeping time absolute - it changes the speed of light for different observers. On a space-time graph, transforming coordinates from inertial

frames (linear) onto the line $x=0$ represents a shift in reference, boosting into a different perspective. In other words, because Galilei transformed space coordinates while keeping time coordinates absolute, Galilean transformation can be visualized by squeezing all the coordinate space to the right or left - resembling a parallelogram.

The flaw becomes clear - all coordinates, **including those of light**, transform, resulting in a different speed for different reference points. This obviously contradicts Einstein's conclusions on special relativity. As a solution, Hendrik Lorentz discovered a transformation that kept the speed of light absolute, plotting the time observed by a boosted point of reference to an earlier time (scaling it by a factor of γ) rather than keeping it constant. To visualize a Lorentz transformation:

To keep the slope (velocity) constant for light, all coordinates are scaled along the two light lines. In the case above, coordinates are squeezed along the right light line and elongated through the left. Thus, transformation occurs diagonally - squeezed to the left and squeezed downwards. By definition, light's vector is the eigenvector of Lorentz transformations - they are invariant.

The mathematical summary of Lorentz transformations further aid understanding. Considering a reference point of S at rest, an

inertial frame S' moving at a constant velocity V , and an event E characterized by coordinates $E(d,y,z,t)$, the coordinates of (x',y',z',t') observing the same event E in the S' frame can be derived. Event E happens at a distance of d from reference point S (observed by a resting reference point). For reasons discussed, measurements of both time and length differ by reference point - scaled by the Lorentz factor γ . For length contraction occurs only along the x -axis, y' and z' remains unchanged from its Galilean forms - y and z , respectively. x , the length of $S'E$ measured by S , is the difference of lengths SE and SS' :

$$x = (d - Vt) \quad (\text{Eq. 3.1})$$

Reference frames S' and S experience a relative velocity of magnitude V with each other. Thus, measurements get contracted by a factor of γ^{-1} in the other frame.

Giving:

$$x = \frac{x'}{\gamma} \quad (\text{Eq. 3.2})$$

Plugging in 3.2 into 3.1 yields:

$$x' = \gamma(d - Vt)$$

Similar to the logic of 3.1, d' is the sum lengths SS' and $S'E$ viewed from S' . Giving:

$$d' = Vt' + x'$$

Substituting for x' gives:

$$d = \gamma(Vt' + \gamma(d - Vt))$$

Organizing t' in terms of t

$$t' = \gamma \left(t - \frac{V}{c^2} d \right)$$

For this transformation is restricted to movement along the x -axis, a more general form is necessary to be derived. A convenient substitution happens by the following:

$$\begin{aligned} d' &= \frac{d}{\gamma} \\ x^0 &= ct \\ x^1 &= d \\ x^2 &= y \\ x^3 &= z \\ \beta &= \frac{v}{c} \end{aligned}$$

Here, x and β simply denote physical quantities, holding no particular significance. Rewriting the Lorentz transformation in terms of the substitutions:

$$\begin{aligned} x'^0 &= ct' = c\gamma \left(t - \frac{v}{c^2} d \right) = \gamma \left(ct - \frac{v}{c} d \right) = \gamma(x^0 - \beta x^1) \\ x'^1 &= x' = \gamma(d - Vt) = \gamma \left(d - \frac{V}{c} ct \right) = \gamma(x^1 - \beta x^0) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned}$$

In a convenient matrix form:

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

Can be summarized as

$$x'^\mu = \sum_{v=0}^3 \Lambda_v^\mu x^v$$

Where the Λ_v^μ is the value ascribed to the μ th row, v th column of the Lorentz transformation matrix. In cases with more complex reference frames, the Lorentz transformation matrix Λ becomes more complicated. A Four-vector is known as a 4 dimensional vector that transforms under the Lorentz transformation.

Unlike dot products with three components, when finding the dot product of two 4-vectors, its zeroth components have a negative sign:

$$a' \cdot b' = -a'^0 b'^0 + a'^1 b'^1 + a'^2 b'^2 + a'^3 b'^3$$

This is known as the four-dimensional scalar product. Just as how scalar products are invariant under rotations, four-dimensional scalar products remain unchanged under Lorentz transformations. Substituting appropriately for the scalar product above gives:

$$\gamma^2 (\beta^2 - 1) (a^0 b^0 - a^1 b^1) + a^2 b^2 + a^3 b^3$$

Substituting the lorentz factor γ and symbol β :

$$\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 \left(\left(\frac{v}{c}\right)^2 - 1\right) (a^0 b^0 - a^1 b^1) + a^2 b^2 + a^3 b^3$$

$$= \frac{1}{1-\frac{v^2}{c^2}} \left(1 - \frac{v^2}{c^2}\right) (a^0 b^0 - a^1 b^1) + a^2 b^2 + a^3 b^3$$

$$= a^0 b^0 - a^1 b^1 + a^2 b^2 + a^3 b^3$$

Thus proving invariance for four-dimensional scalar products under Lorentz transformations.

Conventionally, to minimize quantitative errors in calculations, a covariant vector a_μ is introduced. Covariant vectors are different from contravariant vectors (denoted with a higher index) only by their zeroth component - signs are opposite:

$$a_\mu = (a_0, a_1, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3)$$

Components of a_μ can be expressed by the following summation:

$$x_\mu = \sum_{v=0}^3 g_{\mu v} x^v$$

Where $g_{\mu v}$ is the matrix:

$$g_{\mu v} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, x_μ is the Minkowski metric, a convenient tool to denote four-dimensional scalar products. Like its denoting value suggests, the zeroth component (x_0 and x^0) is called the temporal index - substituting the *time* component ct . Similarly, 1st~3rd components are called the spatial indices - substituting x, y , and z coordinates. By the nature of the Minkowski metric, raising or lowering temporal

indices cause a negative sign ($a_0 = -a^0$), while raising or lowering spatial indices cause no change.

Representing a four vector's

four-dimensional scalar product with itself in terms of contravariant and covariant vectors:

$$a^\mu a_\mu = (-a^0)(a^0) + (a^1)^2 + (a^2)^2 + (a^3)^2$$

The quantity $a^\mu a_\mu$ above can be positive (*spacelike*), negative (*timelike*), or zero (*lightlike*). Spacelike vectors have their spatial terms dominate, timelike vectors have their temporal terms dominate, and lightlike vectors do not have a dominating term.

$$a^\mu a_\mu > 0 \rightarrow a^{1^2} + a^{2^2} + a^{3^2} > a^{0^2}$$

$$a^\mu a_\mu < 0 \rightarrow a^{1^2} + a^{2^2} + a^{3^2} < a^{0^2}$$

$$a^\mu a_\mu = 0 \rightarrow a^{1^2} + a^{2^2} + a^{3^2} = a^{0^2} \rightarrow a^\mu = \langle ct, ct, y, z \rangle$$

In Galilean and Newtonian mechanics, the addition of velocity came by intuition - simple addition. Consider a train moving at $0.9c$ to the right when a mischievous man throws a ball at $0.9c$ towards the direction of the train. According to Galilean velocity addition, relative velocity of the ball viewed from the ground will equal to $1.8c$, deviating greatly from special relativity. Using the lorentz transformations developed previously, vector addition can be written like the following (d is substituted with x to prevent confusion with differentiation):

$$dx' = \gamma(dx - Vdt)$$

$$dt' = \gamma\left(dt - \frac{V}{c^2}dx\right)$$

$$u' = \frac{dx'}{dt'} = \frac{dx - Vdt}{dt - \frac{V}{c^2}dx} = \frac{\frac{dx}{dt} - V}{1 - \frac{V}{c^2}\frac{dx}{dt}} = \frac{u - V}{1 - \frac{V}{c^2}u}$$

$$u = \frac{u' + V}{1 + \frac{V}{c^2}u'}$$

Where u and V are velocities. Going back to the example of the high speed train, setting u' as the train's velocity, V as the ball's velocity relative to the train, u gives 2.98×10^8 , very close to the speed of light. Notice, when substituted $u'=c$, $V=c$, that u still gives c .

Minkowski diagrams

In classical mechanics, motion is described, or "mapped", using position (y-axis) vs time (x-axis) graphs, having velocity equal to the slope. The opposite is done in special relativity - motion is graphed on a x^0 (ct) vs position plane, having velocity equal to the slope's reciprocal. Such diagrams in special relativity are called Minkowski diagrams. For velocities over the value c are not allowed, slopes on the Minkowski diagram cannot be less than 1. Trajectories that movements follow are called world lines, all of which occur in between lines of slope 1 and -1.

The real practicality of Minkowski diagrams comes with the plotting the ct' - x' axes onto the ct - x coordinate plane. For instance, consider two reference frames A and B - A is stationary while B moves at velocity v away from A along the x -axis. Since all events that lie on the ct' axis are stationary with respect to frame B, points on line ct' must follow along B with velocity v . Thus, the ct' axis can be plotted on a line that makes a slope

of v^{-1} with the x axis, or v with the ct axis. To plot the x' axis, the fact that the speed of light is consistent for all frames will be used. As discussed, since the light's world line bisects the angle between the ct and x axes, the light's world line on the ct' - x' plane must also bisect the angle between the ct' and x axes to have the c value constant. Thus, the ct' axis will make an angle of $\theta = \arctan(v)$ with the ct axis, and the x' axis will

make the same angle θ with the x axis. Visualized: Here, drawing a parallelogram to plot an event on the light's world line, points can be seen being

stretched along the line diagonally with greater ease. Notice that the top right corner of all parallelograms stretch out with greater velocities:

The twin paradox

Armed with the understanding of Lorentz transformations and Minkowski diagrams, quantitative conclusions come with ease. Visiting a famous problem, titled the “Twin Paradox”, the effects of special relativity and the application of Lorentz transformations become evident.

Consider two twins - one is an astronaut, the other is a geologist. The astronaut twin gets on his spaceship on his 20th birthday while his geologist twin remains on Earth. The astronaut twin travels 5 years, measured by the astronaut twin’s clock, takes a turn, and returns home for 5 years at a constant velocity of $0.9c$. What would be the difference of the twins’ ages?

Following intuition, when the astronaut twin returns home, he would have aged 10 years, precisely on the time of his 30th birthday. However, his twin brother would have aged by a factor of $\gamma \approx 2.3$, about 43 years old. Thus, the difference in their ages would be 13 years.

The “paradoxical” nature of this problem comes when considering relative velocity - both twins perceive each other moving away at $0.9c$. It seems, by the nature of the problem, that both twins should have aged equally for their times dilate by the same factor.

The discrepancy of intuition and calculation comes with the return of the astronaut twin. The change in direction causes the twin to

accelerate - no longer acts as an inertial frame.

Phrasing is important from here because acceleration does not provide an accurate reason to why the paradox occurs. However, the astronaut twin’s change in direction does cause him to shift reference frames - his outbound and return trips are different. In other words, time “tics” differently for the two parts of the trip. Drawing a Minkowski diagram for the astronaut twin:

Since the time axis transforms for the astronaut twin, a “time gap” will occur between the onbound and the return journeys. This gap is the age difference between the two twins after return. At the turn, the gap will be unaccounted for by the astronaut twin, thus the astronaut twin will be younger (he has experienced “less” time). Before and after the turn, time is accounted for both twins.

Said perhaps in a more comprehensive way, even if the two twins are blinded, both of them can tell whether they are in motion or not since, as discussed in chapter 1, acceleration can be perceived. With this realization, the twin paradox problem turns ordinary. If the astronaut twin can be clearly distinguished from the geologist twin by movement, the astronaut twin’s time will be slowed by a factor of γ when viewed from the frame of the geologist twin. Returning to the problem, 10 years for the astronaut twin will be dilated by a factor of

$\gamma \approx 2.3$ for the geologist - the geologist twin will experience 23 years. Thus, the difference in their ages will be 13 years.

To further investigate the *exact* aging process with the twin paradox, consider a new case in which each of the two twins send a signal of light for every year they perceive. Since the speed of light is the same for both twins, both twins can calculate how their times compare with each other. Everytime the astronaut twin receives a signal, he can know that a time of (1 year) + (time it takes for light to travel from Earth to the spaceship) must have passed for the geologist twin since receiving the last signal. Calculations can be conducted similarly every time the geologist twin receives a signal. To visualize this:

Earth → Space

Space → Earth

It becomes clear that the astronaut twin perceives his brother aging slowly for the first half of his trip and quickly for the other half. Conversely, the geologist twin will perceive his brother aging slowly for most of his trip. This way, the twin paradox is also verified graphically.

4. RELATIVISTIC MECHANICS

Velocity

Consider an airplane traveling from Los Angeles to Korea. The captain announces that the plane is moving at speed $3/5c$. This value is measured by an observer on ground, thus ordinary speed u has a value of:

$$u = \frac{dL}{dt}$$

Where dL is an infinitesimal change in distance, and dt is an infinitesimal change in time as measured by an observer stationary on the ground. Here, *proper velocity* η is defined as :

$$\eta = \frac{dL}{d\tau}$$

Where τ is time measured by an observer on the train. Though this notation seems against intuition or meaningless at first for combining two different measurement in two different reference frames (dL on ground, $d\tau$ on plane), writing η in terms of u :

$$\eta = \frac{dL}{d\tau} = \frac{u dt}{d\tau}$$

$$\gamma d\tau = dt$$

$$\eta = \gamma u$$

η has an advantage over u in the sense that it transforms with greater ease. For velocity is a physical quantity, not of time, η is a **spatial** index for a 4-vector transformation.

$$\eta^\mu = \frac{dx^\mu}{d\tau}$$

With temporal component:

$$\eta^0 = \frac{dx^0}{d\tau} = \frac{d(ct)}{d\tau} = c \frac{dt}{d\tau} = \gamma c$$

For being a four vector, η 's transformations are known immediately:

$$\eta'^0 = \gamma(\eta^0 - \beta\eta^1)$$

$$\eta'^1 = \gamma(\eta^1 - \beta\eta^0)$$

$$\eta'^2 = \eta^2$$

$$\eta'^3 = \eta^3$$

And, as discussed:

$$\eta'^\mu = \sum_{\nu=0}^3 \Lambda_{\nu}^{\mu} \eta^{\nu}$$

η 's advantage becomes conspicuous here. Since proper time τ does not depend on velocity, $d\tau$ in the denominator of η is invariant. In other words, the numerator dL is the sole quantity that transforms with η . This is clearly not true with u where both dL and dt have to transform.

In comparison to η , u 's transformation can be derived following a similar procedure to velocity addition:

$$u_x = \frac{dx'}{dt'} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$u_y = \frac{dy'}{dt'} = \frac{u_y}{1 - \frac{u_y v}{c^2}}$$

$$u_z = \frac{dz'}{dt'} = \frac{u_z}{1 - \frac{u_z v}{c^2}}$$

Momentum

Momentum, in traditional classical mechanics, is defined as a quantity taking the multiple of mass and its velocity. Under relativity, however, as discussed, momentum is defined as:

$$\phi = m\eta = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Again, momentum ϕ is a spatial component of a 4-vector. Finding the temporal component:

$$\phi^0 = m\eta^0 = m\gamma c = \frac{mc}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Energy

Einstein, with his paper, defined energy as $\phi^0 c$:

$$E = \phi^0 c = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Here, *rest energy* will give the most famous equation in history:

$$v = 0 \rightarrow E_{rest} = mc^2$$

In a closed system, both relativistic energy and momentum is still conserved. Thus, total energy will equal to the sum of rest energy and energy expended to *move* - kinetic:

$$E = E_{kinetic} + E_{rest}$$

Giving:

$$E_{kinetic} = mc^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right)$$

The reason why an object cannot travel over the speed of light becomes clear now. As v approaches

the speed of light, $E_{kinetic}$ becomes infinitely large, thus requiring an infinite amount of energy to continue the motion.

A highly powerful mathematical conclusion comes with the relativistic energy-momentum relation [4]. The derivation is as follows:

$$E^2 = \frac{m^2 c^4}{1 - \frac{v^2}{c^2}}$$

$$p^2 = \frac{m^2 v^2}{1 - \frac{v^2}{c^2}} \rightarrow p^2 c^2 = \frac{m^2 v^2 c^2}{1 - \frac{v^2}{c^2}}$$

$$\frac{E^2 - p^2 c^2}{E^2 - p^2 c^2} = \frac{\frac{m^2 c^4}{1 - \frac{v^2}{c^2}} - \frac{m^2 v^2 c^2}{1 - \frac{v^2}{c^2}}}{\frac{m^2 c^4}{1 - \frac{v^2}{c^2}}} = \frac{m^2 c^4 \left(1 - \frac{v^2}{c^2}\right)}{m^2 c^4} = m^2 c^4$$

This result is powerful for allowing the calculation of energy and momentum with information of only one variable - velocity is not needed.

If both momentum and energy are still proportional to mass in special relativity, a massless particle can be mistaken to impart no impulse or possess zero energy. However, as Max Planck discovered in 1900 [4], photons, massless elementary particles responsible for the particle-like nature of light, delivered energy that was proportional to its frequency:

$$E = hf$$

Where E is energy, f is frequency and h is planck's constant with the value:

$$h = 6.62607004 \times 10^{-34} \text{ m}^2 \text{ kg s}^{-1}$$

Planck's constant was found purely experimentally.

For a wave's velocity v can be written as:

$$v = \lambda f$$

Where λ is wavelength, a photon's energy can be rewritten as:

$$E = h \frac{c}{\lambda}$$

Compton scattering

To observe an application of relativistic mechanics, suppose a photon collides with a stationary electron.

Since momentum solely exists along the x-axis, the net momentum in the y-direction after the collision must be zero - y-momentum of the two particles are equal. Thus, where

$$\begin{aligned} p_{\mu} &= \text{momentum of particle } \mu \\ E_{\mu} &= \text{energy of particle } \mu \\ e^{-} &= \text{electron} \\ \gamma &= \text{photon} \\ h &= \text{planck's constant} \\ \lambda_{\mu} &= \text{wavelength of particle } \mu \\ f_{\mu} &= \text{frequency of particle } \mu \\ a_0 &= \text{initial value of } a \end{aligned}$$

The following can be written using the relation

$$E=pc:$$

$$p_{\gamma} \sin(\theta) = p_{e^{-}} \sin(\phi)$$

Yielding:

$$E_{\gamma} = p_{\gamma} c \rightarrow \frac{E_{\gamma}}{c} \sin(\theta) = p_{e^{-}} \sin(\phi)$$

For a similar reason, net momentum in the x-direction must also be conserved. Using the trigonometric relations above:

$$\frac{E_{\gamma 0}}{c} = p_{\gamma} \cos(\theta) + p_e \cos(\phi) = \frac{E_{\gamma}}{c} \cos(\theta) + p_e \sqrt{1 - \left(\frac{E_{\gamma}}{cp_e} \sin(\theta) \right)^2}$$

Some algebraic working:

$$\begin{aligned} E_{\gamma 0} &= E_{\gamma} \cos(\theta) + p_e c \sqrt{1 - \left(\frac{E_{\gamma}}{cp_e} \sin(\theta) \right)^2} \\ (E_{\gamma 0} - E_{\gamma} \cos(\theta))^2 &= p_e^2 c^2 - p_e^2 c^2 \left(\frac{E_{\gamma}}{cp_e} \sin(\theta) \right)^2 \\ p_e^2 c^2 &= (E_{\gamma 0} - E_{\gamma} \cos(\theta))^2 + E_{\gamma}^2 \sin^2(\theta) \end{aligned}$$

Since energy is also conserved:

$$\begin{aligned} E_{\gamma 0} + mc^2 &= E_{\gamma} + E_e \rightarrow E_e^2 - p_e^2 c^2 = m^2 c^4 \rightarrow E_{\gamma 0} + mc^2 = E_{\gamma} + \sqrt{m^2 c^4 + p_e^2 c^2} \\ E_{\gamma 0} + mc^2 &= E_{\gamma} + \sqrt{m^2 c^4 + (E_{\gamma 0} - E_{\gamma} \cos(\theta))^2 + E_{\gamma}^2 \sin^2(\theta)} \end{aligned}$$

Expressing E_{γ} in terms of $E_{\gamma 0}$, using the relation $E = h(c/\lambda)$:

$$E_{\gamma} = \frac{1}{\frac{1}{E_{\gamma 0}} + \frac{1 - \cos(\theta)}{mc^2}} = hf_{\gamma} = \frac{hc}{\lambda_{\gamma}} \rightarrow \lambda_{\gamma} = \lambda_{\gamma 0} + \frac{h}{mc} (1 - \cos(\theta))$$

This is called the Compton scattering effect - the description of how photons interact with other particles. When a photon collides with a particle, its wavelength increases, decreasing in frequency. The Compton effect is observable in fields of nuclear physics and astrophysics, and it is also known as one of the primary examples that supported light's particle nature.

Force

To proceed into relativistic *dynamics*, consider a particle subjected to a constant force of magnitude F . Using a relativistic version of Newton's first law (ϕ instead of p for momentum):

$$\frac{d\phi}{dt} = F$$

$$\phi = Ft = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$v = \frac{\frac{F}{m}t}{\sqrt{1 + \left(\frac{Ft}{mc} \right)^2}} = \dot{x}$$

$$x = \int_0^t \left(\frac{\frac{F}{m}T}{\sqrt{1 + \left(\frac{FT}{mc} \right)^2}} \right) dT = \frac{mc^2}{F} \left[\sqrt{1 + \left(\frac{Ft}{mc} \right)^2} - 1 \right]$$

As it is clear here, a particle follows a hyperbolic trajectory when acted upon a constant force - differing from the parabolas objects draw for non-relativistic regions.

To observe lorentz transformations of force vectors:

$$\begin{aligned} F_x' &= \frac{dp_x'}{dt'} = \frac{\gamma(m d\eta^1 - m\beta d\eta^0)}{\gamma \left(dt - \frac{\beta}{c} dx \right)} \\ &= \frac{\gamma(dp_x - m\beta dp^0)}{\gamma \left(dt - \frac{\beta}{c} dx \right)} = \frac{\frac{dp_x}{dt} - \beta \frac{dp^0}{dt}}{1 - \frac{\beta}{c} \frac{dx}{dt}} \\ &= \frac{F_x - \frac{\beta}{c} \frac{dE}{dt}}{1 - \frac{\beta}{c} v_x} \end{aligned}$$

Momentum in the y & z directions are invariant under transformations for relativistic effects only occur along the direction of motion. Giving:

$$\begin{aligned} F_y' &= \frac{dp_y'}{dt'} = \frac{dp_y}{\gamma \left(dt - \frac{\beta}{c} dx \right)} = \frac{F_y}{\gamma \left(1 - \frac{\beta}{c} v_x \right)} \\ F_z' &= \frac{F_z}{\gamma \left(1 - \frac{\beta}{c} v_x \right)} \end{aligned}$$

It is clear here, that Newton's third law of motion breaks down in relativistic regions. When mass A applies a force of F on mass B, in either A or B's

reference frames, a force of $-F$ will act back on A - the third law still applies in the two reference frames. However, to a moving frame of reference, by the relativity of simultaneity, F and $-F$ will act on different times, no longer supporting the law of action and reaction. For instance, if the observer were to run from A to B at a speed close to c , $-F$ will act before F .

As it is evident above, ordinary force gives an untidy result for transforming by both its denominator and the numerator. Thus, similar to the discussion of velocity, *proper* force is easier to deal with for it only transforms by its numerator. Proper force is also known as the Minkowski force K and is a plain 4-vector:

$$K^\mu = \frac{dp^\mu}{d\tau}$$

With spatial components $\mu=1,2,3$, a relationship between the Minkowski force K and the ordinary force F gives:

$$K^{1,2,3} = \frac{dp^{1,2,3}}{d\tau} = \frac{dt}{d\tau} \frac{dp^{1,2,3}}{dt} = \frac{dt}{d\tau} F_{x,y,z} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} F_{x,y,z}$$

Although it may be more convenient, Minkowski forces are used less frequently for a particle's path in ordinary time is of greater interest.

5. A BRIEF ACCOUNT ON MULTIVARIABLE CALCULUS

The del operator

An understanding of multivariable calculus operations is essential in observing special relativity's blend with electromagnetism and electrostatics. As its name suggests, multivariable

calculus studies infinitesimal changes and sums of functions composed of two or more variables. For the sake of convenience, only 3D examples will be used from here. This by no means restricts operations to three dimensional functions. The del vector, denoted using the greek alphabet nabla (inverted Δ), is defined as the following:

$$\nabla = \left[\begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right]$$

Again, for higher dimensions, extra components will be in the del vector following the same trend - $\partial/\partial w, \partial/\partial k, \partial/\partial p \dots$ The *gradient* of vector E is known as

$$\nabla E = \left[\begin{array}{c} \frac{\partial E}{\partial x} \\ \frac{\partial E}{\partial y} \\ \frac{\partial E}{\partial z} \end{array} \right]$$

The gradient is the derivative of a multivariable function - the gradient vector points to the direction of steepest ascent. To build some intuition, the partial derivatives in the del vector can be thought of as slicing a function in x, y , and z directions and performing derivatives with the sliced pieces. Thus, at a point (q_1, q_2, q_3) , the "slope" will equal to the sum of the "slope vectors" in the x, y , and z directions defined at (q_1, q_2, q_3) .

Divergence

The divergence of vector E is the dot product of E and the del vector:

$$\nabla \cdot E = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

The divergence, as its name suggests, calculates how much a function moves away or towards a certain point. Again, looking at the equation, divergence is equal to the scalar sum of the derivatives in x,y, and z directions - vectors that reach out or in. If the divergence at a certain point is:

$$\begin{aligned}\nabla \cdot E > 0 &\rightarrow \text{more vectors pointing out} \\ \nabla \cdot E = 0 &\rightarrow \text{equal number of vectors pointing in and out} \\ \nabla \cdot E < 0 &\rightarrow \text{more vectors pointing in}\end{aligned}$$

When divergence is positive, the point is commonly called a source - it "produces" vector fields. Similarly, when the divergence is negative, the point is called a sink for it intakes vector fields.

Curl

The curl of a vector is the cross product with the del vector:

$$\nabla \times E = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{bmatrix} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{i} - \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \hat{j} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{k}$$

The curl of a vector observes how much the function *rotates* around a point. Unlike divergence, curl is a vector that defines infinitesimal rotations at a point. If the magnitude of curl is:

$$\begin{aligned}\nabla \times E > 0 &\rightarrow \text{rotating clockwise} \\ \nabla \times E = 0 &\rightarrow \text{no rotation} \\ \nabla \times E < 0 &\rightarrow \text{rotating counterclockwise}\end{aligned}$$

Integration

In single-variable calculus, integrals are mathematical operations that represent a summation of infinitesimal components. With 3D objects, surface and volume integrals notate the addition of small pieces of a function along its surface and volume, respectively:

Surface integral

Volume integral

$$\iint_S$$

$$\iiint_V$$

The number of integral signs indicate the number of dimensional components required to characterize the quantity. A surface is two dimensional, a volumetric body is three dimensional.

The divergence theorem

The divergence theorem states that:

$$\iint_S \vec{E} \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{E} dV$$

To cultivate an intuitive sense, imagine a volume of fluid flowing with the presence of a vector field - the fluid is displaced to wherever the field points with a speed indicative from the vectors' magnitude. Suppose a hollow object with a permeable surface is placed at a point in the vector field. The divergence theorem describes that the amount of fluid that exits through an object's surface is equal to the amount of fluid that enters the object's volume. As an informal proof, consider a vector field with components $E = \langle A, B, C \rangle$ passing through a cube that has one of its vertices on the origin.

Flux is known as a quantity that measures the number of vector field lines that passes through a surface perpendicularly. Flux of vector E through

surface S can be written with the following surface integral:

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

Since total flux through the cube is equal to the sum of fluxes through all faces, the sum of all infinitesimal blocks of dimensions dx , dy , and dz subjected to vector $\langle dA, dB, dC \rangle$ must give the total flux, thus showing the divergence theorem:

$$\sum_{\mu=1}^3 \Phi_{\mu} = \Phi_1 + \Phi_2 + \Phi_3$$

$$\Phi_1 = \text{flux through faces 1 \& 2} = dA \, dy \, dz$$

$$\Phi_2 = \text{flux through faces 3\&4} = dB \, dx \, dz$$

$$\Phi_3 = \text{flux through faces 5\&6} = dC \, dx \, dz$$

$$\begin{aligned} \sum_{\mu=1}^3 \Phi_{\mu} &= dA \, dy \, dz + dB \, dx \, dz + dC \, dx \, dz \\ &= \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) dx \, dy \, dz = \nabla \cdot \mathbf{E} \, dV \end{aligned}$$

$$\sum_{\mu=1}^3 \Phi_{\mu} = \iint_S \mathbf{E} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{E} \, dV$$

Stokes theorem

Stokes theorem states:

$$\int_C \vec{E} \cdot d\vec{r} = \iint_S \nabla \times \vec{E}$$

To topologically comprehend the theorem, consider a fluid flowing clockwise on a divisible square:

The large square is then cut into four equal sized squares subjected to the same fluid flow:

As evident in the diagram, fluid flow on the inside "cross" negates each other for they always have another *pair* running in the opposite direction.

Since the only streams that do not cancel lie on the outer boundary, the original flow pattern is considered unaffected. With deeper thought, this pattern will be consistent with even more square cuts - the square can be cut an infinite number of times and still maintain its net outer flow. Thus, said in another way, vector $\mathbf{E}$ that runs along closed loop C :

$$\int_C \vec{E} \cdot d\vec{r}$$

is equal to the sum of infinitesimal loops that render the total area the loop encloses:

$$\iint_S \nabla \times \vec{E} \, d\vec{S}$$

These two elegant, powerful theorems are fundamental in the derivation and comprehension of the four electrodynamic equations. As it will be explored, these tools built, disproved, and fixed the basis of electromagnetism and statics with the addition of relativistic studies.

6. MAXWELL'S EQUATIONS

By the time when electrodynamics was believed to be “completed”, several physicists started noticing fundamental flaws that were inexplicable with existing studies. The most notable of these physicists was James Clerk Maxwell. Maxwell identified the inconsistency of the existing equations, fixed it appropriately, and communicated his findings in his *Treatise on Electricity and Magnetism* in 1873 [2]. For the sake of understanding, it is imperative to investigate pre-Maxwell equations.

Gauss's law

Consider a conductive sphere of radius R with charge q . The electric field at a distance r from the sphere's center, where $r > R$, is:

$$E = k \frac{q}{r^2} \hat{r}$$

as given by Coulomb's law. k is known as coulomb's constant, a quantity equal to $1/4\pi\epsilon_0$. ϵ_0 is electrical permittivity of free space - a measure of how well a vacuum supports electric fields.

$$k = 8.98 \times 10^9 \text{ N m}^2 \text{C}^{-2}$$

A surface is needed to perform the surface and volume integrals over the E-field the sphere generates. Such surface is called a *Gaussian* surface, and in this case, a sphere of radius r will be used:

Gaussian Surface

Applying the divergence theorem:

$$\iint_S E \cdot da = \iiint_V \nabla \cdot E dV = \iint_S k \frac{q}{r^2} \hat{r} \cdot da$$

Since the specified Gaussian surface is a sphere, the sum of all surface components above will equal to $4\pi r^2$:

$$\iint_S k \frac{q}{r^2} \hat{r} \cdot da = 4\pi r^2 k \frac{q}{r^2} = 4\pi k q$$

Charge can be written with the following volume integral in terms of charge density ρ :

$$q = \iiint_V \rho dV$$

Substituting this integral for q and equating with the original divergence theorem relation:

$$\iiint_V \nabla \cdot E dV = 4\pi k \iiint_V \rho dV$$

Since both integrals integrate over dV , the following relation can only be true if the integrands are equal:

$$\nabla \cdot E = 4\pi k \rho = \frac{\rho}{\epsilon_0}$$

This remarkable result is known as Gauss's flux law of electric fields.

Faraday's law

If the divergence of an electric field is directly proportional to charge density and inversely proportional to electric permittivity, what would its curl be? Faraday's law states that voltage ϵ is equal to the change of magnetic flux per unit time:

$$\epsilon = - \frac{d\Phi_B}{dt}$$

By the definition of fluxes, Φ_B equals to

$$\Phi_B = \iint_S \vec{B} \cdot d\vec{A}$$

Thus:

$$-\frac{d\Phi_B}{dt} = -\frac{d}{dt} \left(\iint_S \vec{B} \cdot d\vec{A} \right)$$

Here, Kirchoff's voltage law states that voltage can also be written as the line integral:

$$\int_C \vec{E} \cdot d\vec{l}$$

Applying stokes theorem, the integral above must equal:

$$\iint_S \nabla \times \vec{E} \cdot d\vec{A}$$

Equating this with ϵ :

$$\iint_S \nabla \times \vec{E} \cdot d\vec{A} = -\frac{d}{dt} \left(\iint_S \vec{B} \cdot d\vec{A} \right)$$

Giving:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This representation of Faraday's law is known as the Maxwell-Faraday law of electromagnetic induction.

Gauss's law of magnetism

Similar to how Faraday's original law has to be taken as an experimental fact, the impossibility of magnetic monopoles is also taken for granted. Magnets are observed to split into two

separate magnets when sliced through - north poles and south poles cannot exist independently: For this reason, the divergence of a magnetic field must be zero. Though it may not be obvious, since both sources and sinks exist together, the system cannot produce or intake a net number of vector fields:

As observable, magnetic fields are always fully "absorbed" by the S pole. Thus, a surface fully containing a magnet cannot have a divergence of a magnitude greater than zero - there's an equal number of vector lines that exit and return to the surface:

$$\nabla \cdot \vec{B} = 0$$

This statement is known as Gauss's law for magnetic fields.

Ampere's law

To calculate the curl of a magnetic field, consider a straight wire in a closed circuit carrying current of magnitude I :

The strength of the magnetic field at point P distance r away from the wire is given as:

$$B = \frac{\mu_0 I}{2\pi r}$$

(Eq. 6.1)

Thus, all points on the *Amperian Loop* shown above will experience a magnetic field of (Eq. 6.1) teslas. (Eq. 6.1) is a result provided by Biot-Savart's law of steady currents and

$$B = \frac{\mu_0 I}{4\pi} \int \frac{dS \times \hat{\eta}}{\eta^2}$$

magnetism:

Where μ_0 is magnetic permittivity of free space, the measure of how well a vacuum supports a magnetic field. η and S are vectors that are directed to the point of consideration and along the wire, respectively. The derivation of the result (Eq. 6.1) is trivial to explore for the purpose of this section. Using (Eq. 6.1), the line integral of a magnetic field:

$$\int_l B \cdot dl$$

Since the sum of components along the Amperian loop is equal to $2\pi r$:

$$B \int_l dl = \frac{\mu_0 I}{2\pi r} 2\pi r = \mu_0 I$$

Similar to the derivation Gauss's law of electrostatics, I can be represented with the following integral with integrand as current density J :

$$I = \iint_S J \cdot dS$$

The negative sign is placed since the quantity dQ/dt will always be negative

Substituting results in Ampere's law:

$$\int_l B \cdot dl = \iint_S \nabla \times B \cdot dS = \mu_0 \iint_S J \cdot dS$$

$$\nabla \times B = \mu_0 J$$

Ampere's law describes the formation of circular magnetic fields around a current-carrying wire.

7. CHARGE CONTINUITY

The four Pre-Maxwell equations were later found to be *inconsistent* with the continuity equation of charge. Consider an object of volume V and surface S bearing a charge of magnitude Q . If charges leave the surface, what relation can be developed with the system? Since Q can be represented with the volume integral:

$$Q = \int_V \rho dV$$

Taking the derivative of this expression with respect to t , current I can be found, the rate at which Q leaves the object:

$$I = -\frac{dQ}{dt} = -\frac{d}{dt} \left(\int_V \rho dV \right)$$

when charges leave the sphere - current is always a positive value.

Bringing the time derivative into the integral:

$$\int_V -\frac{d\rho}{dt} dV$$

Since current that runs across a surface can also be written as:

$$\oint_S J \cdot dS$$

Where J is current density and the integral is the summation of infinitesimal fragments of the sphere's surface dS . By the divergence theorem:

$$\oint_S J \cdot dS = \int_V \nabla \cdot J dV$$

Equating this with the original expression for current:

$$\int_V -\frac{d\rho}{dt} dV = \int_V \nabla \cdot J dV$$

$$\nabla \cdot J = -\frac{d\rho}{dt}$$

It becomes clear that Ampere's law fails with the application of the continuity equation that is to be consistent for all vector fields of J . Charge count is conserved for all frames of reference regardless of velocity.

What does change, however, is charge density. Consider a copper wire with equal numbers of electrons and protons. Following Coulomb's law, when a positive charge moving with a speed of v along the wire is introduced, no forces will act for all electrostatic forces will cancel within the wire. Even with current, the net charge in the wire intuitively seems constant. However, it is an observed fact that moving charges are subjected to a force next to a current-carrying wire.

Length contraction along the charges accounts for the forces a moving charge experiences. When current of magnitude I is applied as shown above, electrons begin moving to the right with velocity v_e . According to relativistic velocity summation, the electron's speed in the moving charge's reference frame will be:

$$V_- = \frac{v - v_e}{1 - \frac{v_e v}{c^2}}$$

Similarly, the positive charges in the wire will seem to travel to the left at velocity v :

$$V_+ = \frac{0 - v}{1 - \frac{0v}{c^2}} = -v$$

Suppose that charge distribution was equal to λ_0 for both electrons and protons in the wire at rest. The wire is subjected to length contraction when the total charge stays constant. This will *increase* charge distribution. Thus:

$$\lambda_+ = \frac{\lambda_0}{\sqrt{1 - \frac{V_+^2}{c^2}}} \quad \lambda_- = \frac{\lambda_0}{\sqrt{1 - \frac{V_-^2}{c^2}}}$$

Since V_+ is clearly greater than V_- , charge distribution is greater with the electrons. Thus, the wire is perceived to be negative in the frame of the moving positive charge.

Due to charge continuity and distribution contribution, Gauss's law of electrostatics and Ampere's law were noticed to be inconsistent. As for the other two, statements were developed purely experimentally. As it is discernible in the following chapter, With the built intuition and understanding, the failure of the two laws and how they were repaired by Maxwell is relatively easy to perceive.

8. MAXWELL'S ADDITION TO EXISTING EQUATIONS

Gauss's law of electrostatics simply scales its charge density by a factor of γ for the same reason investigated in the previous chapter - dimensions of the surface contracts:

$$\rho_0 = \frac{\sum Q}{V_0}$$

$$\rho' = \frac{\sum Q}{\frac{V_0}{\gamma}} = \gamma \rho_0$$

To be exact, Gauss's law is still valid under relativistic conditions for the relation that the divergence of an electric field is linearly proportional to charge density never changes. However, it is crucial to note the change in charge density in investigating the transformation of electric and magnetic fields.

To investigate the inconsistency of Ampere's law, take the divergence of the curl of a magnetic field:

$$\nabla \cdot (\nabla \times B) = \mu_0 (\nabla \cdot J)$$

Since the divergence of a curl is always equal to zero, Ampere's law states that:

$$\nabla \cdot J = 0$$

for all current density vectors. This clearly contradicts the charge continuity equation. The only way the statement above can be true is when charges are stationary within a wire. When no current flows, charge density will be constant, thus the value:

$$-\frac{d\rho}{dt}$$

Will equal zero.

To amend this inconsistency, Maxwell added a term called *displacement current*:

$$\mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

To formulate what is known as Ampere-Maxwell equation:

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

This repairment was developed with more scrutiny than algebraic derivation. Maxwell noticed that taking the divergence now resulted in:

$$\nabla \cdot (\nabla \times B) = \mu_0 (\nabla \cdot J) + \mu_0 \epsilon_0 \frac{\partial (\nabla \cdot E)}{\partial t}$$

$$= -\mu_0 \frac{d\rho}{dt} + \mu_0 \epsilon_0 \frac{\partial \left(\frac{\rho}{\epsilon_0} \right)}{\partial t} = 0$$

Showing consistency. It is noticed that the result relies on Gauss's law of electrostatics. Even with relativistic effects, $\gamma \rho_0$ will still cancel the first term on the right hand side.

Maxwell went on to notice that there were non-trivial results even when charge density and current density equaled zero. Taking the time derivative of the curl of a magnetic field:

$$\frac{\partial}{\partial t} (\nabla \times B) = \frac{\partial}{\partial t} (\mu_0 J) + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\frac{\partial E}{\partial t} \right) = 0 + \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

Evaluating:

$$-\nabla \times \nabla \times E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial^2 E}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 E$$

As it turns out, the equation above characterized a wave, it was a *wave equation*. This particular equation gives its wave a speed of:

$$\frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Which, remarkably, is equal to the speed of light: 299 792 458 m/s. Maxwell's discovery led to the direct relationship between light and electrodynamics.

9. FIELD TRANSFORMATIONS

Consider three inertial frames of reference - S_0 , S , and S' . S_0 is at rest, S moves to the right with velocity v as measured from S_0 , and S' moves to the right with velocity v_{rel} with respect to S . A square-plate capacitor of side length a moves to the right with frame S and S' , as shown above.

x' : measured in frame S'
 x : measured in frame S_0
 v_{rel} : velocity of S' relative to S

If the two square plates bear charges of magnitude Q , charge density observed in the frame of S_0 will be contracted.

Giving:

$$\sigma = \frac{Q}{aa'} = \frac{Q}{a \frac{a}{\gamma_0}} = \gamma_0 \sigma_0$$

Where σ_0 is surface charge density at rest:

$$\sigma_0 = \frac{Q}{a^2}$$

γ_0 would be:

$$\gamma_0 = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}}$$

Following, Gauss's law, S_0 will observe an electric field in S of magnitude:

$$E_z = \frac{\sigma}{\epsilon_0} = \frac{\gamma_0 \sigma_0}{\epsilon_0}$$

Since charges move along right with the capacitor, inducing current, a magnetic field will also be detectable by a S_0 observer. Following the right hand rule, the field will point in the positive y -direction. Magnitude is determined by Ampere's law:

$$B_y = \mu_0 v_0 \gamma_0 \sigma_0$$

S' will move at a velocity v' with respect to S_0 , given by the relativistic velocity sum with v_0 :

$$v' = \frac{v_0 + v_{rel}}{1 + \frac{v_0 v_{rel}}{c^2}}$$

γ' would be:

$$\gamma' = \frac{1}{\sqrt{1 - \frac{v'^2}{c^2}}}$$

All relations would be in the same format just with the use of γ' instead of γ_0 and v' instead of v_0 . Thus, to write E_z and B_y in terms of E_z' and B_y' :

$$E_z = \frac{\sigma}{\epsilon_0} = \frac{\gamma_0 \sigma_0}{\epsilon_0}$$

$$B_y = \mu_0 v_0 \gamma_0 \sigma_0$$

Using the relation $\sigma = \gamma_0 \sigma_0$

$$\begin{aligned} E_z' &= \frac{\gamma' \sigma_0}{\epsilon_0} = \frac{\frac{\sigma}{\gamma_0} \gamma'}{\epsilon_0} = \frac{\gamma' \sigma}{\gamma_0 \epsilon_0} \\ B_y' &= \frac{\gamma' \mu_0 v' \sigma}{\gamma_0} \\ \frac{\gamma'}{\gamma_0} &= \frac{\sqrt{1 - \frac{v_0^2}{c^2}}}{\sqrt{1 - \frac{v'^2}{c^2}}} = \frac{1 + \frac{v_{rel} v_0}{c^2}}{\sqrt{1 - \frac{v_{rel}^2}{c^2}}} = \gamma \left(1 + \frac{v_{rel} v_0}{c^2} \right) \\ E_z' &= \gamma \left(1 + \frac{v_{rel} v_0}{c^2} \right) \frac{\sigma}{\epsilon_0} = \gamma \left(E_z + \frac{v_{rel} v_0}{c^2} \frac{\sigma}{\epsilon_0} \right) = \gamma \left(E_z + \frac{v_{rel}}{c^2} \frac{B_y}{\mu_0 \epsilon_0} \right) \\ B_y' &= \gamma \left(1 + \frac{v_{rel} v_0}{c^2} \right) \mu_0 v' \sigma = \gamma (E_z \epsilon_0 \mu_0 v_{rel} + B_y) \end{aligned}$$

Since:

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

Field transformations of E_z and B_y can be established in simpler terms:

$$\begin{aligned} E_z' &= \gamma (E_z + v_{rel} B_y) \\ B_y' &= \gamma \left(\frac{v_{rel}}{c^2} E_z + B_y \right) \end{aligned}$$

To find transformations of E_y and B_z , the capacitor can simply be “flipped” to have the positive plate facing the positive y direction:

This completes the transformation of electric and magnetic fields:

$$\begin{aligned} E_x' &= E_x \\ E_y' &= \gamma (E_y - v_{rel} B_z) \\ E_z' &= \gamma (E_z + v_{rel} B_y) \\ B_x' &= B_x \\ B_y' &= \gamma \left(\frac{v_{rel}}{c^2} E_z + B_y \right) \\ B_z' &= \gamma \left(B_z - \frac{v_{rel}}{c^2} E_y \right) \end{aligned}$$

10. CONCLUDING STATEMENT

This field developed by Albert Einstein in the 20th century is one of the most diverse, elegant, and applicable approaches to physics, and it truly deserves greater attention even in elementary courses. Fundamental concepts in various fields are outlined beautifully with the language of special relativity - there are countless other examples not dealt in this paper that supports this statement. I would like to end this paper with a note that this discussion is not complete. Especially with electrodynamics, reviewing various sources led me to the realization that relativistic effects on natural constants are not dealt in detail. Of course, free space constants will not change under relativity. However, in the case of the atmosphere, electric and magnetic permittivity is to change when a moving observer views the system - air molecules will contract along the direction of motion [1]. Since the apparent speed of light (through a medium) is not universal, the Lorentz transformations may require modifications. Although this effect was conceptualized in a few papers [5], the idea was never discussed in context of re-defining such fundamental postulates. Shen's work on this subject [5], for instance, supports that

when a time-harmonic electromagnetic wave travels in between two inertial frames moving with a non-zero relative velocity, electric and magnetic permittivity transform in a direction *perpendicular* to the poynting vector of the electromagnetic wave, which is highly counterintuitive since relativistic effects are relevant to motion parallel to the velocity vector. Such inquiries are interesting considerations for future studies.

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