RINGS OF RECTANGULAR MATRICES UNDER SANDWICH OPERATION

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ABSTRACT. Let $M_{mn} = M_{mn}(R)$ denote the set of all $m \times n$ matrices over a ring R , and fix some $n \times m$ matrix A . An associative operation $*$ may be defined on M_{mn} by $X * Y = XAY$ for all $X, Y \in M_{mn}$ and the resulting sandwich ring $M_{mn}^A(R)$ is known as the Munn ring which are a fundamental tool in the representation theory of finite semigroups. In this article we consider the matrix ring $M_{mn}^{J_1}(R)$ where R is a ring with unity and J_1 the $n \times m$ matrix whose first entry is 1 and all other entries are zero and discuss some of its structural properties like idempotent elements Green's relations.

1. INTRODUCTION

Let R be a ring and $I = \{1, \dots, m\}$ and $\Lambda = \{1, \dots, n\}$ be index sets, P be a $\Lambda \times I$ matrix over R . Let $M(R; I, \Lambda; P)$ be the additive group of all $I \times \Lambda$ matrices over R . Define a product $*$ in $M(R; I, \Lambda; P)$ by $A * B = APB$ ($A, B \in M(R; I, \Lambda; P)$). Then $M(R; I, \Lambda; P)$ is a ring, which we call the Munn $I \times \Lambda$ matrix ring over R with sandwich matrix P , briefly, the Munn ring is named after Douglas Munn. Munn-rings play a key role in representation theory of finite semigroups.

1.1. Preliminaries. A binary operation on a set S is a mapping $*$: $S \times S \rightarrow S$. A groupoid is a non-empty set S together with a binary operation $*$ on S , if the binary operation of the groupoid is associative then it is a semigroup.

An element e in a semigroup S is called a left [right] identity element of a in S if $ea = a$ [$ae = a$]. Similarly we can define left [right] zero of an element. For an element a in a semigroup S , the smallest left ideal containing a is $Sa \cup \{a\}$ or S^1a is called the principal left ideal generated by a . The equivalence relation $\mathcal{L}$ on S is defined on S by $a \mathcal{L} b$ if and only if $S^1a = S^1b$. Similarly we say that $a \mathcal{R} b$ if and only if $aS^1 = bS^1$. The following lemma is due to J.A. Green.

Lemma 1. *Let a, b be elements of a semigroup S . Then*

- *$a \mathcal{L} b$ if and only if there exists $x, y \in S^1$ such that $xa = b$ and $yb = a$*

2010 *Mathematics Subject Classification.* 20M10.

Key words and phrases. Munn Ring, Green's Relations, Band.

- $a\mathcal{R}b$ if and only if there exists $x, y \in S^1$ such that $ax = b$ and $by = a$

The following is lemma 2.1 of [3]

Lemma 2. *The equivalence relations $\mathcal{L}$ and $\mathcal{R}$ on any semigroup commutes.*

Definition 1. *The relation $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ is the smallest equivalence relation $\mathcal{L} \vee \mathcal{R}$ containing both $\mathcal{L}$ and $\mathcal{R}$ and is known as $\mathcal{D}$ relation. The relation $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ is also an equivalence relation known as Green's $\mathcal{H}$ relation.*

Definition 2. *An non-empty set R together with two binary operations $+$ and $\cdot$ called addition and multiplication respectively, is a ring if it has the following three properties.*

- (1) $(R, +)$ is an abelian group,
- (2) $(R, \cdot)$ is a semigroup and
- (3) addition distributes over multiplication.

If the ring $(R, \cdot)$ has a multiplicative identity, it is unique and is denoted by 1_R and is called the identity [unity] of R . For a ring R with 1, an element $u \in R$ is said to be a unit or invertible if there exists $v \in R$ such that $uv = vu = 1$, such a v is unique and is the multiplicative inverse of u denoted u^{-1} .

An element $a \in R$ is said to be a left[right] zero-divisor if there exist $b \neq 0$ such that $a \cdot b = 0$, [$b \cdot a = 0$]. An element $a \in R$ is said to be a zero-divisor if a is either a left zero-divisor or a right zero-divisor

Definition 3. *An element $a \in R$ is nilpotent if there is a positive integer n such that $a^n = 0$, where a^n stands for $a.a.a...a$ (n -factors). An element $a \in R$ is an idempotent if $a^2 = a$.*

The set of all idempotents in R is denoted by $E(R)$. Let R be a ring, and $a \in R$, the principal left(right) ideal $(a)_l$ [$(a)_r$] generated by a is $\mathbb{Z}a + Ra$ [$\mathbb{Z}a + aR$], where $\mathbb{Z}$ is the ring of integers. If R is having unity, then $(a)_l = Ra$, $(a)_r = aR$.

The **Green's Equivalence relations** $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}$ relations in a ring R is defined as follows:

$$a\mathcal{L}b \iff (a)_l = (b)_l$$

$$a\mathcal{R}b \iff (a)_r = (b)_r$$

$$a\mathcal{J}b \iff (a) = (b)$$

where $(a), (b)$ are two sided ideals generated by a and b respectively.

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2. THE RING $M_{m,n}^{J_1}$ OVER A FIELD F .

Let $M_{m,n}^{J_1}$ be the set of all $m \times n$ matrices over a field F with matrix multiplication defined by $AB = AJ_1B$ where J_1 is the $n \times m$ matrix whose first entry is one and all other entries are zero. For $A = (a_{ij})$, $B = (b_{ij}) \in M_{m,n}^{J_1}$, we have

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} 1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & \cdots & a_{11}b_{1n} \\ \cdots & \cdots & \cdots \\ a_{m1}b_{11} & \cdots & a_{m1}b_{1n} \end{bmatrix}.$$

Clearly $M_{m,n}^{J_1}$ is a ring without unity.

In the following we consider the case of 2×3 matrices $M_{2,3}^{J_1}(F)$ and describe its idempotents as well as Green's equivalences.

Proposition 1. *If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ in $M_{2,3}^{J_1}$, then A is an idempotent if and only if $a_{1,1}a_{1,j} = a_{1,j}$ and $a_{2,1}a_{1,j} = a_{2,j}$ for $j = 1, 2, 3$.*

Proof. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ in $M_{2,3}^{J_1}$ be an idempotent. So

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11}a_{11} & a_{11}a_{12} & a_{11}a_{13} \\ a_{21}a_{11} & a_{21}a_{12} & a_{21}a_{13} \end{bmatrix}.$$

Hence $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ is an idempotent when

$$\begin{bmatrix} a_{11}a_{11} & a_{11}a_{12} & a_{11}a_{13} \\ a_{21}a_{11} & a_{21}a_{12} & a_{21}a_{13} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

So it is easily seen that a matrix $A \in M_{2,3}^{J_1}$ is an idempotent when $a_{i1}a_{1j} = a_{ij}$ for $i = 1, 2$ and $j = 1, \dots, 3$. $\square$

Lemma 3. *Let A and B be two matrices of $M_{m,n}^{J_1}$ then*

- (1) $A \mathcal{L} B$ if and only if the rows of B are multiple of the first row of A .
- (2) $A \mathcal{R} B$ if and only if the columns of B are multiple of the first column of A .

Proof. We have $a\mathcal{L}b$ if and only if there exists $X, Y \in M_{mn}^{J_1}$ such that $XJA = B$ and $YJB = A$. That is

$$\begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

$$\begin{bmatrix} x_{11}a_{11} & \cdots & x_{11}a_{1n} \\ \cdots & \cdots & \cdots \\ x_{m1}a_{11} & \cdots & x_{m1}a_{1n} \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

ie., rows of B are multiples of the first row of A .

Similarly, it is seen that
$$\begin{bmatrix} y_{11}b_{11} & \cdots & y_{11}b_{1n} \\ y_{m1}b_{11} & \cdots & y_{m1}b_{1n} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

Hence the columns of A are multiples of first column of B .

□

Example 1. Consider all 2×3 matrices on $\mathbb{Z}_2$ and $J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then

the idempotent matrices in $M_{23}^{J_1}(\mathbb{Z}_2)$, are

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & a_{12} & a_{13} \\ 1 & a_{12} & a_{13} \end{bmatrix},$$

where $a_{12}, a_{13} \in \mathbb{Z}_2$.

In the light of the above example it is easy to see that one can enumerate all the idempotent elements of any ring $M_{mn}^{J_1}(\mathbb{B})$ where $\mathbb{B}$ is a multiplicative band (ie., a semigroup of idempotents).

Remark 1. The ring $M_{n,m}^{J_1}(R)$ does not have a unity. The left [right] zero divisors in $M_{n,m}^{J_1}$ are those non-zero A, B such that $AJB = 0$ [$BJA = 0$]. But $AJB = 0$ implies

$$AJB = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & \cdots & a_{11}b_{1n} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & \cdots & a_{21}b_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}b_{11} & a_{n1}b_{12} & a_{n1}b_{13} & \cdots & a_{n1}b_{1n} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

That is the first column of A and first row of B are zero divisors of R .

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