Location of the Zeros of Certain Polynomials

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Abstract

In this paper we generalize Aziz and Mohammad's result that the zeros of a polynomial

$$P(z) = \sum_{j=0}^n a_j z^j \text{ with } ta_j \geq a_{j-1}, j = 2, 3, \dots, n \text{ for some } t > 0 \text{ of modulus greater than } \frac{t(n-1)}{n}$$

are simple, and other similar results to polynomials with lesser restrictive conditions on their coefficients.

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1. Introduction

Regarding the multiplicity of certain zeros of a polynomial, Aziz and Mohammad [1] proved the following result.

Theorem A. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 > 0.$$

Then all the zeros of $P(z)$ of modulus greater than or equal to $\frac{n}{n+1}$ are simple.

In [2], Aziz and Mohammad gave a generalization as well a refinement of Theorem A as follows.

Theorem B. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ such that for some $t > 0$,

$ta_j \geq a_{j-1} > 0, j = 2, 3, \dots, n, a_0$ may be a real or complex number. Then all the zeros of $P(z)$ of modulus greater than $\frac{t(n-1)}{n}$ are simple.

Bairagi, Jain, Mishra and Saha [3] extended Theorem B to polynomials with complex coefficients and proved the following result.

Theorem C. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ such that for some $t > 0$,

$$t|a_j| \geq |a_{j-1}|, j = 1, 2, \dots, n \text{ and for some real } \alpha, \beta, |Arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1, 2, \dots, n.$$

Then $P(z)$ cannot have a zero of order ≥ 2 with modulus greater than

$$\frac{t(n-1)}{n} \left\{ \cos \alpha + \sin \alpha + \frac{2 \sin \alpha}{n|a_n|} \sum_{j=0}^{n-2} \left(\frac{n}{t(n-1)} \right)^{n-1-j} (j+1) |a_{j+1}| \right\}.$$

In other words, all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n} \left\{ \cos \alpha + \sin \alpha + \frac{2 \sin \alpha}{n|a_n|} \sum_{j=0}^{n-2} \left(\frac{n}{t(n-1)} \right)^{n-1-j} (j+1) |a_{j+1}| \right\}$$

are simple.

In the same paper they also proved the following result.

Theorem D. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ with $\operatorname{Re} a_j = \alpha_j, \operatorname{Im} a_j = \beta_j$,

$j=0,1,\dots,n$. If for some $t > 0$

$$t\alpha_j \geq \alpha_{j-1} \geq 0, j = 2,3,\dots,n \quad \text{and} \quad \alpha_n > 0,$$

then $P(z)$ cannot have a zero of order ≥ 2 with modulus greater than

$$\frac{t(n-1)}{n} \left\{ 1 + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-j} (j+1) |\beta_{j+1}| \right\}.$$

In other words, all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n} \left\{ 1 + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-j} (j+1) |\beta_{j+1}| \right\}$$

are simple.

Gulzar and Rubia [6] proved the following generalizations of Theorems C and D.

Theorem E. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ such that for some $t > 0, k \geq 1$,

$$k|a_n| \geq \frac{1}{t}|a_{n-1}| \geq \frac{1}{t^2}|a_{n-2}| \geq \dots \geq \frac{1}{t^{n-1}}|a_1| \geq \frac{1}{t^n}|a_0|$$

$$\text{and for some real } \alpha, \beta, \quad |\operatorname{Arg} a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1, 2, \dots, n.$$

Then all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n|a_n|} \left\{ k|a_n|(\cos \alpha + \sin \alpha + 1) - |a_n| + \frac{2 \sin \alpha}{n} \sum_{j=0}^{n-2} \left(\frac{n}{t(n-1)} \right)^{n-1-j} (j+1) |a_{j+1}| \right\}$$

are simple.

Theorem F. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ with $\operatorname{Re} a_j = \alpha_j, \operatorname{Im} a_j = \beta_j$,

$j=0,1,\dots,n$. If for some $t > 0$ and $k \geq 1$,

$$k\alpha_n \geq \frac{1}{t}\alpha_{n-1} \geq \frac{1}{t^2}\alpha_{n-2} \geq \dots \geq \frac{1}{t^{n-1}}\alpha_1 \geq \frac{1}{t^n}\alpha_0 \geq 0, \quad \text{and} \quad \alpha_n > 0,$$

then $P(z)$ cannot have a zero of order ≥ 2 with modulus greater than

$$\frac{t(n-1)}{n} \left\{ 2k - 1 + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-j} (j+1) |\beta_{j+1}| \right\}.$$

In other words, all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n} \left\{ 2k - 1 + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-j} (j+1) |\beta_{j+1}| \right\}$$

are simple.

2. Main Results

In this paper we prove the following generalizations of Theorems E and F.

Theorem 1. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ such that for some $t > 0, k_1 \geq 1, k_2 \geq 1$,

$$k_1 |a_n| \geq \frac{1}{t} |a_{n-1}| \geq \frac{1}{t^2} |a_{n-2}| \geq \dots \geq k_2 \cdot \frac{1}{t^{n-\lambda}} |a_\lambda| \geq \frac{1}{t^{n-\lambda+1}} |a_{\lambda-1}| \geq \dots \geq \frac{1}{t^{n-1}} |a_1| \geq \frac{1}{t^n} |a_0|.$$

Then all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n} \left\{ k_1 (\cos \alpha + \sin \alpha + 1) - 1 + \frac{2k_2 a^{n-1-\lambda}}{n|a_n|} (\lambda + 1) |a_{\lambda+1}| \sin \alpha + \frac{(k_2 - 1) a^{n-1-\lambda}}{n|a_n|} |a_{\lambda-1}| \right. \\ \left. + \frac{2 \sin \alpha}{n|a_n|} \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-2} a^{n-1-j} (j+1) |a_{j+1}| \right\}$$

are simple.

Remark 1. For $k_1 = k, k_2 = 1$, Theorem 1 reduces to Theorem E.

Theorem 2. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ with $\operatorname{Re} a_j = \alpha_j, \operatorname{Im} a_j = \beta_j$,

$j=0, 1, \dots, n$. If for some $t > 0$ and $k \geq 1$,

$$k_1 \alpha_n \geq \frac{1}{t} \alpha_{n-1} \geq \frac{1}{t^2} \alpha_{n-2} \geq \dots \geq k_2 \cdot \frac{1}{t^{n-\lambda}} \alpha_\lambda \geq \frac{1}{t^{n-\lambda+1}} \alpha_{\lambda-1} \geq \dots \geq \frac{1}{t^{n-1}} \alpha_1 \geq \frac{1}{t^n} \alpha_0 \geq 0, \alpha_n > 0,$$

then $P(z)$ cannot have a zero of order ≥ 2 with modulus greater than

$$\frac{t(n-1)}{n} \left\{ (2k_1 - 1) + \frac{(k_2 - 1) a^{n-1-\lambda}}{n \alpha_n} (\lambda + 1) |\alpha_{\lambda+1}| + \frac{2}{n \alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-1} (j+1) |\beta_{j+1}| \right\}.$$

In other words, all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n} \left\{ (2k_1 - 1) + \frac{(k_2 - 1) a^{n-1-\lambda}}{n \alpha_n} (\lambda + 1) |\alpha_{\lambda+1}| + \frac{2}{n \alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-1} (j+1) |\beta_{j+1}| \right\}$$

are simple.

Remark 2. For $k_1 = k, k_2 = 1$, Theorem 1 reduces to Theorem F.

3. Lemmas

For the proofs of the above results we need the following lemmas.

Lemma 1. For any two complex numbers z_1, z_2 with $|z_1| \geq |z_2|$ and $|\operatorname{Arg} z_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1, 2$, for some real α, β ,

$$|z_1 - z_2| \leq (|z_1| - |z_2|) \cos \alpha + (|z_1| + |z_2|) \sin \alpha.$$

Lemma 1 is due to Govil and Rahman [4].

Lemma 2. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that for some real α, β ,

$$|Arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1, 2, \dots, n,$$

and

$$k_1 |a_n| \geq |a_{n-1}| \geq \dots \geq k_2 |a_\lambda| \geq |a_{\lambda-1}| \geq \dots \geq |a_1| \geq |a_0|.$$

Then $P(z)$ has all its zeros in

$$|z| \leq \frac{1}{|a_n|} \left\{ k_1 |a_n| (\cos \alpha + \sin \alpha + 1) - |a_n| + 2k_2 |a_\lambda| \sin \alpha + (k_2 - 1) |a_\lambda| \right\} + 2 \sin \alpha \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-1} |a_j|$$

Proof of Lemma2. Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \dots + (a_1 - a_0)z + a_0 \\ &= -a_n z^{n+1} - (k_1 - 1)a_n z^n + (k_1 a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots \\ &\quad + (a_{\lambda+1} - k_2 a_\lambda)z^{\lambda+1} + (k_2 a_\lambda - a_\lambda)z^{\lambda+1} + (k_2 a_\lambda - a_{\lambda-1})z^\lambda - (k_2 a_\lambda - a_\lambda)z^\lambda \\ &\quad \dots + (a_1 - a_0)z + a_0 \end{aligned}$$

For $|z| > 1$, we have by using the hypothesis and Lemma 1,

$$\begin{aligned} |F(z)| &\geq |a_n| |z|^n \left[|z + k_1 - 1| - \frac{1}{|a_n|} \left\{ |k_1 a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \dots + \frac{|a_{\lambda+1} - k_2 a_\lambda|}{|z|^{n-\lambda-1}} + \frac{(k_2 - 1)|a_\lambda|}{|z|^{n-\lambda-1}} \right\} \right. \\ &\quad \left. - \frac{|k_2 a_\lambda - a_{\lambda-1}|}{|z|^{n-\lambda}} + \dots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \right] \\ &> |a_n| |z|^n \left[|z + k_1 - 1| - \frac{1}{|a_n|} \left\{ |k_1 a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + |a_{\lambda+1} - k_2 a_\lambda| + (k_2 - 1)|a_\lambda| \right\} \right. \\ &\quad \left. + |k_2 a_\lambda - a_{\lambda-1}| + \dots + |a_1 - a_0| + |a_0| \right] \\ &\geq |a_n| |z|^n \left[|z + k_1 - 1| - \frac{1}{|a_n|} \left\{ (k_1 |a_n| - |a_{n-1}|) \cos \alpha + (k_1 |a_n| + |a_{n-1}|) \sin \alpha \right. \right. \\ &\quad \left. \left. + (|a_{n-1}| - |a_{n-2}|) \cos \alpha + (|a_{n-1}| + |a_{n-2}|) \sin \alpha + \dots \right. \right. \\ &\quad \left. \left. + (|a_{\lambda+1}| - k_2 |a_\lambda|) \cos \alpha + (|a_{\lambda+1}| + k_2 |a_\lambda|) \sin \alpha + \dots \right. \right. \\ &\quad \left. \left. + (k_2 - 1)|a_\lambda| + (k_2 |a_\lambda| - |a_{\lambda-1}|) \cos \alpha + (k_2 |a_\lambda| + |a_{\lambda-1}|) \sin \alpha \right. \right. \\ &\quad \left. \left. + \dots + (|a_1| - |a_0|) \cos \alpha + (|a_1| + |a_0|) \sin \alpha + |a_0| \right\} \right] \\ &= |a_n| |z|^n \left[|z + k_1 - 1| - \frac{1}{|a_n|} \left\{ k_1 |a_n| (\cos \alpha + \sin \alpha) + 2k_2 |a_\lambda| \sin \alpha + (k_2 - 1) |a_\lambda| \right\} \right. \\ &\quad \left. - |a_0| (\cos \alpha - \sin \alpha - 1) + 2 \sin \alpha \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-1} |a_j| \right] \\ &\geq |a_n| |z|^n \left[|z + k_1 - 1| - \frac{1}{|a_n|} \left\{ k_1 |a_n| (\cos \alpha + \sin \alpha) + 2k_2 |a_\lambda| \sin \alpha + (k_2 - 1) |a_\lambda| \right\} \right. \\ &\quad \left. + 2 \sin \alpha \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-1} |a_j| \right] \\ &> 0 \end{aligned}$$

if

$$|z + k_1 - 1| > \frac{1}{|a_n|} \left\{ k_1 |a_n| (\cos \alpha + \sin \alpha) + 2k_2 |a_\lambda| \sin \alpha + (k_2 - 1) |a_\lambda| \right\} + 2 \sin \alpha \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-1} |a_j|$$

This shows that the zeros of $F(z)$ with modulus greater than 1 lie in

$$|z + k_1 - 1| \leq \frac{1}{|a_n|} \left\{ k_1 |a_n| (\cos \alpha + \sin \alpha) + 2k_2 |a_\lambda| \sin \alpha + (k_2 - 1) |a_\lambda| \right\} + 2 \sin \alpha \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-1} |a_j|$$

Since the zeros of $F(z)$ with modulus less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of $F(z)$ and hence $P(z)$ lie in

$$|z + k_1 - 1| \leq \frac{1}{|a_n|} \left\{ k_1 |a_n| (\cos \alpha + \sin \alpha) + 2k_2 |a_\lambda| \sin \alpha + (k_2 - 1) |a_\lambda| \right\} + 2 \sin \alpha \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-1} |a_j|$$

Thus for any zero z of $P(z)$, we have

$$\begin{aligned} |z| &= |z + k_1 - 1 - (k_1 - 1)| \\ &\leq |z + k_1 - 1| + |k_1 - 1| \\ &\leq \frac{1}{|a_n|} \left\{ k_1 |a_n| (\cos \alpha + \sin \alpha) + 2k_2 |a_\lambda| \sin \alpha + (k_2 - 1) |a_\lambda| \right\} + (k_1 - 1) \\ &= \frac{1}{|a_n|} \left\{ k_1 |a_n| (\cos \alpha + \sin \alpha + 1) - |a_n| + 2k_2 |a_\lambda| \sin \alpha + (k_2 - 1) |a_\lambda| \right\} + 2 \sin \alpha \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-1} |a_j| \end{aligned}$$

In other words, all the zeros of $P(z)$ lie in

$$|z| \leq \frac{1}{|a_n|} \left\{ k_1 |a_n| (\cos \alpha + \sin \alpha + 1) - |a_n| + 2k_2 |a_\lambda| \sin \alpha + (k_2 - 1) |a_\lambda| \right\} + 2 \sin \alpha \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-1} |a_j|$$

Lemma 3. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that for some real α, β ,

$$|\operatorname{Arg} a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1, 2, \dots, n,$$

and

$$k_1 |a_n| \geq |a_{n-1}| \geq a^2 |a_{n-2}| \geq \dots \geq k_2 a^{n-\lambda} |a_\lambda| \geq a^{n-\lambda+1} |a_{\lambda-1}| \dots \geq a^{n-1} |a_1| \geq a^n |a_0|.$$

Then $P(z)$ has all its zeros in

$$|z| \leq \frac{1}{a|a_n|} \left\{ k_1|a_n|(\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2a^{n-\lambda}|a_\lambda|\sin\alpha + (k_2 - 1)a^{n-\lambda}|a_\lambda| \right\} + 2\sin\alpha \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-1} a^{n-j}|a_j|$$

Proof of Lemma 3. Using the conditions of Lemma 2, we can say that the polynomial $P(z/a)$ satisfies the conditions of Lemma 1 and therefore has all its zeros in

$$|z| \leq \frac{1}{|a_n|} \left\{ k_1|a_n|(\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2a^{n-\lambda}|a_\lambda|\sin\alpha + (k_2 - 1)a^{n-\lambda}|a_\lambda| \right\} + 2\sin\alpha \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-1} a^{n-j}|a_j|$$

Accordingly the polynomial $P(z) = P(az/a)$ has all its zeros in

$$|z| \leq \frac{1}{a|a_n|} \left\{ k_1|a_n|(\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2a^{n-\lambda}|a_\lambda|\sin\alpha + (k_2 - 1)a^{n-\lambda}|a_\lambda| \right\} + 2\sin\alpha \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-1} a^{n-j}|a_j|$$

Lemma 4. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial with $\operatorname{Re} a_j = \alpha_j, \operatorname{Im} a_j = \beta_j, j = 0, 1, \dots, n$

such that for some $k \geq 1$,

$$k_1\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \dots \geq k_2\alpha_\lambda \geq \alpha_{\lambda-1} \geq \dots \alpha_1 \geq \alpha_0 \geq 0, \text{ and } \alpha_n > 0.$$

Then $P(z)$ has all its zeros in

$$|z| \leq \frac{1}{\alpha_n} \left\{ (2k_1 - 1)\alpha_n + (k_2 - 1)|\alpha_\lambda| + 2 \sum_{j=0}^n |\beta_j| \right\}.$$

Proof of Lemma 4. Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \dots + (a_1 - a_0)z + a_0 \\ &= -\alpha_n z^{n+1} - (k_1 - 1)\alpha_n z^n + (k_1\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \dots \\ &\quad + (\alpha_{\lambda+1} - k_2\alpha_\lambda)z^{\lambda+1} + (k_2\alpha_\lambda - \alpha_\lambda)z^{\lambda+1} + (k_2\alpha_\lambda - \alpha_{\lambda-1})z^\lambda - (k_2\alpha_\lambda - \alpha_\lambda)z^\lambda \\ &\quad \dots + (\alpha_1 - \alpha_0)z + \alpha_0 - i\beta_n z^{n+1} + i \sum_{j=1}^n (\beta_j - \beta_{j-1})z^j \end{aligned}$$

For $|z| > 1$, we have by using the hypothesis,

$$|F(z)| \geq |\alpha_n| |z|^n \left[|z| - \frac{1}{|\alpha_n|} \left\{ (k_1 - 1)|\alpha_n| + |k_1\alpha_n - \alpha_{n-1}| + \frac{|\alpha_{n-1} - \alpha_{n-2}|}{|z|} + \dots + \frac{|\alpha_{\lambda+1} - k_2\alpha_\lambda|}{|z|^{n-\lambda-1}} \right. \right. \\ \left. \left. + \frac{(k_2 - 1)|\alpha_\lambda|}{|z|^{n-\lambda-1}} + \frac{|k_2\alpha_\lambda - \alpha_{\lambda-1}|}{|z|^{n-\lambda}} + \dots + \frac{|\alpha_1 - \alpha_0|}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} + |\beta_n| \right. \right. \\ \left. \left. + \sum_{j=1}^n \left(\frac{|\beta_j| + |\beta_{j-1}|}{|z|^{n-j}} \right) \right\} \right]$$

$$\begin{aligned}
 &> |\alpha_n| |z|^n \left[|z| - \frac{1}{|\alpha_n|} \left\{ (k_1 - 1)|\alpha_n| + |k_1 \alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + |\alpha_{\lambda+1} - k_2 \alpha_\lambda| \right\} \right] \\
 &> |\alpha_n| |z|^n \left[|z| - \frac{1}{|\alpha_n|} \left\{ (k_1 - 1)|\alpha_n| + |k_1 \alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + |\alpha_{\lambda+1} - k_2 \alpha_\lambda| \right. \right. \\
 &\quad \left. \left. + (k_2 - 1)|\alpha_\lambda| + |k_2 \alpha_\lambda - \alpha_{\lambda-1}| + \dots + |\alpha_1 - \alpha_0| + |\alpha_0| + 2 \sum_{j=0}^n |\beta_j| \right\} \right] \\
 &= |\alpha_n| |z|^n \left[|z| - \frac{1}{\alpha_n} \left\{ (2k_1 - 1)\alpha_n + (k_2 - 1)|\alpha_\lambda| + 2 \sum_{j=0}^n |\beta_j| \right\} \right] \\
 &> 0
 \end{aligned}$$

if

$$|z| > \frac{1}{\alpha_n} \left\{ (2k_1 - 1)\alpha_n + (k_2 - 1)|\alpha_\lambda| + 2 \sum_{j=0}^n |\beta_j| \right\}.$$

This shows that the zeros of F(z) with modulus greater than 1 lie in

$$|z| \leq \frac{1}{\alpha_n} \left\{ (2k_1 - 1)\alpha_n + (k_2 - 1)|\alpha_\lambda| + 2 \sum_{j=0}^n |\beta_j| \right\}.$$

Since the zeros of F(z) with modulus less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of F(z) and hence P(z) lie in

$$|z| \leq \frac{1}{\alpha_n} \left\{ (2k_1 - 1)\alpha_n + (k_2 - 1)|\alpha_\lambda| + 2 \sum_{j=0}^n |\beta_j| \right\}.$$

Lemma 5. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ with $\operatorname{Re} a_j = \alpha_j, \operatorname{Im} a_j = \beta_j$,

$j=0,1,\dots,n$. If for some $a > 0$ and $k_1, k_2 \geq 1$,

$$k_1 \alpha_n \geq a \alpha_{n-1} \geq a^2 \alpha_{n-2} \geq \dots \geq k_2 a^{n-\lambda} \alpha_\lambda \geq a^{n-\lambda+1} \alpha_{\lambda-1} \geq \dots \geq a^{n-1} \alpha_1 \geq a^n \alpha_0 \geq 0, \alpha_n > 0$$

Then P(z) has all its zeros in

$$|z| \leq \frac{1}{a \alpha_n} \left\{ (2k_1 - 1)\alpha_n + (k_2 - 1)a^{n-\lambda} |\alpha_\lambda| + 2 \sum_{j=0}^n a^{n-j} |\beta_j| \right\}.$$

Proof of Lemma 5. Using the conditions of Lemma 4, we can say that the polynomial P(z/a) satisfies the conditions of Lemma 3 and therefore has all its zeros in

$$|z| \leq \frac{1}{\alpha_n} \left\{ (2k_1 - 1)\alpha_n + (k_2 - 1)|\alpha_\lambda| + 2 \sum_{j=0}^n |\beta_j| \right\}.$$

Accordingly the polynomial P(z) = P(az/a) has all its zeros in

$$|z| \leq \frac{1}{a \alpha_n} \left\{ (2k_1 - 1)\alpha_n + (k_2 - 1)a^{n-\lambda} |\alpha_\lambda| + 2 \sum_{j=0}^n a^{n-j} |\beta_j| \right\}.$$

4. Proofs of Theorems

Proof of Theorem 1. For $P(z) = \sum_{j=0}^n a_j z^j$, we have

$$P'(z) = \sum_{j=0}^{n-1} (j+1)a_{j+1}z^j = \sum_{j=0}^{n-1} b_j z^j = kna_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \dots + 2a_2z + a_1.$$

Since

$$k_1|a_n| \geq \frac{1}{t}|a_{n-1}| \geq \frac{1}{t^2}|a_{n-2}| \geq \dots \geq k_2 \cdot \frac{1}{t^{n-\lambda}}|a_\lambda| \geq \frac{1}{t^{n-\lambda+1}}|a_{\lambda-1}| \geq \dots \geq \frac{1}{t^{n-1}}|a_1| \geq \frac{1}{t^n}|a_0|$$

and

$$\frac{n-1}{n} \geq \frac{j}{j+1}, j = 1, 2, \dots, n-1,$$

We have

$$k_1|b_{n-1}| \geq a|b_{n-2}| \geq a^2|b_{n-3}| \geq \dots \geq k_2 a^{n-\lambda-1}|b_\lambda| \geq a^{n-\lambda}|b_{\lambda-1}| \geq \dots \geq a^{n-1}|b_0|,$$

$$\text{where } a = \frac{n}{t(n-1)}.$$

Hence by Lemma 2, it follows that all the zeros of the polynomial $P'(z)$ of degree $(n-1)$ lie in

$$|z| \leq \frac{1}{a|b_{n-1}|} \left\{ k_1|b_{n-1}|(\cos \alpha + \sin \alpha + 1) - |b_{n-1}| + 2k_2 a^{n-1-\lambda}|b_\lambda| \sin \alpha + (k_2 - 1)a^{n-1-\lambda}|a_{\lambda-1}| \right\} + 2 \sin \alpha \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-2} a^{n-1-j}|b_j|$$

i.e.

$$|z| \leq \frac{t(n-1)}{n} \left\{ k_1(\cos \alpha + \sin \alpha + 1) - 1 + \frac{2k_2 a^{n-1-\lambda}}{n|a_n|}(\lambda+1)|a_{\lambda+1}| \sin \alpha + \frac{(k_2 - 1)a^{n-1-\lambda}}{n|a_n|}|a_{\lambda-1}| \right\} + \frac{2 \sin \alpha}{n|a_n|} \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-2} a^{n-1-j}(j+1)|a_{j+1}|$$

In other words, it follows that all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n} \left\{ k_1(\cos \alpha + \sin \alpha + 1) - 1 + \frac{2k_2 a^{n-1-\lambda}}{n|a_n|}(\lambda+1)|a_{\lambda+1}| \sin \alpha + \frac{(k_2 - 1)a^{n-1-\lambda}}{n|a_n|}|a_{\lambda-1}| \right\} + \frac{2 \sin \alpha}{n|a_n|} \sum_{\substack{j=0 \\ j \neq \lambda}}^{n-2} a^{n-1-j}(j+1)|a_{j+1}|$$

are simple.

Proof of Theorem 2. For $P(z) = \sum_{j=0}^n a_j z^j$, we have

$$P'(z) = \sum_{j=0}^{n-1} (j+1)a_{j+1}z^j = \sum_{j=0}^{n-1} b_j z^j.$$

Since

$$k_1 \alpha_n \geq \frac{1}{t} \alpha_{n-1} \geq \frac{1}{t^2} \alpha_{n-2} \geq \dots \geq k_2 \cdot \frac{1}{t^{n-\lambda}} \alpha_\lambda \geq \frac{1}{t^{n-\lambda+1}} \alpha_{\lambda-1} \geq \dots \geq \frac{1}{t^{n-1}} \alpha_1 \geq \frac{1}{t^n} \alpha_0 \geq 0,$$

and

$$\frac{n-1}{n} \geq \frac{j}{j+1}, j = 1, 2, \dots, n-1,$$

We have

$$k_1 \operatorname{Re} b_{n-1} \geq a \operatorname{Re} b_{n-2} \geq \dots \geq a^{n-2} \operatorname{Re} b_1 \geq a^{n-1} \operatorname{Re} b_0,$$

$$\text{where } a = \frac{n}{t(n-1)}.$$

Hence by Lemma 4, it follows that all the zeros of the polynomial $P'(z)$ of degree $(n-1)$ lie in

$$|z| \leq \frac{t(n-1)}{n} \left\{ (2k_1 - 1) + \frac{(k_2 - 1)a^{n-1-\lambda}}{n\alpha_n} (\lambda + 1) |\alpha_{\lambda+1}| + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-1} (j+1) |\beta_{j+1}| \right\}.$$

In other words, it follows that all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n} \left\{ (2k_1 - 1) + \frac{(k_2 - 1)a^{n-1-\lambda}}{n\alpha_n} (\lambda + 1) |\alpha_{\lambda+1}| + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-1} (j+1) |\beta_{j+1}| \right\}$$

are simple.

Conflict of interest: The authors declare that they have no conflict of interest.

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