Numerical solution of nonlinear Fredholm integro-differential equations using Chebyshev polynomials

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ABSTRACT

The aim of this work is to present a numerical approximation based on Chebyshev polynomials of the first kind to solve nonlinear Fredholm integro-differential equations. The presented method reduces the main problem to a nonlinear system of algebraic equations. Numerical examples are provided to demonstrate the validity and applicability of the proposed method.

Key words: Nonlinear Fredholm integro-differential equations, Chebyshev polynomials, Collocation method, Gaussian quadrature, Clenshaw-Curtis Quadrature formula.

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INTRODUCTION

In this paper, we consider nonlinear Fredholm integro-differential equations of the type

$$\begin{cases} y'(x) = \int_0^1 K(x,t)F(y(t))dt + g(x), & 0 \leq x \leq 1, \\ y(0) = y_0 \end{cases} \quad (1)$$

Where $K \in L^2[0,1]^2$ and $g \in L^2[0,1]$ are given and y is an unknown function to be determined.

Integro-differential equations are one of the most useful mathematical tools in pure and applied mathematics, engineering and physics. Many mathematical formulations of physical phenomena contain integro-differential equations, which arise in different fields such as fluid dynamics, heat transfer, diffusion processes, neutron diffusion, biological models, nanohydrodynamics, economics, models of population growth, wind ripple in the desert and

many other disciplines lead to these equations. Integro-differential equations are usually difficult to solve analytically, so it is required to obtain an efficient approximation solution.

Recently, many different methods have been developed for determining the numerical solution of integro-differential equations. Such as, El-Gendi [12] used the Chebyshev polynomials for solving differential, integral and integro-differential equations, Brunner [9] proposed an approximate solution for Volterra integro-differential equations. In addition, the Lagrange interpolation method is presented for numerically solving integro-differential equations in [16]. However, the papers [3, 2, 5] are concerned with Chebychev collocation method for solving linear integral equations, Fredholm-Volterra integro-differential equations and nonlinear Hammerstein integral equations. Furthermore, the authors in [13], applied Hybrid Legendre polynomials and Block-Pulse functions approach to solve Volterra-Fredholm integro-differential equations. In a recent research, the Fredholm integro-differential equation is solved by using the Bernstein polynomials [14]. Barzkar et al. [6] discussed the Chebychev Wavelets method for solving Fredholm integral equations and Fredholm integro-differential equations. Different methods used orthogonal approximation by the means of many kinds of polynomials, see [15, 18, 17, 7, 1, 4, 8] and also [10, 11]. In the present paper, we develop and apply the Chebyshev polynomials of the first kind to find the approximate solution of nonlinear Fredholm integro-differential equations.

Chebyshev Polynomials

The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n , defined by the relation

$$T_n(x) = \cos(n \cos^{-1} x), \quad n \geq 0.$$

Also, they are derived from the fundamental recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \dots$$

which together with the initial conditions

$$T_0(x) = 1, \quad T_1(x) = x.$$

These polynomials are orthogonal on $[-1, 1]$ with respect to the weight function

$$\omega(x) = (1 - x^2)^{-\frac{1}{2}}$$

$$\int_{-1}^1 T_i(x) T_j(x) (1 - x^2)^{-\frac{1}{2}} dx = \begin{cases} \pi, & i = j = 0, \\ \frac{\pi}{2} \delta_{ij}, & i, j > 0 \end{cases}$$

Functions Approximations

A function $y(x) \in L^2[-1, 1]$ may be expanded by Chebyshev polynomials of the first kind as

$$y(x) = \sum_{r=0}^{\infty} a_r T_r(x), \quad (2)$$

where $a_r = \langle y(x), T_r(x) \rangle$, in which $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2[-1, 1]$. If we consider truncated series in (2), we obtain

$$y(x) \approx \sum_{r=0}^n a_r T_r(x) = A^T T(x),$$

where

$$A = [a_0, a_1, a_2, \dots, a_n]^T,$$

and

$$T(x) = [T_0(x), T_1(x), T_2(x), \dots, T_n(x)]^T.$$

ILLUSTRATION OF THE METHOD

In this section, we describe our method. Consider the one-dimensional nonlinear Fredholm integro-differential equations given in Eq(1). At first, we have

$$\begin{aligned} F(y(t)) &= F\left(\int_0^t y'(x) dx + y_0\right), \\ &= F\left(\int_0^t \left(\int_0^1 K(x, s)F(y(s)) ds + g(x)\right) dx + y_0\right), \\ &= F\left(\int_0^t \int_0^1 K(x, s)F(y(s)) ds dx + \int_0^t g(x) dx + y_0\right). \end{aligned} \quad (3)$$

Then, we approximate $F(y(t))$ by a linear combination as

$$F(y(t)) = \sum_{r=0}^n a_r T_r(t) = A^T T(t). \quad (4)$$

After that, we substitute the approximation (4) into Eq. (3) leads to

$$A^T T(t) = F\left(\int_0^t \int_0^1 K(x, s)A^T T(s) ds dx + \int_0^t g(x) dx + y_0\right). \quad (5)$$

We collocate Eq. (5) at points $t_i = \frac{1}{2}\left(1 - \cos\left(\frac{i\pi}{n}\right)\right), i = 0, 1, 2, \dots, n$

$$A^T T(t_i) = F\left(\int_0^{t_i} \int_0^1 K(x, s)A^T T(s) ds dx + \int_0^{t_i} g(x) dx + y_0\right). \quad (6)$$

In order to use Gaussian integration formula for Eq.(6), we convert the intervals $[0, 1]$ and $[0, t_i]$ into interval $[-1, 1]$ by using the following transformation, respectively

$$\eta_1 = 2s - 1, \quad s \in [0, 1],$$

$$\eta_2 = \frac{2}{t_i}x - 1, \quad x \in [0, t_i].$$

Let

$$H(x, s) = K(x, s)A^T T(s),$$

Eq.(6) becomes

$$\begin{aligned} A^T T(t_i) &= F\left[\frac{t_i}{2} \int_{-1}^1 \int_{-1}^1 H\left(\frac{t_i}{2}(\eta_2 + 1), \frac{1}{2}(\eta_1 + 1)\right) d\eta_1 d\eta_2 \right. \\ &\quad \left. + \frac{t_i}{2} \int_{-1}^1 g\left(\frac{t_i}{2}(\eta_2 + 1)\right) d\eta_2 + y_0\right]. \end{aligned} \quad (7)$$

By using the Clenshaw-Curtis quadrature [11] formula, we have

$$A^T T(t_i) = F\left[\sum_{j=0}^n \sum_{k=0}^n \omega_k \omega_j H\left(\frac{t_i}{2}(\eta_j + 1), \frac{1}{2}(\eta_k + 1)\right) + \frac{t_i}{2} \sum_{j=0}^n g\left(\frac{t_i}{2}(\eta_j + 1)\right) + y_0\right] \quad (8)$$

for $i = 0, 1, \dots, n$ and where

$$\omega_k = \frac{4}{n} \sum_{\substack{i=0 \\ i \text{ even}}}^n \frac{1}{1-i^2} \cos\left(i \frac{k\pi}{n}\right),$$

and double prime means that the first and last terms are halved. Eq.(8) generates a nonlinear system of $(n + 1)$ equations that can be solved by usual iterative Newton's method to obtain unknown vector.

Then, $y'(x)$ can be approximated as

$$\begin{aligned} y'(x) &= \int_{-1}^1 K(x, t) F(y(t)) dt + g(x), \\ &= \frac{1}{2} \int_{-1}^1 K\left(x, \frac{\eta+1}{2}\right) A^T T\left(\frac{\eta+1}{2}\right) d\eta + g(x), \\ &= \frac{1}{2} \sum_{i=0}^n \omega_i K\left(x, \frac{\eta_i+1}{2}\right) A^T T\left(\frac{\eta_i+1}{2}\right) + g(x). \end{aligned} \quad (9)$$

Also, we approximate $y(t)$ as follows

$$y(t) = \frac{1}{2} \sum_{i=0}^n \omega_i \int_0^t K\left(x, \frac{\eta_i+1}{2}\right) A^T T\left(\frac{\eta_i+1}{2}\right) dx + \int_0^t g(x) dx + y_0. \quad (10)$$

NUMERICAL EXAMPLES

In this section, three examples are provided to illustrate the efficiency of the proposed method.

Example 1. Consider the following integro-differential equation [7, 14]

$$\begin{cases} y'(x) = 1 - \frac{x}{3} + \int_0^1 xy^2(t) dt, \\ y(0) = 0. \end{cases}$$

the exact solution is $y(t) = t$. A comparison between the proposed method and the methods in [7, 14] is shown in Table1. It is easy to see that the approximate solution obtained by the present method has good agreement with the exact solution.

Example 2. Consider the following integro-differential equation [17]

$$\begin{cases} y'(x) = \int_0^1 \cos(x-t) y^2(t) dt + g(x), \\ y(0) = 0. \end{cases}$$

with exact solution $y(t) = \sin(t)$. Also,

$$g(x) = \cos(x) - \frac{2}{3} \sin(x) + \frac{1}{2} \sin(x-1) + \frac{1}{4} \sin(x+1) - \frac{1}{12} \sin(x-3).$$

In Table 2, we have compared the absolute error of the current method with the results obtained by Lagrange interpolation in [17]. The conclusion is that the proposed method is more accurate rapidly convergent than the method discussed in [17].

Example 3. As the last example, consider the following integro-differential equation

$$\begin{cases} y'(x) = e^x - \frac{e^3 - 1}{3} e^{-x^2} + \int_0^1 e^{2t-x^2} y(t) dt, \\ y(0) = 1, \end{cases}$$

with exact solution $y(t) = e^t$. Table 3 shows the absolute error.

Table1. Absolute error for Example 1.

x	Method of [14]	Method of [7]	The proposed method
	$n = 7$	$n = 3$	$n = 3$
0.0	$3.20e - 009$	$4.02e - 010$	0
0.2	$7.18e - 010$	$4.91e - 010$	$5.86e - 019$
0.4	$1.42e - 010$	$5.99e - 010$	$2.34e - 018$
0.6	$4.07e - 011$	$7.32e - 010$	$5.27e - 018$
0.8	$9.10e - 010$	$8.94e - 010$	$9.38e - 018$
1.0	$3.70e - 009$	$1.49e - 010$	$1.46e - 017$

Table2. Absolute error for Example 2.

x	Method of [17]	The proposed method	
	$n = 6$	$n = 6$	$n = 14$
0.0	0	0	0
0.2	$6e - 006$	$3.64e - 007$	$4.68e - 017$
0.4	$1e - 005$	$8.84e - 007$	$1.27e - 016$
0.6	$2e - 005$	$1.54e - 006$	$2.37e - 016$
0.8	$4e - 005$	$2.30e - 006$	$3.74e - 016$
1.0	$5e - 005$	$3.15e - 006$	$5.30e - 016$

Table3. Absolute error for Example 3.

x	The proposed method		
	$n = 5$	$n = 10$	$n = 15$
0.0	0	0	0
0.2	$6.28e - 005$	$3.96e - 010$	$1.06e - 016$
0.4	$1.21e - 004$	$7.62e - 010$	$2.03e - 016$
0.6	$1.70e - 004$	$1.07e - 009$	$2.86e - 016$
0.8	$2.09e - 004$	$1.32e - 009$	$3.52e - 016$
1.0	$2.38e - 004$	$1.50e - 009$	$3.99e - 016$

CONCLUSION

In this paper, a collocation method based on the Chebyshev polynomials of the first kind is proposed to numerically solve a class of Fredholm integro-differential equations. This method

is tested on some examples and the results are satisfactory. The method can be developed and applied to other classes of integro-differential equations.

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