Conditioned Cancellation Modules

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Abstract

The purpose of this paper is to investigate semiprime cancellation modules. We introduce the concept of semiprime cancellation module generalizing cancellation ideal. We show that an ideal A of R is semiprime cancellation ideal if and only if AM is semiprime cancellation module under the condition M is cancellation module various properties and characterizations of semiprime cancellation module are considered. We also give description for the trace of semiprime cancellation module

Keywords:- semiprime ideal, Cancellation module, semiprime cancellation module, Trace of module, Multiplication module, Flat module.

Introduction 1

Let R be a commutative ring and M a unital R -module. Gilmer [1, p.60] has been defined the concept of cancellation ideal to be the ideal I of R which satisfies the following:-

Whenever $AI = BI$ with A and B are ideals of R implies $A = B$

Mijbass in [2] has been generalized this concept to modules. He has been defined the cancellation modules as follows:

An R -module M is called a cancellation module whenever $AM = BM$ with A and B are ideals of R implies $A = B$.

In this work we shall introduce the concept of semiprime cancellation module by using some restrictions on the ideals A and B in the above definition, namely we shall say that

An R -module M is called semiprime cancellation, whenever $AM = BM$ with A is a semiprime ideal of R and B is any ideal of R implies $A = B$.

An ideal A of a ring R is said to be semiprime if $A \cap B = BA$ for all ideal B of R .

Clearly, the class of cancellation modules contains the class of semiprime cancellation modules and we can give an example to show that this inclusion is properly. However we shall give conditions under which the two classes are equivalent, see proposition (1.9).

This paper consists of two sections our principal aim in the

first section is to study the semiprime cancellation modules. Moreover we study the relationships between cancellation modules and semiprime cancellation module. Also, we discuss the property of semiprime cancellation in each of the module and its trace, where we prove that a module is semiprime cancellation if its trace is a semiprime cancellation ideal, see corollary (1.12).

Semiprime Cancellation Modules2

In this section we introduce the concept of semiprime cancellation modules with some examples and basic properties about this concept. Also, we investigate semiprime cancellation module by using the trace of the module. Finally, we study the relation between semiprime cancellation modules and some types of modules.

Definition (2.1):-

An R -module M is called semiprime cancellation whenever $AM = BM$, with A is a semiprime ideal of R and B is any ideal of R , implies $A = B$.

Examples and Remarks (2.2):-

(1) Z_6 as a Z_{12} -module is semiprime cancellation module

It is clear that (3) is pure ideal of Z_{12} and $(3)Z_6 = (9)Z_6$. Then $(3) = (9)$.

(2) Every cancellation module is semiprime cancellation module, but the converse is not true in general, for example: Z_6 as a Z_{12} -module. See above example number 1. But Z_6 is not cancellation Z_{12} -module, since an $nR(Z_6) = (6)$ is not faithful and hence Z_6 is not cancellation by [2, remark (1-4), p.8].

(4) Z_5 as a Z_{15} -module is semiprime cancellation module. Since, (3) is semiprime ideal of Z_{15} and $(3)Z_5 = (6)Z_5$. Then $(3) = (6)$. Also, (5) is semiprime ideal of Z_{15} and $(5)Z_5 = (10)Z_5$.

Then $(5) = (10)$.

Recall that the element m in an R -module M (where R is an integral domain) is called torsion element if there exists $0 \neq r \in R$ such that $rm = 0$. And m is called a non-torsion element if $rm \neq 0, \forall 0 \neq r \in R$, [2]

For cyclic modules we have the following result:-

Proposition (2.3):- Every cyclic module generated by a non-torsion element is semiprime cancellation.

Proof: Let $M = \langle m \rangle$, where m is a non-torsion element and $A\langle m \rangle = B\langle m \rangle$, where A is pure ideal of R and B is any ideal of R . $am \in B\langle m \rangle$ for all $a \in A$, then $am = bm$, where $b \in B$, implies $am - bm = 0$. Therefore $(a - b)m = 0$, but m is a non-torsion element, then $a - b = 0$, which implies $a = b$. Therefore $A \subseteq B$.

Similarly $B \subseteq A$, and hence $A = B$.

We shall show by an example that the condition M is generated by a non-torsion element in proposition (2.3) can not be dropped.

Example (2.4):-

Let $M = Z_2$ as a Z -module, it is clear that $Z_2 = (1)$ and

is a torsion element in Z_2 and Z_2 is not semiprime cancellation Z_4 -module.

In the following theorem we give some characterizations of semiprime cancellation modules.

Theorem (2.5) :- Let M be an R -module. Then the following statements are equivalent:-

(1) M is semiprime cancellation module.

(2) If $AM \subseteq BM$, such that A is any ideal of R and B is a semiprime ideal of R , then $A \subseteq B$.

(3) If $\langle a \rangle M \subseteq BM$, such that $a \in R$ and B is a semiprime ideal of R , then $a \in B$

(4) $(AM:M) = A$ for all semiprime ideals A of R

(5) $(AM:BM) = (A:B)$, for all ideals B of R and for all semiprime ideals A of R

Proof: (1) $\Rightarrow$ (2) suppose that M is semiprime cancellation module and $AM \subseteq BM$, where B is a semiprime ideal of R and A is any ideal of R . Now, $BM = AM + BM = (A + B)M$, then $B = A + B$ implies $A \subseteq B$.

(2) $\Rightarrow$ (3) Let $\langle a \rangle M \subseteq BM$. Then $\langle a \rangle \subseteq B$ by (2). Hence $a \in B$.

(3) $\Rightarrow$ (4) Let $x \in (AM:M)$. Then $xM \subseteq AM$ by (3) $x \in A$. Hence $(AM:M) \subseteq A$.

On the other side if $x \in A$, then $xM \subseteq AM$. Therefore $x \in (AM:M)$ and hence $(AM:M) = A$.

(4) $\Rightarrow$ (5) Let $x \in (AM:M)$. Then $x \in ((AM:M):B)$ since $(AM:M) = A$ by (4), implies $x \in (AM:BM)$ [4, proposition (2.3), p.38]

Now if $x \in (AM:BM) = ((AM:M):B)$ and since $(AM:M) = A$ by (4).

) $\cup$ Then $x \in (A:B)$. Therefore $(AM:BM) = (A:B)$

(5) $\Rightarrow$ (1) Let $AM = BM$ and A is a semiprime ideal of R , B is any ideal of R . Then $(AM:BM) = R$ implies $(A:B) = R$. Therefore $B \subseteq A$.

Similarly $A \subseteq B$. Then $A = B$. Hence M is semiprime cancellation module

In order to give another characterization of semiprime cancellation module we need to recall the definition of tensor product of modules and some related lemma: A tensor product of two R -modules A and B means a pair (M, T) where M is an R -module and $T: A \times B \rightarrow M$ is a bilinear map, such that for any R -module X and any bilinear map $g: A \times B \rightarrow X$, there exists a unique homeomorphism $h: M \rightarrow X$

such that $h \circ T = g$ [3].

The following proposition gives a necessary and sufficient condition for a module to be semiprime cancellation.

Proposition (2.6) :-

Let M be an R -module. Then M is semiprime cancellation module if and only if $\text{ann}_R(M \otimes R/A) = A$ for all semiprime ideals A of R .

Proof: Let M be semiprime cancellation module. And by [2, proposition (1.11), p.12], we get $M \otimes R/A \cong M/AM$, where A is semiprime ideal of R . Therefore $\text{ann}_R(M \otimes R/A) = (AM:M) = A$ by theorem (1.5, (4))

Now suppose $\text{ann}_R(M \otimes R/A) = A$ for all semiprime ideal A of R . $M \otimes R/A \cong M/AM$ by [2, proposition (1.11), p.12]. Then $\text{ann}_R(M/AM) = \text{ann}_R(M \otimes R/A) = A$. But $\text{ann}_R(M/AM) = (AM:M)$. Therefore $(AM:M) = A$, M is semiprime cancellation module by theorem (1.5, (4))

Proposition (2.7):-

If M is an R -module and N is a homeomorphic image of M , which is semiprime cancellation R -module, then M is semiprime cancellation module.

Proof: Let $AM = BM$, where A is pure ideal of R and B is any ideal of R . And let $\varphi: M \rightarrow N$ be an epimorphism such that N is semiprime cancellation module. Then $\varphi(AM) = \varphi(BM)$. But $\varphi(AM) = A\varphi(M) = \varphi(BM) = B\varphi(M)$. Then $AN = BN$. But N is semiprime cancellation R -module. Then $A = B$ which completes the proof.

From proposition (1.7), we get the following result :-

Corollary (2.8) :-

If M has a semiprime direct summand, then M is also semiprime cancellation R -module.

3.Trace of Semi Cancellation Modules

In this section we study the trace of semiprime cancellation module. Some results about this study is submitted.

The following proposition gives a condition under which the property of semiprime cancellation and cancellation are equivalents.

Proposition (3.1) :-

Let M be an R -module. Then M is a cancellation module if and only if M is faithful semiprime cancellation module.

Proof: It is know that, every cancellation module is semiprime cancellation module, and every cancellation module is faithful module, [2, remark (1.4), p.8]

Conversely: suppose that M is faithful semiprime cancellation module. Let $AM = BM$, where A and B are two ideals in R .

If A is pure ideal of R and B is any ideal of R , implies $A = B$ (since M is semiprime cancellation module)

If A is not semiprime ideal of R and B is any ideal of R , $AM - BM = 0 \Rightarrow A - B \subseteq \text{ann}(M) = 0$. Hence $A = B$.

Now, we have the following.

Proposition (3.2) :-

Let M be a cancellation R -module and A be an ideal in R . Then AM is semiprime cancellation module if and only if A is semiprime cancellation ideal.

Proof: Suppose that AM is semiprime cancellation module. To prove that A is semiprime cancellation ideal. Let $BA = CA$, where B is a pure ideal of R and C is any ideal of R . Since, $BAM = CAM$, implies $C = B$. Therefore A is semiprime cancellation ideal.

Conversely, suppose that A is semiprime cancellation ideal and $BAM = CAM$, where B is semiprime ideal of R and C is any ideal of R . Then $BA = CA$ (since M is semiprime cancellation module), implies $B = C$ (since A is semiprime cancellation ideal). Therefore AM is semiprime cancellation module.

Proposition (3.3) :-

Let M and N be two R -modules and $L = \sum \phi_\alpha(M)$ be a submodule of N , where the sum is taken for any subset of $\text{Hom}(M, N)$, such that L is semiprime cancellation module. Then M is semiprime cancellation module.

Proof: Let $AM = BM$, where A is a semiprime ideal of R and B is any ideal of R . Then $\phi_\alpha(AM) = \phi_\alpha(BM)$, for each $\phi_\alpha \in \text{Hom}(M, N)$

implies

$\sum_{\phi_\alpha \in \text{Hom}(M, N)} \phi_\alpha(AM) = \sum_{\phi_\alpha \in \text{Hom}(M, N)} \phi_\alpha(BM)$.But $\phi_\alpha(AM) = A\phi_\alpha(M) = B\phi_\alpha(M)$
.Then $A\sum_{\phi_\alpha \in \text{Hom}(M, N)} \phi_\alpha(M) = B\sum_{\phi_\alpha \in \text{Hom}(M, N)} \phi_\alpha(M)$.

Therefore $AL = BL$, implies $A = B$ (since L is semiprime cancellation submodule)

Corollary (3.4) :-

If M is an R -module and $T(M)$ is semiprime cancellation ideal of R , then M is semiprime cancellation module.

Proof: The result is clear by using the definition of $T(M)$ and proposition (3.3) .

Corollary (3.5) :-

If M is an R -module and $T(M)$ is multiplication ideal of R , which contain a non-zero divisor element, then M is semiprime cancellation module.

Proof: Let $a \in T(M)$ and a is a non-zero divisor. $T(M)$ is a multiplication ideal of R , so there exists an ideal J of R , such that: $\langle a \rangle = JT(M)$. Implies $T(M)$ is an invertible ideal of R [4, proposition (6.3), p.125]. Therefore $T(M)$ is a cancellation ideal [5, p.879]. Implies $T(M)$ is semiprime cancellation module. Then M is semiprime cancellation module by corollary (1.12).

Corollary (3.6) :-

Let M be an R -module, such that $T(M)$ is semiprime cancellation ideal. Then $M^* = \text{Hom}(M, R)$ is semiprime cancellation R -module.

Proof: Let $aM^* \subseteq BM^*$, such that B is a semiprime ideal of R .

Now, $a f \in aM^* \subseteq BM^*$, $\forall f \in M^*$. Thus $a f \in BM^*$, implies

$a f = \sum_{i=1}^n b_i f_i(m)$, where $b_i \in B$ and $f_i \in M^*$, $\forall m \in M$.

But $T(M)$ is semiprime cancellation ideal. Then $a \in B$ by theorem ((2.3), (3)) and hence M^* is semiprime cancellation module.

Recall that a ring R is called regular (Von-Neumann) if for each element $a \in R$, there exists an element $r \in R$ such that $a = a^2r$

($a = a^2r$ if R is commutative), [6]

An R -module M is flat if for each injective homomorphism $f: \bar{N} \rightarrow N$ from one R -module into the homomorphism $1_M \otimes f: M \otimes \bar{N} \rightarrow_R M \otimes N$ is injective, where 1_M is the identity isomorphism of M , [6].

Now, we give the following result:-

Proposition (3.7):-

Let M be a module over a regular ring R such that AM is faithful for all ideal A of R . Then M is semiprime cancellation module.

Proof: Let $AM = BM$, where A is semiprime ideal of R and B is any ideal of R . From a ring R is regular by [8, proposition(1.10)].

we get $A^2M = B^2M$. Then $A(AM) = B(BM)$

which implies $A(AM) = B(AM)$ and hence $A - B \subseteq \text{ann}(AM) = 0$. (since AM is a faithful for all ideal A of R). Thus $A = B$

Therefore M is semiprime cancellation module.

.following corollary As an application of proposition (2.15), we give the

Corollary (3.8) :-

Let M be a flat R -module and AM is faithful for all ideal A of R . then M is semiprime cancellation module .

Proof: It is obvious according to [8, proposition (1.12), p.9] and proposition (2.15).

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