Orthogonal Generalized Symmetric Higher bi-Derivations and Generalized Higher Derivations on Semiprime Γ -Rings .

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Abstract

In this paper we introduce the concept of orthogonal generalized symmetric higher bi-derivations and generalized higher derivations on Γ -rings and we study some lemmas and theorems of orthogonality on semiprime Γ -rings.

Keyword : Semiprime Γ -ring , symmetric bi-derivations, higher symmetric bi-derivations, higher derivations, generalized higher derivations, orthogonal generalized symmetric higher bi-derivations, orthogonal generalized higher derivations .

1-Introduction

Let M and Γ be two additive abelian groups, M is called a Γ ring if the following conditions are satisfied for any $x, y, z \in M$ and $\alpha, \beta \in \Gamma$:

- (i) $x\alpha y \in M$
- (ii) $x\alpha(y + z) = x\alpha y + x\alpha z$
- $x(\alpha + \beta)y = x\alpha y + x\beta y$
- $(x + y)\alpha z = x\alpha z + y\alpha z$
- (iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$

The notion of a Γ -ring was first introduced by **Nobusawa** 1964 [9] and generalized by **Barnes** 1966 [2] as above definition. It is well known that every ring is Γ -ring . M is called prime if $x\Gamma M\Gamma y = (0)$ implies that $x=0$ or $y=0$ and its said to be semiprime if $x\Gamma M\Gamma x = (0)$ implies that $x=0$ for all $x, y \in M$ [7], also M is said to be n -torsion free if $nx=0$, $x \in M$ implies that $x=0$ where n is positive integer.

In [6] **F .J. Jing** defined a derivation on Γ -ring as follows : "An additive mapping $d: M \rightarrow M$ is said to be derivation on M if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$ " .

M. Sapanci and A. Nakajima in [5] are defined a Jordan derivation on Γ -ring as follows: "An additive mapping $d: M \rightarrow M$ is said to be Jordan derivation on Γ -ring if $d(x\alpha x) = d(x)\alpha x$

$+x\alpha d(x)$ for all $x \in M$ and $\alpha \in \Gamma$. It is clear that every derivation of a Γ -ring M is Jordan derivation of M .

Y. Ceven and M. Ozturk in [3] are defined a generalized derivation on Γ -ring as follows : "An additive mapping $D:M \rightarrow M$ is said to be generalized derivation on M if there exists a derivation $d:M \rightarrow M$ such that $D(x\alpha y) = D(x)\alpha y + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$ ", also defined a Jordan generalized derivation on Γ -ring as follows: "An additive mapping $D:M \rightarrow M$ is said to be Jordan generalized derivation if there exists a Jordan derivation $d:M \rightarrow M$ such that $D(x\alpha x) = D(x)\alpha x + x\alpha d(x)$ for all $x \in M$ and $\alpha \in \Gamma$ ".

M. Ashraf and M. Jamal in [1] are introduced the definition of orthogonal derivation on Γ -ring as follows : "Let d and g be two derivations on M are said to be orthogonal if $d(x)\Gamma M\Gamma g(y) = (0) = g(y)\Gamma M\Gamma d(x)$ for all $x, y \in M$ ", also Ashraf and Jamal are defined the orthogonal generalized derivation on Γ -ring as follows : "Let D and G be two generalized derivations on M is said to be orthogonal if $D(x)\Gamma M\Gamma G(y) = (0) = G(y)\Gamma M\Gamma D(x)$ for all $x, y \in M$ ".

In [10] **M. A. Ozturk, M. Sapanci, M. Soyuturk and K. H. Kim** are defined a symmetric bi-derivation on Γ -ring M as follows : "A mapping $d:M \times M \rightarrow M$ is said to be symmetric if $d(x, y) = d(y, x)$ for all $x, y \in M$ ". A mapping $f:M \rightarrow M$ defined by $f(x) = d(x, x)$, where $d:M \times M \rightarrow M$ is a symmetric mapping, is called the trace of d and the trace f of d satisfies the relation $f(x+y) = f(x) + f(y) + 2d(x, y)$ for all $x, y \in M$. A symmetric bi-additive mapping on $M \times M$ into M is said to be symmetric bi-derivation on M if $d(x\alpha y, z) = d(x, z)\alpha y + x\alpha d(y, z)$ for all $x, y, z \in M$, $\alpha \in \Gamma$ and d is said to be Jordan bi-derivation on M if $d(x\alpha x, y) = d(x, y)\alpha x + x\alpha d(x, y)$ for all $x, y \in M$, $\alpha \in \Gamma$, and authors in [10] introduced the notion of generalized bi-derivation and Jordan generalized bi-derivation on Γ -ring as follows: "A symmetric bi-additive mapping $D:M \times M \rightarrow M$ is said to be generalized bi-derivation if there exists $d:M \times M \rightarrow M$ bi-derivation such that $D(x\alpha y, z) = D(x, z)\alpha y + x\alpha d(y, z)$ for all $x, y, z \in M$, $\alpha \in \Gamma$, and D is said to be Jordan generalized bi-derivation if there exists a Jordan bi-derivation $d:M \times M \rightarrow M$ such that $D(x\alpha x, y) = D(x, y)\alpha x + x\alpha d(x, y)$ for all $x, y \in M$, $\alpha \in \Gamma$ ".

S.M. Salih and A.M. Marir in [8] and [9] are introduced the concept of higher bi-derivation on Γ -ring M as follows : "Let $D = (d_i)_{i \in \mathbb{N}}$ be a family of bi-additive mapping on $M \times M$ into M is said to be higher bi-derivation if $d_n(x\alpha y, z\alpha w) = \sum_{i+j=n} d_i(x, z)\alpha d_j(y, w)$ for all $x, y, z, w \in M$, $\alpha \in \Gamma$ ", and $D = (d_i)_{i \in \mathbb{N}}$ be a family of bi-additive mapping on $M \times M$ into M is said to be Jordan bi-derivation if $d_n(x\alpha x, y\alpha y) = \sum_{i+j=n} d_i(x, y)\alpha d_j(x, y)$ for all $x, y \in M$, $\alpha \in \Gamma$, and authors in [8] are defined the generalized higher bi-derivation on Γ -ring M as follows: "Let $D = (D_i)_{i \in \mathbb{N}}$ be a family of bi-additive mapping on $M \times M$ into M is said to be generalized

higher bi-derivation if there exists a higher bi-derivation $d_n: M \times M \rightarrow M$ such that $D_n(x\alpha y, z\alpha w) = \sum_{i+j=n} D_i(x, z)\alpha d_j(y, w)$ for all $x, y, z, w \in M$, $\alpha \in \Gamma$, and $D = (D_i)_{i \in \mathbb{N}}$ be a family of bi-additive mapping on $M \times M$ into M is said to be Jordan generalized higher bi-derivation if there exists $d_n: M \times M \rightarrow M$ a Jordan higher bi-derivation such that $D_n(x\alpha x, y\alpha y) = \sum_{i+j=n} D_i(x, y)\alpha d_j(x, y)$ for all $x, y \in M$, $\alpha \in \Gamma$.

In this paper the our aim is to introduce the concept of orthogonal generalized symmetric higher bi-derivations and generalized higher derivations on Γ -rings and we present and study some properties of it.

2- Orthogonal generalized symmetric higher bi-derivations and generalized higher derivations on Γ -rings .

In this section we introduce and study the concept of orthogonal generalized symmetric higher bi-derivations and generalized higher derivations on semiprime Γ -rings and we introduce and prove some lemmas and theorem used in our work.

Definition (2.1):

Let $D = (D_i)_{i \in \mathbb{N}}$ be generalized symmetric higher bi-derivations and $G = (G_i)_{i \in \mathbb{N}}$ be generalized higher derivations on Γ -ring M . Then D_n and G_n are said to be orthogonal if for every $x, y, z \in M$ and $n \in \mathbb{N}$:

$$D_n(x, y)\Gamma M\Gamma G_n(z) = (0) = G_n(z)\Gamma M\Gamma D_n(x, y)$$

Where $D_n(x, y)\Gamma M\Gamma G_n(z) = \sum_{i=1}^n D_i(x, y)\alpha m\beta G_i(z) = 0$ for all $m \in M$ and $\alpha, \beta \in \Gamma$.

The following is an example of orthogonal generalized higher bi-derivations and generalized higher derivations on a Γ -ring M .

Example (2.2):

Let d_n be symmetric higher bi-derivations and g_n be higher derivations on a Γ -ring M for all $n \in \mathbb{N}$. Put $M' = M \times M$ and $\Gamma' = \Gamma \times \Gamma$, we defined d_n' be symmetric higher bi-derivations and g_n' be higher derivations on M' into itself such that $d_n'((x, y)) = (d_n(x), 0)$ and $g_n'((x, y)) = (0, g_n(y))$ for all $(x, y) \in M'$. More over if (D_n, d_n) is generalized symmetric higher bi-derivations and (G_n, g_n) is generalized higher derivations on M , we defined D_n' and G_n' on M' into itself such that $D_n'((x, y)) = (D_n(x), 0)$ and $G_n'((x, y)) = (0, G_n(y))$ for all $(x, y) \in M'$ and $n \in \mathbb{N}$. Then (D_n', d_n') is generalized symmetric higher bi-derivations and (G_n', g_n') is generalized higher derivations on M' such that D_n' and G_n' are orthogonal.

Lemma (2.3):[5]

Let M be a 2-torsion free semiprime Γ -ring and a, b the elements of M . If for all $\alpha, \beta \in \Gamma$, then the following conditions are equivalent:

- (i) $a\alpha M\beta b = 0$
- (ii) $b\alpha M\beta a = 0$
- (iii) $a\alpha M\beta b + b\alpha M\beta a = 0$

If one of these conditions is fulfilled, then $a\alpha b = b\alpha a = 0$.

Lemma (2.4) :[4]

Let M be a 2-torsion free semiprime Γ -ring and a, b the elements of M such that $a\alpha M\beta b + b\alpha M\beta a = 0$ for all $a, \beta \in \Gamma$, then $a\alpha M\beta b = b\alpha M\beta a = 0$.

Lemma (2.5):

Let R be a semiprime ring. Suppose that D_n is bi-additive mappings on $M \times M$ into M and G_n is additive mappings on M into M for all $n \in \mathbb{N}$ satisfies $D_n(x, y)\Gamma M\Gamma G_n(x) = (0)$ for all $x, y \in M$ and $n \in \mathbb{N}$. Then $D_n(x, y)\Gamma M\Gamma G_n(z) = (0)$ for all $x, y, z \in M$ and $n \in \mathbb{N}$.

Proof :

Suppose that $D_n(x, y)\Gamma M\Gamma G_n(x) = (0)$

$$D_n(x, y)\Gamma M\Gamma G_n(x) = \sum_{i=1}^n D_i(x, y)\alpha m\beta G_i(x) = 0 \text{ for all } m \in M \text{ and } \alpha, \beta \in \Gamma \dots (1)$$

Replace x by $x + z$ in (1) for all $z \in M$ we get $\sum_{i=1}^n D_i(x + z, y)\alpha m\beta G_i(x + z) = 0$

$$\sum_{i=1}^n [D_i(x, y) + D_i(z, y)]\alpha m\beta [G_i(x) + G_i(z)] = 0$$

$$\sum_{i=1}^n D_i(x, y)\alpha m\beta G_i(x) + D_i(x, y)\alpha m\beta G_i(z) + D_i(z, y)\alpha m\beta G_i(x) + D_i(z, y)\alpha m\beta G_i(z) = 0$$

By equation (1) we get

$$\sum_{i=1}^n D_i(x, y)\alpha m\beta G_i(z) + D_i(z, y)\alpha m\beta G_i(x) = 0$$

$$\sum_{i=1}^n D_i(x, y)\alpha m\beta G_i(z) = - \sum_{i=1}^n D_i(z, y)\alpha m\beta G_i(x) \dots (2)$$

Multiplication (2) by $\delta t \gamma \sum_{i=1}^n D_i(x, y)\alpha m\beta G_i(z)$ for all $t \in M$ and $\delta, \gamma \in \Gamma$ we get

$$\sum_{i=1}^n D_i(x, y)\alpha m\beta G_i(z) \delta t \gamma \sum_{i=1}^n D_i(x, y)\alpha m\beta G_i(z) = 0$$

Since M is semiprime Γ -ring we get

$$\sum_{i=1}^n D_i(x, y)\alpha m\beta G_i(z) = 0$$

Thus $D_n(x, y)\Gamma M\Gamma G_n(z) = (0)$

Lemma (2.6):

Let M be a 2-torsion free semiprime Γ -ring such that $a\alpha y\beta z = a\beta y\alpha z$, a generalized symmetric higher bi-derivations D_n associated with symmetric higher bi-derivations d_n and G_n be generalized higher derivations associated with higher derivations g_n for all $n \in \mathbb{N}$. Then D_n and G_n are orthogonal iff $G_n(x)\alpha D_n(z, y) = 0$ for all $x, y, z \in M, \alpha, \beta \in \Gamma$ and $n \in \mathbb{N}$.

Proof :

Suppose that $G_n(x)\alpha D_n(z, y) = 0$

$$\sum_{i=1}^n G_i(x)\alpha D_i(z, y) = 0 \quad \dots(1)$$

Replace x by $x\beta w$ in (1) for all $w \in M$ we get

$$\sum_{i=1}^n G_i(x\beta w)\alpha D_i(z, y) = 0$$

$$\sum_{i=1}^n G_i(x)\beta g_i(w)\alpha D_i(z, y) = 0 \quad \dots(2)$$

Replace $g_i(w)$ by m in (2) for all $m \in M$ we get

$$\sum_{i=1}^n G_i(x)\beta m\alpha D_i(z, y) = 0$$

$$G_n(x)\Gamma M\Gamma D_n(z, y) = (0)$$

Thus D_n and G_n are orthogonal

Conversely, suppose that D_n and G_n are orthogonal

$$G_n(x)\Gamma M\Gamma D_n(z, y) = (0)$$

$$\sum_{i=1}^n G_i(x)\alpha m\beta D_i(z, y) = 0$$

By Lemma (2.3) we get

$$\sum_{i=1}^n G_i(x)\alpha D_i(z, y) = 0$$

Thus $G_n(x)\alpha D_n(z, y) = 0$.

Theorem (2.7) :

Let M be a 2-torsion free semiprime Γ -ring such that $a\alpha y\beta z = a\beta y\alpha z$, a generalized symmetric higher bi-derivations D_n associated with symmetric higher bi-derivations d_n and G_n be generalized higher derivations associated with higher derivations g_n for all $n \in \mathbb{N}$. Then D_n and G_n are orthogonal if and only if $g_n(x)\alpha D_n(y, z) = 0$ or $G_n(x)\alpha d_n(z, y) = 0$ for all $x, y, z \in M, \alpha, \beta \in \Gamma$ and $n \in \mathbb{N}$.

Proof :

Suppose that $g_n(x)\alpha D_n(z, y) = 0$

$$g_n(x)\alpha D_n(z, y) = \sum_{i=1}^n g_i(x)\alpha D_i(z, y) = 0 \quad \dots(1)$$

Replace x by $x\beta w$ in (1) for all $w \in M$ we get

$$\sum_{i=1}^n g_i(x\beta w)\alpha D_i(z, y) = 0$$

$$\sum_{i=1}^n g_i(x)\beta g_i(w)\alpha D_i(z, y) = 0 \quad \dots(2)$$

Replace $g_i(x)$ by $G_i(x)$ in (2) we get

$$\sum_{i=1}^n G_i(x)\beta g_i(w)\alpha D_i(z, y) = 0$$

By Lemma (2.3) we get

$$\sum_{i=1}^n G_i(x)\alpha D_i(z, y) = 0$$

$$G_n(x)\alpha D_n(z, y) = 0$$

By Lemma (2.6) we get D_n and G_n are orthogonal

Similarly we if $G_n(x)\alpha d_n(z, y) = 0$ we get D_n and G_n are orthogonal

Conversely , suppose that D_n and G_n are orthogonal

By Lemma (2.6) we get

$$G_n(x)\alpha D_n(z, y) = 0$$

$$\sum_{i=1}^n G_i(x)\beta D_i(z, y) = 0 \quad \dots(3)$$

Replace z by $w\beta z$ in (3) for all $w \in M$ we get

$$\sum_{i=1}^n G_i(x)\alpha D_i(w\beta z, y) = 0$$

$$\sum_{i=1}^n G_i(x)\alpha D_i(w, y)\beta d_i(z, y) = 0$$

By Lemma (2.3) we get

$$\sum_{i=1}^n G_i(x)\alpha d_i(z, y) = 0$$

$$\text{Hence } G_n(x)\beta d_n(z, y) = 0$$

And replace x by $w\beta x$ in (3) we get

$$\sum_{i=1}^n G_i(w\beta x)\alpha D_i(z, y) = 0$$

$$\sum_{i=1}^n G_i(w)\beta g_i(x)\alpha D_i(z, y) = 0 \dots(4)$$

Multiplication (4) by $g_i(x)\alpha D_i(z, y)\delta$ for all $\delta \in \Gamma$ we get

$$\sum_{i=1}^n g_i(x)\alpha D_i(z, y)\delta G_i(w)\beta g_i(x)\alpha D_i(z, y) = 0$$

Since M is semiprime we get

$$\sum_{i=1}^n g_i(x)\alpha D_i(z, y) = 0$$

$$\text{Hence } g_n(x)\alpha D_n(z, y) = 0$$

3- Main Results :

In this section we present and study basic results of orthogonal generalized symmetric higher bi-derivations and generalized higher derivations on Γ -ring M .

Theorem (3.1) :

Let M be a 2-torsion free semiprime Γ -ring such that $\alpha\alpha\beta z = \alpha\beta y\alpha z$, a generalized symmetric higher bi-derivations D_n associated with symmetric higher bi-derivations d_n and G_n be generalized higher derivations associated with higher derivations g_n for all $n \in \mathbb{N}$. Then D_n and G_n are orthogonal if and only if $G_n(x)\alpha D_n(z, y) = G_n(x)\alpha d_n(z, y) = 0$ for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$ and $n \in \mathbb{N}$.

Proof :

Suppose that D_n and G_n are orthogonal

By Lemma (2.6) we get

$$G_n(x)\alpha D_n(z, y) = 0 \dots(1)$$

And by Theorem (2.7) we get

$$G_n(x)\alpha d_n(z, y) = 0 \dots(2)$$

$$\text{From (1) and (2) we get } G_n(x)\alpha D_n(z, y) = G_n(x)\alpha d_n(z, y) = 0$$

Conversely, suppose that $G_n(x)\alpha D_n(z, y) = 0$

By Lemma (2.6) we get D_n and G_n are orthogonal

Now, if $G_n(x)\alpha d_n(z, y) = 0$

By Theorem (2.7) we get D_n and G_n are orthogonal

Theorem (3.2) :

Let M be a 2-torsion free semiprime Γ -ring such that $\alpha\gamma\beta z = \alpha\beta\gamma z$, a generalized symmetric higher bi-derivations D_n associated with symmetric higher bi-derivations d_n and G_n be generalized higher derivations associated with higher derivations g_n for all $n \in \mathbb{N}$. If D_n and G_n are orthogonal then the following relations hold for all $x, y, z \in M, \alpha \in \Gamma$ and $n \in \mathbb{N}$:

- i. d_n and G_n are orthogonal and $G_n(x)\alpha d_n(z, y) = 0$
- ii. g_n and D_n are orthogonal and $g_n(x)\alpha D_n(z, y) = 0$
- iii. d_n and g_n are orthogonal symmetric higher bi-derivations
- iv. $G_n d_n = 0$ and $g_n D_n = 0$
- v. $G_n D_n = 0$

Proof : (i)

Suppose that D_n and G_n are orthogonal

By Theorem (2.7) we get

$$G_n(x)\alpha d_n(z, y) = 0 \quad \dots(1)$$

$$\sum_{i=1}^n G_i(x)\alpha d_i(z, y) = 0 \quad \dots(2)$$

Replace x by $x\beta w$ in (2) $w \in M$ and $\beta \in \Gamma$ we get

$$\sum_{i=1}^n G_i(x\beta w)\alpha d_i(z, y) = 0$$

$$\sum_{i=1}^n G_i(x)\beta g_i(w)\alpha d_i(z, y) = 0 \quad \dots(3)$$

Replace $g_i(w)$ by m in (3), $m \in M$ we get

$$\sum_{i=1}^n G_i(x)\beta m\alpha d_i(z, y) = 0 \quad \dots(4)$$

$$G_n(x)\Gamma M\Gamma d_n(z, y) = (0)$$

Hence d_n and G_n are orthogonal

Proof:(ii)

Similarly way used in the proof of (i)

Proof: (iii)

By Lemma (2.6) we get

$$G_n(x)\alpha D_n(z, y) = 0$$

$$\sum_{i=1}^n G_i(x)\alpha D_i(z, y) = 0 \quad \dots(1)$$

Replacing x by $w\beta x$ and z by $w\beta z$ in (1) we get

$$\sum_{i=1}^n G_i(w\beta x)\alpha D_i(w\beta z, y) = 0$$

$$\sum_{i=1}^n G_i(w)\beta g_i(x)\alpha D_i(w, y)\beta d_i(z, y) = 0 \quad \dots(2)$$

Replace $D_i(w, y)$ by m in (2) for all $m \in M$ we get

$$\sum_{i=1}^n G_i(w)\beta g_i(x)\alpha m\beta d_i(z, y) = 0 \quad \dots(3)$$

Multiplication (3) by $g_i(x)\alpha m\beta d_i(z, y)\delta$ we get

$$\sum_{i=1}^n g_i(x)\alpha m\beta d_i(z, y)\delta G_i(w)\beta g_i(x)\alpha m\beta d_i(z, y) = 0$$

Since M is semiprime we get

$$\sum_{i=1}^n g_i(x)\alpha m\beta d_i(z, y) = 0$$

$$g_n(x)\Gamma M \Gamma d_n(z, y) = (0)$$

Hence d_n and g_n are orthogonal symmetric higher bi-derivations

Proof : (iv)

Since by (ii) $G_n(x)\alpha d_n(z, y) = 0$

$G_n(G_n(x)\alpha d_n(z, y)) = 0$, for all $m \in M$

$$\sum_{i=1}^n G_i(G_i(x)\alpha d_i(z, y)) = 0 \quad \dots(1)$$

Replace x by $x\beta w$ in (1) we get

$$\sum_{i=1}^n G_i(G_i(x\beta w)\alpha d_i(z, y)) = 0$$

$$\sum_{i=1}^n G_i(G_i(x)\beta g_i(w)\alpha d_i(z, y)) = 0$$

$$\sum_{i=1}^n G_i(G_i(x))\beta g_i(g_i(w))\alpha g_i(d_i(z, y)) = 0 \quad \dots(2)$$

Replace $(G_i(y))$ by $(d_i(z, y))$ in (2) we get

$$\sum_{i=1}^n G_i(d_i(z, y))\beta g_i(g_i(w))\alpha g_i(d_i(z, y)) = 0 \quad \dots(3)$$

Replace $g_i(d_i(z, y))$ by $G_i(d_i(z, y))$ in (3)

$$\sum_{i=1}^n G_i(d_i(z, y))\beta g_i(g_i(w))\alpha G_i(d_i(z, y)) = 0$$

Since M is semiprime we get

$$\sum_{i=1}^n G_i(d_i(z, y)) = 0$$

$$G_n(d_n(z, y)) = 0$$

Thus $G_n d_n = 0$

And by (iii) $g_n(x) \alpha D_n(z, y) = 0$

$$g_n(g_n(x) \alpha D_n(z, y)) = 0$$

$$\sum_{i=1}^n g_i(g_i(x) \alpha D_i(z, y)) = 0 \quad \dots(4)$$

Replace x by $x\beta w$ in (4) we get

$$\sum_{i=1}^n g_i(g_i(x\beta w) \alpha D_i(z, y)) = 0$$

$$\sum_{i=1}^n g_i(g_i(x) \beta g_i(w) \alpha D_i(z, y)) = 0$$

$$\sum_{i=1}^n g_i(g_i(x)) \beta g_i(g_i(w)) \alpha g_i(D_i(z, y)) = 0 \quad \dots(5)$$

Replace $(g_i(x))$ by $(D_i(z, y))$ in (5) we get

$$\sum_{i=1}^n g_i(D_i(z, y)) \beta g_i(g_i(w)) \alpha g_i(D_i(z, y)) = 0$$

Since M is semiprime we get

$$\sum_{i=1}^n g_i(D_i(z, y)) = 0$$

$$g_n(D_n(z, y)) = 0$$

Thus $g_n D_n = 0$

Proof: (v)

Since by D_n and G_n are orthogonal

$$D_n(x, y) \Gamma M \Gamma G_n(z) = (0)$$

$$G_n(D_n(x, y) \Gamma M \Gamma G_n(z)) = (0)$$

$$\sum_{i=1}^n G_i(D_i(x, y) \alpha m \beta G_i(z)) = 0$$

$$\sum_{i=1}^n G_i(D_i(x, y)) \alpha g_i(m) \beta g_i(G_i(z)) = 0 \quad \dots(6)$$

Replace $g_i(G_i(z))$ by $G_i(D_i(x, y))$ in (6) we get

$$\sum_{i=1}^n G_i(D_i(x, y)) \alpha g_i(m) \beta G_i(D_i(x, y)) = 0$$

Since M is semiprime we get

$$\sum_{i=1}^n G_i(D_i(x, y)) = 0$$

$$G_n(D_n(x, y)) = 0$$

Thus $G_n D_n = 0$.

Theorem (3.3) :

Let R be a 2-torsion free semiprime ring such that $\alpha\alpha\gamma\beta z = \alpha\beta\gamma\alpha z$, a generalized symmetric higher bi-derivations D_n associated with symmetric higher bi-derivations d_n and G_n be generalized higher derivations associated with higher derivations g_n for all $n \in \mathbb{N}$. Then D_n and G_n are orthogonal if and only if for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ $G_n(x) \alpha d_n(z, y) + g_n(x) \alpha D_n(z, y) = 0$.

Proof :

Suppose that D_n and G_n are orthogonal

By Theorem (2.7) we get

$$G_n(x) \alpha d_n(z, y) = 0 \quad \dots(1)$$

And by Lemma (2.6) we get

$$G_n(x) \alpha D_n(y, z) = 0$$

$$\sum_{i=1}^n G_i(x) \alpha D_i(z, y) = 0 \quad \dots(2)$$

Replace x by $t\beta x$ in (2) $t \in M$ we get

$$\sum_{i=1}^n G_i(t\beta x) \alpha D_i(z, y) = 0$$

$$\sum_{i=1}^n G_i(t) \beta g_i(x) \alpha D_i(z, y) = 0 \quad \dots(3)$$

Multiplication (3) by $g_i(x) \alpha D_i(z, y) \delta$ we get

$$\sum_{i=1}^n g_i(x) \alpha D_i(z, y) \delta G_i(t) \beta g_i(x) \alpha D_i(z, y) = 0$$

Since M is semiprime we get

$$\sum_{i=1}^n g_i(x) \alpha D_i(z, y) = 0$$

$$g_n(x) \alpha D_n(z, y) = 0 \quad \dots(4)$$

From (1) and (4) we get

$$G_n(x)\alpha d_n(z, y) + g_n(x)\alpha D_n(z, y) = 0$$

Conversely, suppose that $G_n(x)\alpha d_n(z, y) + g_n(x)\alpha D_n(z, y) = 0$

$$\sum_{i=1}^n G_i(x)\alpha d_i(z, y) + g_i(x)\alpha D_i(z, y) = 0 \quad \dots(5)$$

Replace x by $x\beta t$ in (5) we get

$$\sum_{i=1}^n G_i(x\beta t)\alpha d_i(z, y) + g_i(x\beta t)\alpha D_i(z, y) = 0$$

$$\sum_{i=1}^n G_i(x)\beta g_i(t)\alpha d_i(z, y) + g_i(x)\beta g_i(t)\alpha D_i(z, y) = 0 \quad \dots(6)$$

Replacing $d_i(z, y)$ by $D_i(z, y)$ and $g_i(x)$ by $G_i(x)$ in (6) we get

$$\sum_{i=1}^n G_i(x)\beta g_i(t)\alpha D_i(z, y) + G_i(x)\beta g_i(t)\alpha D_i(z, y) = 0 \quad \dots(7)$$

$$2(\sum_{i=1}^n G_i(x)\beta g_i(t)\alpha D_i(z, y)) = 0$$

Since M is 2-torsion free we get

$$\sum_{i=1}^n G_i(x)\beta g_i(t)\alpha D_i(z, y) = 0 \quad \dots\dots(8)$$

Replace $g_i(t)$ by m we get

$$G_n(x)\Gamma M \Gamma D_n(z, y) = (0)$$

Hence D_n and G_n are orthogonal.

Theorem (3.4):

Let M be a 2-torsion free semiprime Γ -ring such that $a\alpha\beta z = a\beta y\alpha z$, a generalized symmetric higher bi-derivations D_n associated with symmetric higher bi-derivations d_n and G_n be generalized higher derivations associated with higher derivations g_n for all $n \in \mathbb{N}$. Then D_n and G_n are orthogonal if and only if $G_n(x)\alpha D_n(z, y) = 0$ for all $x, y, z \in M, \alpha, \beta \in \Gamma, n \in \mathbb{N}$ and $G_n d_n = g_n d_n = 0$.

Proof :

Suppose that D_n and G_n are orthogonal

By Lemma (2.6) we get

$$G_n(x)\alpha D_n(z, y) = 0 \quad \dots(1)$$

And by Theorem (2.7)(ii) we get

$$G_n(x)\alpha d_n(z, y) = 0$$

$$G_n(G_n(x)\alpha d_n(z, y)) = 0$$

$$\sum_{i=1}^n G_i(G_i(x)\alpha d_i(z, y)) = 0 \quad \dots(2)$$

Replace x by $x\beta t$ in (2), $t \in R$ we get

$$\sum_{i=1}^n G_i(G_i(x\beta t)\alpha d_i(z, y)) = 0$$

$$\sum_{i=1}^n G_i(G_i(x)\beta g_i(t)\alpha d_i(z, y)) = 0$$

$$\sum_{i=1}^n G_i(G_i(x))\beta g_i(g_i(t))\alpha g_i(d_i(z, y)) = 0 \quad \dots(3)$$

Replace $G_i(x)$ by $d_i(z, y)$ in (3) we get

$$\sum_{i=1}^n G_i(d_i(z, y))\beta g_i(g_i(t))\alpha g_i(d_i(z, y)) = 0 \quad \dots(4)$$

Replace $g_i(d_i(z, y))$ by $G_i(d_i(z, y))$ in (4) we get

$$\sum_{i=1}^n G_i(d_i(z, y))\beta g_i(g_i(t))\alpha G_i(d_i(z, y)) = 0$$

Since M is semiprime we get

$$\sum_{i=1}^n G_i(d_i(z, y)) = 0$$

$$G_n(d_n(z, y)) = 0$$

$$G_n d_n = 0 \quad \dots(5)$$

Also by Theorem (3.2)(iii) we get

$$g_n(x)\Gamma M \Gamma d_n(z, y) = (0)$$

$$g_n(g_n(x)\Gamma M \Gamma d_n(z, y)) = (0)$$

$$\sum_{i=1}^n g_i(g_i(x)\alpha m \beta d_i(z, y)) = 0, \quad \text{for all } m \in M$$

$$\sum_{i=1}^n g_i(g_i(x))\alpha g_i(m)\beta g_i(d_i(z, y)) = 0 \quad \dots(6)$$

Replace $g_i(x)$ by $d_i(z, y)$ in (6) we get

$$\sum_{i=1}^n g_i(d_i(z, y))\alpha g_i(m)\beta g_i(d_i(z, y)) = 0$$

Since M is semiprime we get

$$\sum_{i=1}^n g_i(d_i(z, y)) = 0$$

$$g_n(d_n(z, y)) = 0$$

$$g_n d_n = 0 \quad \dots(7)$$

From (1), (5) and (7) we get

$$G_n(x)D_n(z, y) = 0 \text{ and } G_nd_n = g_nd_n = 0$$

Conversely, suppose that $G_n(x)\alpha D_n(z, y) = 0 \dots(8)$

And $G_nd_n = 0$

$$(G_nd_n)(x\alpha z, y) = 0$$

$$\sum_{i=1}^n G_i(d_i(x\alpha z, y)) = 0$$

$$\sum_{i=1}^n G_i(d_i(x, y)\alpha d_i(z, y)) = 0$$

$$\sum_{i=1}^n G_i(d_i(x, y))\alpha g_i(d_i(z, y)) = 0 \dots(9)$$

Replacing $(d_i(x, y))$ by x and $g_i(d_i(z, y))$ by $d_i(z, y)$ in (9) we get

$$\sum_{i=1}^n G_i(x)\alpha d_i(z, y) = 0$$

$$G_n(x)\alpha d_n(z, y) = 0 \dots(10)$$

From (8) and (10) we get

$$G_n(x)\alpha D_n(z, y) = G_n(x)\alpha d_n(z, y) = 0$$

By Theorem (3.1) we get D_n and G_n are orthogonal.

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