On the Location of the Zeros of Certain Polynomials

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Abstract:

In this paper we generalize Bairagi, Jain, Mishra and Saha's result that the zeros of a polynomial

$$P(z) = \sum_{j=0}^n a_j z^j \text{ with } t|a_j| \geq |a_{j-1}|, j = 2, 3, \dots, n \text{ for some } t > 0 \text{ and } |\operatorname{Arg} a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1, 2, \dots, n \text{ for}$$

some real α, β , of modulus greater than

$$\frac{t(n-1)}{n} \left\{ \cos \alpha + \sin \alpha + \frac{2 \sin \alpha}{n|a_n|} \sum \left(\frac{n}{t(n-1)} \right)^{n-1-j} (j+1) |a_{j+1}| \right\}$$

are simple, and other similar results to polynomials with lesser restrictive conditions on their coefficients.

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1. Introduction

About the order of certain zeros of a polynomial, Aziz and Mohammad [1] proved the following result.

Theorem A. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 > 0.$$

Then all the zeros of $P(z)$ of modulus greater than or equal to $\frac{n}{n+1}$ are simple.

In [2], Aziz and Mohammad gave a generalization as well a refinement of Theorem A as follows.

Theorem B. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ such that for some $t > 0$,

$ta_j \geq a_{j-1} > 0, j = 2, 3, \dots, n, a_0$ may be a real or complex number. Then all the zeros of $P(z)$ of modulus greater than $\frac{t(n-1)}{n}$ are simple.

Bairagi, Jain, Mishra and Saha [3] extended Theorem B to polynomials with complex coefficients and proved the following result.

Theorem C. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ such that for some $t > 0$,

$$t|a_j| \geq |a_{j-1}|, j = 1, 2, \dots, n \text{ and for some real } \alpha, \beta, |\operatorname{Arg} a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1, 2, \dots, n.$$

Then $P(z)$ cannot have a zero of order ≥ 2 with modulus greater than

$$\frac{t(n-1)}{n} \left\{ \cos \alpha + \sin \alpha + \frac{2 \sin \alpha}{n|a_n|} \sum_{j=0}^{n-2} \left(\frac{n}{t(n-1)} \right)^{n-1-j} (j+1) |a_{j+1}| \right\}.$$

In other words, all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n} \left\{ \cos \alpha + \sin \alpha + \frac{2 \sin \alpha}{n|a_n|} \sum_{j=0}^{n-2} \left(\frac{n}{t(n-1)} \right)^{n-1-j} (j+1) |a_{j+1}| \right\}$$

are simple.

In the same paper they also proved the following result.

Theorem D. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ with $\operatorname{Re} a_j = \alpha_j, \operatorname{Im} a_j = \beta_j$,

$j=0,1,\dots,n$. If for some $t > 0$

$$t\alpha_j \geq \alpha_{j-1} \geq 0, j = 2,3,\dots,n \quad \text{and} \quad \alpha_n > 0,$$

then $P(z)$ cannot have a zero of order ≥ 2 with modulus greater than

$$\frac{t(n-1)}{n} \left\{ 1 + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-j} (j+1) |\beta_{j+1}| \right\}.$$

In other words, all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n} \left\{ 1 + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-j} (j+1) |\beta_{j+1}| \right\}$$

are simple.

2. Main Results

In this paper we prove the following generalizations of Theorems C and D.

Theorem 1. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ such that for some $t > 0, k \geq 1$,

$$k|a_n| \geq \frac{1}{t}|a_{n-1}| \geq \frac{1}{t^2}|a_{n-2}| \geq \dots \geq \frac{1}{t^{n-1}}|a_1| \geq \frac{1}{t^n}|a_0|$$

and for some real α, β , $|\operatorname{Arg} a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1,2,\dots,n$.

Then all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n|a_n|} \left\{ k|a_n|(\cos \alpha + \sin \alpha + 1) - |a_n| + \frac{2 \sin \alpha}{n} \sum_{j=0}^{n-2} \left(\frac{n}{t(n-1)} \right)^{n-1-j} (j+1) |a_{j+1}| \right\}$$

are simple.

Remark 1. For $k=1$, Theorem 1 reduces to Theorem C.

Theorem 2. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ with $\operatorname{Re} a_j = \alpha_j, \operatorname{Im} a_j = \beta_j$,

$j=0,1,\dots,n$. If for some $t > 0$ and $k \geq 1$,

$$k\alpha_n \geq \frac{1}{t}\alpha_{n-1} \geq \frac{1}{t^2}\alpha_{n-2} \geq \dots \geq \frac{1}{t^{n-1}}\alpha_1 \geq \frac{1}{t^n}\alpha_0 \geq 0, \quad \text{and} \quad \alpha_n > 0,$$

then $P(z)$ cannot have a zero of order ≥ 2 with modulus greater than

$$\frac{t(n-1)}{n} \left\{ 2k - 1 + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-j} (j+1) |\beta_{j+1}| \right\}.$$

In other words, all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n} \left\{ 2k - 1 + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-j} (j+1) |\beta_{j+1}| \right\}$$

are simple.

Remark 2. For $k=1$, Theorem 2 reduces to Theorem D.

Applying Theorem 2 to the polynomial $-iP(z)$, we get the following result.

Corollary 1. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ with $\operatorname{Re} a_j = \alpha_j, \operatorname{Im} a_j = \beta_j$,

$j=0,1,\dots,n$. If for some $t > 0$ and $k \geq 1$,

$$k\beta_n \geq \frac{1}{t} \beta_{n-1} \geq \frac{1}{t^2} \beta_{n-2} \geq \dots \geq \frac{1}{t^{n-1}} \beta_1 \geq \frac{1}{t^n} \beta_0 \geq 0, \quad \text{and } \beta_n > 0,$$

then $P(z)$ cannot have a zero of order ≥ 2 with modulus greater than

$$\frac{t(n-1)}{n} \left\{ 2k - 1 + \frac{2}{n\beta_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-j} (j+1) |\alpha_{j+1}| \right\}.$$

In other words, all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n} \left\{ 2k - 1 + \frac{2}{n\beta_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-j} (j+1) |\alpha_{j+1}| \right\}$$

are simple.

3. Lemmas

For the proofs of the above results we need the following lemmas.

Lemma 1. For any two complex numbers z_1, z_2 with $|z_1| \geq |z_2|$ and $|\operatorname{Arg} z_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j=1,2$,

for some real α, β ,

$$|z_1 - z_2| \leq (|z_1| - |z_2|) \cos \alpha + (|z_1| + |z_2|) \sin \alpha.$$

Lemma 1 is due to Govil and Rahman [4].

Lemma 2. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that for some real α, β ,

$$|\operatorname{Arg} a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j=1,2,\dots,n,$$

and

$$k|a_n| \geq |a_{n-1}| \geq \dots \geq |a_1| \geq |a_0|.$$

Then $P(z)$ has all its zeros in

$$|z| \leq \frac{1}{|a_n|} \left\{ k|a_n| (\cos \alpha + \sin \alpha + 1) - |a_n| + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| \right\}.$$

Proof of Lemma 2. Consider the polynomial

$$F(z) = (1-z)P(z)$$

$$\begin{aligned}
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \dots + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} - (k-1)a_n z^n + (ka_n - a_{n-1})z^n + \dots + (a_1 - a_0)z + a_0
\end{aligned}$$

For $|z| > 1$, we have by using the hypothesis and Lemma 1,

$$\begin{aligned}
|F(z)| &\geq |a_n| |z|^n \left[|z+k-1| - \frac{1}{|a_n|} \left\{ |ka_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \dots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \right\} \right] \\
&> |a_n| |z|^n \left[|z+k-1| - \frac{1}{|a_n|} \left\{ |ka_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \dots + |a_1 - a_0| + |a_0| \right\} \right] \\
&\geq |a_n| |z|^n \left[|z+k-1| - \frac{1}{|a_n|} \left\{ (k|a_n| - |a_{n-1}|) \cos \alpha + (k|a_n| + |a_{n-1}|) \sin \alpha \right. \right. \\
&\quad \left. \left. + (|a_{n-1}| - |a_{n-2}|) \cos \alpha + (|a_{n-1}| + |a_{n-2}|) \sin \alpha + \dots \right. \right. \\
&\quad \left. \left. + (|a_1| - |a_0|) \cos \alpha + (|a_1| + |a_0|) \sin \alpha + |a_0| \right\} \right] \\
&= |a_n| |z|^n \left[|z+k-1| - \frac{1}{|a_n|} \left\{ k|a_n|(\cos \alpha + \sin \alpha) - |a_0|(\cos \alpha + \sin \alpha - 1) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| \right\} \right] \\
&\geq |a_n| |z|^n \left[|z+k-1| - \frac{1}{|a_n|} \left\{ k|a_n|(\cos \alpha + \sin \alpha) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| \right\} \right] \\
&> 0
\end{aligned}$$

if

$$|z+k-1| > \frac{1}{|a_n|} \left\{ k|a_n|(\cos \alpha + \sin \alpha) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| \right\}.$$

This shows that the zeros of $F(z)$ with modulus greater than 1 lie in

$$|z+k-1| \leq \frac{1}{|a_n|} \left\{ k|a_n|(\cos \alpha + \sin \alpha) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| \right\}.$$

Since the zeros of $F(z)$ with modulus less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of $F(z)$ and hence $P(z)$ lie in

$$|z+k-1| \leq \frac{1}{|a_n|} \left\{ k|a_n|(\cos \alpha + \sin \alpha) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| \right\}.$$

Thus for any zero z of $P(z)$, we have

$$\begin{aligned}
|z| &= |z+k-1 - (k-1)| \\
&\leq |z+k-1| + |k-1| \\
&\leq \frac{1}{|a_n|} \left\{ k|a_n|(\cos \alpha + \sin \alpha) + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| \right\} + k-1 \\
&= \frac{1}{|a_n|} \left\{ k|a_n|(\cos \alpha + \sin \alpha + 1) - |a_n| + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| \right\}
\end{aligned}$$

In other words, all the zeros of $P(z)$ lie in

$$|z| \leq \frac{1}{|a_n|} \left\{ k|a_n|(\cos \alpha + \sin \alpha + 1) - |a_n| + 2 \sin \alpha \sum_{j=0}^{n-1} |a_j| \right\}.$$

Lemma 3. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that for some real α, β ,

$$|\operatorname{Arg} a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1, 2, \dots, n,$$

and for some $a > 0$,

$$k|a_n| \geq a|a_{n-1}| \geq a^2|a_{n-2}| \geq \dots \geq a^{n-1}|a_1| \geq a^n|a_0|.$$

Then $P(z)$ has all its zeros in

$$|z| \leq \frac{1}{a|a_n|} \left\{ k|a_n|(\cos \alpha + \sin \alpha + 1) - |a_n| + 2 \sin \alpha \sum_{j=0}^{n-1} a^{n-j} |a_j| \right\}.$$

Proof of Lemma 3. Using the conditions of Lemma 2, we can say that the polynomial $P(z/a)$ satisfies the conditions of Lemma 1 and therefore has all its zeros in

$$|z| \leq \frac{1}{|a_n|} \left\{ k|a_n|(\cos \alpha + \sin \alpha + 1) - |a_n| + 2 \sin \alpha \sum_{j=0}^{n-1} a^{n-j} |a_j| \right\}.$$

Accordingly the polynomial $P(z) = P(az/a)$ has all its zeros in

Lemma 4. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial with $\operatorname{Re} a_j = \alpha_j, \operatorname{Im} a_j = \beta_j, j = 0, 1, \dots, n$

such that for some $k \geq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \dots \geq \alpha_1 \geq \alpha_0 \geq 0, \text{ and } \alpha_n > 0.$$

Then $P(z)$ has all its zeros in

$$|z + k - 1| \leq k + \frac{2}{\alpha_n} \sum_{j=0}^n |\beta_j|.$$

Lemma 4 is a simple deduction from a result of Govil and Mc-tume [5].

Remark 2. For any zero z of $P(z)$ of Lemma 4, we have

$$\begin{aligned} |z| &= |z + k - 1 - (k - 1)| \leq |z + k - 1| + k - 1 \\ &\leq k + \frac{2}{\alpha_n} \sum_{j=0}^n |\beta_j| + k - 1 \\ &= 2k - 1 + \frac{2}{\alpha_n} \sum_{j=0}^n |\beta_j|. \end{aligned}$$

Hence all the zeros of $P(z)$ of Lemma 3 lie in

$$|z| \leq 2k - 1 + \frac{2}{\alpha_n} \sum_{j=0}^n |\beta_j|.$$

Lemma 5. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree $n > 1$ with $\operatorname{Re} a_j = \alpha_j, \operatorname{Im} a_j = \beta_j,$

$j = 0, 1, \dots, n$. If for some $a > 0$ and $k \geq 1$,

$$k\alpha_n \geq a\alpha_{n-1} \geq a^2\alpha_{n-2} \geq \dots \geq a^{n-1}\alpha_1 \geq a^n\alpha_0 \geq 0, \alpha_n > 0$$

Then $P(z)$ has all its zeros in

$$|z| \leq \frac{1}{a} \left\{ 2k - 1 + \frac{2}{\alpha_n} \sum_{j=0}^n a^{n-j} |\beta_j| \right\}.$$

Proof of Lemma 5. Using the conditions of Lemma 4, we can say that the polynomial $P(z/a)$ satisfies the conditions of Lemma 3 and therefore has all its zeros in

$$|z| \leq 2k - 1 + \frac{2}{\alpha_n} \sum_{j=0}^n a^{n-j} |\beta_j|.$$

Accordingly the polynomial $P(z) = P(az/a)$ has all its zeros in

$$|z| \leq \frac{1}{a} \left\{ 2k - 1 + \frac{2}{\alpha_n} \sum_{j=0}^n a^{n-j} |\beta_j| \right\}.$$

4. Proofs of Theorems

Proof of Theorem 1. For $P(z) = \sum_{j=0}^n a_j z^j$, we have

$$P'(z) = \sum_{j=0}^{n-1} (j+1) a_{j+1} z^j = \sum_{j=0}^{n-1} b_j z^j = k n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \dots + 2 a_2 z + a_1.$$

Since

$$k |a_n| \geq \frac{1}{t} |a_{n-1}| \geq \frac{1}{t^2} |a_{n-2}| \geq \dots \geq \frac{1}{t^{n-1}} |a_1| \geq \frac{1}{t^n} |a_0|$$

and

$$\frac{n-1}{n} \geq \frac{j}{j+1}, j = 1, 2, \dots, n-1,$$

We have $t \frac{n-1}{n} (j+1) |a_{j+1}| \geq j |a_j|$, $j = 1, 2, \dots, n-2$ i.e. $|b_j| \geq \frac{n}{t(n-1)} |b_{j-1}|$, $j = 1, 2, \dots, n-2$

and $k |b_{n-1}| \geq \frac{n}{t(n-1)} |b_{n-2}|$.

Thus

$$k |b_{n-1}| \geq |b_{n-2}| \geq a^2 |b_{n-3}| \geq \dots \geq a^{n-1} |b_0|,$$

where $a = \frac{n}{t(n-1)}$.

Hence by Lemma 2, it follows that all the zeros of the polynomial $P'(z)$ of degree $(n-1)$ lie in

$$|z| \leq \frac{t(n-1)}{n |a_n|} \left\{ k |a_n| (\cos \alpha + \sin \alpha + 1) - |a_n| + \frac{2 \sin \alpha}{n} \sum_{j=0}^{n-2} \left(\frac{n}{t(n-1)} \right)^{n-1-j} (j+1) |a_{j+1}| \right\}.$$

In other words, it follows that all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n |a_n|} \left\{ k |a_n| (\cos \alpha + \sin \alpha + 1) - |a_n| + \frac{2 \sin \alpha}{n} \sum_{j=0}^{n-2} \left(\frac{n}{t(n-1)} \right)^{n-1-j} (j+1) |a_{j+1}| \right\}$$

are simple.

Proof of Theorem 2. For $P(z) = \sum_{j=0}^n a_j z^j$, we have

$$P'(z) = \sum_{j=0}^{n-1} (j+1)a_{j+1} z^j = \sum_{j=0}^{n-1} b_j z^j.$$

Since

$$k\alpha_n \geq \frac{1}{t}\alpha_{n-1} \geq \frac{1}{t^2}\alpha_{n-2} \geq \dots \geq \frac{1}{t^{n-1}}\alpha_1 \geq \frac{1}{t^n}\alpha_0 \geq 0,$$

and

$$\frac{n-1}{n} \geq \frac{j}{j+1}, j = 1, 2, \dots, n-1,$$

We have $t \frac{n-1}{n} (j+1)\alpha_{j+1} \geq j\alpha_j, j = 1, 2, \dots, n-2$ i.e $\operatorname{Re} b_j \geq \frac{n}{t(n-1)} \operatorname{Re} b_{j-1}, j = 1, 2, \dots, n-2$

and $k \operatorname{Re} b_{n-1} \geq \frac{n}{t(n-1)} \operatorname{Re} b_{n-2}.$

Thus

$$k \operatorname{Re} b_{n-1} \geq a \operatorname{Re} b_{n-2} \geq \dots \geq a^{n-2} \operatorname{Re} b_1 \geq a^{n-1} \operatorname{Re} b_0$$

where $a = \frac{n}{t(n-1)}.$

Hence by Lemma 4, it follows that all the zeros of the polynomial $P'(z)$ of degree $(n-1)$ lie in

$$|z| \leq \frac{t(n-1)}{n} \left\{ 2k-1 + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-j} (j+1) |\beta_{j+1}| \right\}$$

In other words, it follows that all the zeros of $P(z)$ with modulus greater than

$$\frac{t(n-1)}{n} \left\{ 2k-1 + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left(\frac{n}{t(n-1)} \right)^{n-j} (j+1) |\beta_{j+1}| \right\}.$$

are simple.

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