Steady MHD boundary layer flow of an incompressible micropolar fluid past a moving wedge

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Abstract

A MHD boundary layer flow of a two dimensional micropolar fluid past a moving wedge immersed in the flow, is solved analytically and asymptotically. The governing equations are transformed into a highly non- linear coupled system of ordinary differential equations using similarity transformation. We obtain an exact solution for a particular value of the pressure gradient and Hartmann's number, using the boundary conditions defined on the wall of the wedge which is then extended to all values of the parameter. The values of wall shear stress agree well with those obtained numerically, employing Keller-box method.

Keywords: Steady, Boundary layer, Micropolar, Similarity transformation, Keller-box

1. INTRODUCTION

Magnetohydrodynamics(MHD) is essentially the study of behaviour of electrically conducting fluids under the influence of magnetic field. The motion of conducting material across the magnetic lines of force creates potential differences which cause electric currents to flow and the flow of electric current across a magnetic field creates a body force, called the Lorentz force which influences the fluid flow. Thus MHD is defined and characterized by interdependence of hydrodynamics and electrodynamics. Though the first recorded initiation of the field of MHD is by Hannes Alfven[1], a detailed analysis of important aspects of MHD can be found in a book of Shercliff[2] wherein the central idea is that of the kinematic aspect whereby the fluid motion affects the magnetic field, and dynamical aspect whereby the magnetic field affects the motion.

Some examples of such magnetofluids are plasmas, liquid metals, saltwater, electrolytes etc. Many natural phenomena like eddies currents beneath earth's mantle, sunspots, solar winds, to mention a few are caused by MHD. MHD is also related to engineering problems such as plasma confinement, liquid metal cooling of nuclear reactors, electromagnetic castings, MHD generators for driving submarines aircraft, hypersonic wind tunnel experiments. Given to its many practical applications in various manufacturing and processing industries, many authors have shown interest in MHD boundary layer flows. Author's Literature survey reveals that the magnetohydrodynamic boundary layer flows obeying a power-law model were first investigated by Sarpkaya [3], and it was then followed by many authors. To mention a few, Ibrahim and Terbeche[38] studied laminar boundary layer for conducting power-law non-Newtonian fluids in a transverse magnetic field. Chiam [4] examined the MHD flow of a viscous fluid over a surface stretching with a power-law velocity and came by a numerical solution of the problem by using Runge-Kutta shooting algorithm with Newton Raphson method.

The complexity and diversity of non-Newtonian fluids lays an avenue for various challenging models wherein each one exhibits particular feature of the fluid flow, as a single model can never encapsulate all the characteristics of non-Newtonian fluids. Some important non-Newtonian fluids are micropolar fluid,

Maxwell fluid, Walter-B fluid, Casson fluid, Eyring-Powell fluid[5] and Jeffery fluid. Eringen's micropolar fluid theory[6] describes the fluid physically, as the suspension of rigid,

bar like or spherical particles in a viscous medium. Simulation of micropolar fluid characteristics results in a generalized classical Navier-Stokes equations. Interesting and fetching practical applications in different areas of science engineering and technology especially in biomedical fluid engineering which includes arterial blood flow has attracted many authors to the field of micropolar fluid flows. Lukaszewicz[7] has presented a detailed review on the theory and applications of the micropolar fluid. Peddieson and McNitt[8], Wilson[9] have analysed the boundary layer of micropolar fluid flows. In the present paper we are concerned with micropolar fluids and specifically hydromagnetic conductive micropolar flows. Micropolar fluids are affected by the presence of magnetic terms in the translational momentum equations. The magnetic term does not appear in the micropolar balance equation but this equation is affected due to coupling of the microrotation variable to the translational velocity in the translational momentum equation. As a result of which the magnetic fields have a substantial effect on micropolar flows.

Many researchers investigated MHD effects on boundary layer of micropolar fluid under various physical conditions. Hayat et al[10] studied MHD flow of a micropolar fluid near a stagnation-point towards a non-linear stretching surface. Govardhan and Kishan[11] analyzed unsteady MHD boundary layer flow of micropolar fluid over a stretching sheet, Babu et al[12], studied heat and mass transfer in an unsteady MHD boundary Layer flow of micropolar Fluid over Porous Stretching Sheet under Soret and Dufour effect. Ishak et al[13] studied MHD flow of a micropolar fluid towards a stagnation point on a vertical surface. Ibrahim et al [14] examined unsteady magnetohydrodynamic micropolar fluid flow and heat transfer over a vertical porous plate immersed in a porous medium in the presence of thermal and mass diffusion with a constant heat source. Ashraf and Bashir[15] carried out numerical simulation of MHD stagnation point flow and heat transfer of a micropolar fluid towards a heated shrinking sheet, Ezzat et al[16] studied the effect of thermal relaxation on Combined heat and mass transfer for unsteady MHD flow of perfect conducting micropolar fluid.

A significant amount of work has been published by different authors on convective/conductive fluid flows past a fixed wedge immersed in Newtonian/non-Newtonian fluids under various physical conditions. Watanabe and Pop [18] presented the numerical results for MHD free convection flow of a Newtonian fluid over a wedge in the presence of a magnetic field whereas Kafoussias and Nanousis [19] investigated the MHD laminar boundary-layer flow over a permeable wedge immersed in a Newtonian fluid. Hady and Hassanien [17] examined the effect of a transverse magnetic field on a permeable wedge placed symmetrically with respect to the flow direction in a non-Newtonian fluid has been considered. Later, Yih [20] extended the work of Watanabe and Pop [21], by considering the MHD forced convection flow adjacent to a non-isothermal wedge. The effects of variable magnetic field on the fluid flow and heat transfer characteristics were considered by Cobble [22,23], Helmy [24,25], Devi and Thiyagarajan [26] and very recently by Zhang and Wang [27,28], Amkadni et al. [29] and Hoernel [30]. Papailiou and Lykoudis [31] re-examined the theoretical work done by Lykoudis [32] and established the existence of the similarity solutions for the case of variable magnetic field by an experiment. They noted that similarity solutions exist when the intensity of the

magnetic field changes with $x^{-\frac{1}{4}}$ where x measured along direction of the flow. In Recent work, Kim [33] and Kim and Kim [34] have investigated the steady boundary-layer flow of a micropolar fluid past a fixed wedge with constant surface temperature and constant surface heat flux, respectively. Ishak et al [35] extended the work of Kim and Kim [34] and studied the steady laminar MHD boundary-layer flow past a wedge with constant surface heat flux immersed in an incompressible micropolar fluid in the presence of a variable magnetic field.

The above said authors [33, 34, 35] transformed the governing partial differential equations into non-linear ordinary differential equations using similarity transformations introduced by Falkner and Skan [36].

Boundary layer behavior over a moving solid surface is an important type of flow occurring in several engineering processes. For example, the thermal processing of sheet-like materials is a necessary operation in the production of paper, linoleum, polymeric sheets, wire drawing, drawing of plastic films, metal spinning, roofing shingles, insulating materials and fine fiber matts. In this paper we simulate the two-dimensional laminar MHD boundary layer of incompressible micropolar fluid flow past a moving wedge. Also authors have analyzed different models of boundary layer of micropolar fluid flow using numerical methods and there is no exact analytical solution reported on MHD micropolar fluid flow. Inspired by the work of Kudenatti et al[37] , in this paper we obtain exact solution of the flow past a moving wedge for particular values of physical parameters and extend it to more general values.

2. MATHEMATICAL FORMULATION

A two-dimensional laminar boundary layer flow of an incompressible, viscous, electrically conducting micropolar fluid over a wall of a moving wedge with a constant velocity under the influence of transverse magnetic field, is considered. x -axis is taken parallel to the wedge and y -axis is normal to it. Under usual boundary layer approximations [9] the governing equations for MHD micropolar fluid flow past a wedge moving, in the non-dimensional form with the absence of body forces and body couples, are

Conservation of mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

Conservation of momentum:

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = U \frac{dU}{dx} + (\mu + \chi) \frac{\partial^2 u}{\partial y^2} + \chi \frac{\partial \omega}{\partial y} + \sigma B^2(x)(U - u) \quad (2)$$

Conservation of angular momentum:

$$\rho j \left(u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} \right) = - \frac{\partial}{\partial y} \left(v \frac{\partial \omega}{\partial y} \right) - k \left(2\omega + \frac{\partial u}{\partial y} \right) \quad (3)$$

Conservation of micro-inertia:

$$u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} = 0 \quad (4)$$

With boundary conditions velocity and microrotation are,

at

$$y=0, \quad u=U_w(x), v=0, w=-\frac{1}{2} \frac{\partial u}{\partial y} \quad (5)$$

$$\text{as } \frac{y}{\delta} \rightarrow \infty, \quad u=U, v=0, w=0$$

where u and v are the velocity components in the x and y directions, ν is the kinematic viscosity of the fluid, $U(x)$ is the free stream velocity given by the power law $U(x) = U_\infty x^m$ [7] where x is the distance measured from the onset of the boundary layer U_∞ , m are constants and $U_w(x)$ is the stretching surface velocity which obeys the power-law relation $U_w(x) = U_{0w} x^n$. n is a constant such that $0 \leq n \leq 1$.

In the case when $n=0$, fluid is considered to be strongly concentrated and indicates absence of gyration near the wall. This represents concentrated particle flows which makes the microelements close to the wall surface, unable to rotate. The case $n = \frac{1}{2}$ indicates the vanishing of the antisymmetrical part of the stress tensor and denotes weak concentration. The case $n=1$ models the turbulent boundary-layer flows. We have considered weak concentration of particles at the wedge ($n = 1/2$). In this respect, we follow Kim [33] and Kim and Kim [34] by assuming that close to the surface, the effect of microstructure can be neglected since the suspended particles cannot get closer than their radius to the boundary. The only rotation is due to fluid shear and therefore, we assume that the gyration vector may be equal to the angular velocity, as is the case of weak concentration of the fluid particles at the wedge $n = \frac{1}{2}$.

Introducing the stream function $\psi(x, y)$ with $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$ and adopting the coordinate transformations from the variables (x, y) to the new dimensionless similarity variables ([37] [38]).

$$y = \left(\frac{2\gamma x}{(m+1)U} \right)^{\frac{1}{2}} \eta, \quad \psi = \left(\frac{2\gamma x U}{(m+1)} \right)^{\frac{1}{2}} f(\eta), \quad \omega = U \left(\frac{(m+1)U}{2\gamma x} \right)^{\frac{1}{2}} h(\eta), \quad s = \left(\frac{2\gamma x}{(m+1)U} \right)^{\frac{1}{2}} i, \\ \rho = \left(\mu + \frac{\chi}{2} \right) \left(\frac{2\gamma x}{(m+1)U} \right)^{\frac{1}{2}} i, \quad (7)$$

The boundary layer equations transform to the following non-linear ordinary differential equations

$$(1+k)f''' + ff'' + \frac{2m}{m+1}(1-f'^2) + kh' + M^2(1-f') = 0 \quad (8)$$

$$\left(1 + \frac{k}{2}\right)(ih')' + i \left(fh' - \frac{(3m-1)}{(m+1)} hf' \right) - k(2h + f'') = 0 \quad (9)$$

$$\frac{(m+1)}{2} f i' - (1-m) i f' = 0. \quad (10)$$

The boundary conditions are

$$f(0)=0, \quad f'(0)=-\lambda, \quad i(0)=0, \quad h(0)=-\frac{1}{2}f''(0) \quad (11)$$

$$f'(\infty) \rightarrow 1, \quad h(\infty) \rightarrow 0$$

where $\lambda (= -\frac{U_{0w}}{U_\infty})$ is the ratio of free stream velocity to boundary velocity. If $\lambda > 0$ and $\lambda < 0$ then it corresponds to moving wedge in opposite and in the same direction to the free stream velocity, $M = \left(\sqrt{\frac{2\sigma}{\rho(m+1)U_\infty}} \right) B$ is the Hartmann number (Magnetic parameter) which is the ratio of electromagnetic force to the viscous force, σ is the electrical conductivity, ρ is the fluid density, $k = \frac{\chi}{\mu}$, where $\mu = \rho\gamma$, is the dimensionless viscosity ratio.

3. ANALYTICAL SOLUTION

In the case when $M=0$, the solution of micro inertia density equation(10) satisfying boundary conditions(11) is

$$i = A f^{\frac{2(m-1)}{(m+1)}} \quad (12)$$

where $A = C^{\frac{2}{(m+1)}}$ is a non- dimensional constant of integration. Using the boundary condition $i(0) = 0$ leads to $i(\eta) = 0$ which is a trivial solution, in which case (10) reduces to

$$h = -\frac{1}{2} f'' \quad (13)$$

substituting (12) and (13) in (8) we get

$$\left(1 + \frac{k}{2}\right) f''' + f f'' + \frac{2m}{m+1} (1 - f'^2) = 0 \quad (14)$$

In this paper, we obtain the solution analytically, asymptotically and numerically of this equation with the boundary conditions

$$f(0)=0, \quad f'(0)=-\lambda, \quad f'(\infty) \rightarrow 1 \quad (15)$$

where primes denote differentiation with respect to η and the stream wise pressure gradient is favorable pressure gradient when $m > 0$ and adverse pressure gradient when $m < 0$ whereas Blasius flow over a flat plate when $m = 0$. The flow corresponding to stagnation point when $m = 1$.

Following [42], we seek exact solution of equation (14) with boundary condition(15). Exact Solution of equation(14) for $m = -\frac{1}{3}$ is obtained by integrating equation(14) twice and applying the boundary conditions (15) which results in a Riccati type equation and leads to the solution of (14)

$$f = \eta + \delta - \frac{\delta e^{-\left(1+\frac{k}{2}\right)^{-1}\left(\frac{\eta^2}{2} + \delta\eta\right)}}{1 - \frac{\delta\left(1+\frac{k}{2}\right)^{-\frac{1}{2}}}{2} e^{\frac{\delta^2}{2\left(1+\frac{k}{2}\right)}} \sqrt{\frac{\pi}{2}} \left[\operatorname{erf}\left(\frac{\left(1+\frac{k}{2}\right)^{-\frac{1}{2}}(\eta + \delta)}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{\left(1+\frac{k}{2}\right)^{-\frac{1}{2}}(\delta)}{\sqrt{2}}\right) \right]} \quad (16)$$

Provided $\delta^2 = -\frac{2(1+\lambda)}{\left(1+\frac{k}{2}\right)}$. To obtain an exact analytical solution of the system (14) with

(15) for

all values of m , we rewrite the solution (16) as

$$f(\eta) = \eta + \delta - \frac{\delta}{G(\eta)} \quad (17)$$

where

$$G(\eta) = e^{\frac{\delta^2\left(1+\frac{k}{2}\right)}{2}} - \frac{\delta\left(1+\frac{k}{2}\right)^{-\frac{1}{2}}}{2} e^{\frac{\delta^2}{2\left(1+\frac{k}{2}\right)}} \sqrt{\frac{\pi}{2}} \left[\operatorname{erf}\left(\frac{\left(1+\frac{k}{2}\right)^{-\frac{1}{2}}(\eta + \delta)}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{\left(1+\frac{k}{2}\right)^{-\frac{1}{2}}(\delta)}{\sqrt{2}}\right) \right] e^{\left(1+\frac{k}{2}\right)^{-1}\left(\frac{\eta+\delta}{2}\right)^2} \quad (18)$$

Substituting (17) into (14) and (15), we get

$$\begin{aligned} \left(1+\frac{k}{2}\right) \left(G^2 G''' - 6GG'G'' + 6G'^3 \right) + (\eta + \delta) G^2 G'' - 2(\eta + \delta) GG'^2 \\ - \left(\frac{4m}{m+1} + M^2 \right) G^2 G' + \delta \left(\frac{2}{m+1} \right) G'^2 - \delta GG'' = 0 \end{aligned} \quad (19)$$

with the boundary conditions

$$G(0) = 1, G'(0) = \frac{\delta}{2}, G'(\infty) = 0. \quad (20)$$

The solution of equation(21) for $m = -\frac{1}{3}$ subject to equation(22) is given by equation (17).\\

The error and exponential functions in equation (17) are entire functions. These functions may be expanded in Taylor series about $\eta = 0$ which have an infinite radius of convergence.

Therefore the solution (17) which is in series representation[18] for $m = -\frac{1}{3}$ becomes an important clue for further analysis for general values of m . Thus we let

$$G(\eta) = \sum_{n=0}^{\infty} a_n \eta^n \quad (21)$$

for general m and k . Substituting equation(23) into equation(21) where the coefficients need to be determined. From the first two boundary conditions of equation(21) we have $a_0 = 1$ and

$a_1 = \frac{\left(1 + \frac{k}{2}\right)\delta}{2}$. The coefficient a_2 remains unknown due to the end condition of equation(21).

Equating the coefficients of η to zero we get the coefficients a_n and in general we get the recurrence relation

$$\begin{aligned}
 a_{n+3} = & \frac{-1}{\left(1 + \frac{k}{2}\right)(n+1)(n+2)(n+3)} \left(-\left(1 + \frac{k}{2}\right)\delta \sum_{i=1}^n (i+1)(i+2)a_{n-i}a_{i+2} + \left(\frac{-2}{m+1}\right)(n-i+1)a_{n-i+1}a_{i+1} \right) \\
 & + \sum_{j=0}^{n-1} \left(\sum_{i=0}^{n-j} \left(1 + \frac{k}{2}\right)(j+3)((j+2)(j+1)a_{n-j-i}a_{i+3} + \sum_{i=0}^{n-1-j} (j+1)(j+2)a_i a_{j+2} - 2(i+1)a_{i+1}a_{j+1})a_{n-j-i-1} \right) \\
 & + \sum_{j=0}^n \sum_{i=0}^{n-j} (j+1)\left(-6\left(1 + \frac{k}{2}\right)(j+2)(i+1)a_{n-j-i}a_{i+1}a_{j+2} + 6\left(1 + \frac{k}{2}\right)(i+1)(n-j-i+1)a_{n-j-i+1}a_{i+1}a_{j+1} \right. \\
 & \left. + \left(1 + \frac{k}{2}\right)\delta(j+2)a_{n-j-i}a_i a_{j+2} - 2\left(1 + \frac{k}{2}\right)\delta(i+1)a_{n-j-i}a_{i+1}a_{j+1} - \left(\frac{4m}{m+1} + M^2\right)a_{n-j-i}a_i a_{j+1}) \right)
 \end{aligned} \quad (22)$$

where $n=1,2,3,\dots$ and the coefficients a_n have been expressed in terms of a_2, δ, k, m, M as seen in

$$\begin{aligned}
 a_3 &= \frac{1}{24\left(1 + \frac{k}{2}\right)} \left(2M^2 + \frac{8m}{m+1} + \left(-3 + \frac{2m}{m+1}\right)\delta^2 + 24a_2 \right) \\
 a_4 &= \frac{1}{96\left(1 + \frac{k}{2}\right)} \left(2\left(1 + 3M^2 + \frac{12m}{m+1}\right)\delta^2 + \left(-5 + \frac{4m}{m+1}\right)\delta^4 + 4a_2\left(-2 + 2M^2 + 5\delta^2 + \frac{4m}{m+1}(2 + \delta^2) + 24a_2\right) \right)
 \end{aligned}$$

The unknown constant a_2 which characterizes the coefficient of skin friction must be found such that the end derivative boundary condition at far distance in (15) is satisfied. This is same as determining the value $f''(0)$ of the system (14) and (15) as, from (20) and (21) they are intrinsically related to each other by the following

$$a_2 = \frac{1}{2\left(1 + \frac{k}{2}\right)\delta} \left(f''(0) - \frac{(1+\lambda)^2}{\left(1 + \frac{k}{2}\right)\delta} \right) \quad (23)$$

The coefficients a_n involve two arbitrary constants, namely $f''(0)$ and δ . We match the series (21) for $m = -\frac{1}{3}$ with the series form of the exact solution (16) and we get the same

value $\delta = \sqrt{-\frac{2(1+\lambda)}{\left(1 + \frac{k}{2}\right)}}$. This constant δ plays a crucial role in the present analysis. An

exact solution of the equation (14) exists when the values of arguments in the square root function is positive. The other constants a_2 or $f''(0)$ remains to be determined. Therefore, we have infinite solutions of (14-15) in the form of (21). We determine the constant $f''(0)$ in the following manner. Integrating (14) from $\eta = 0$ to $\eta = \infty$ and using boundary conditions (15), we get

$$\int_0^{\eta_{\infty}} \left(\frac{f' - f'^2 + \frac{2m}{m+1}(1 - f'^2) + M^2(1 - f')}{1 + \frac{k}{2}} \right) d\eta = f''(0) \quad (24)$$

wherein $f''(\eta) \rightarrow 0$ as $\eta \rightarrow \infty$ has been assumed based on the numerical study of Riley and Weidman[44] and also based on most of the other investigations. In equation (24), Skin friction $f''(0)$ appears on both sides as f' term. Hence it should be found iteratively using suitable initial approximations for $f''(0)$. The initial approximation is taken from the known exact solution for $m = -\frac{1}{3}$ and $M=0$ which results in fast convergence of the series and reduces the number of iterations drastically in the above integration. Also, equation (24) ensures the ease of computation by providing the liberty to choose the required number of coefficients (21) and the value of η_{∞} in equation(24). In the case of slow convergence, we use diagonal pade' approximant to sum the series(24) which enlarges the convergence domain and increases rate of convergence to give an accurate enough value of $f''(0)$. The values for skin friction coefficient $f''(0)$ thus obtained are consistent with direct numerical solution of the equation (14) with boundary condition (15) obtained via finite difference based Keller Box method [20]. Therefore we have obtained an exact solution of the equation for all the values of m , M and k . To determine the efficiency of the present method, we compare the values of $f''(0)$ with that of DNS of the problem are given in the table and graphs representing the velocity profiles are given in figures. It is observed from table that results from the above method compare well with the Numerical solutions for all the values of parameters used in the analysis.

4. NUMERICAL SOLUTION

The equation (14) is numerically solved by the method reported by Keller(1970) known as Keller-box method. The method has become popular for obtaining non-similar solutions for boundary layer problems.

It is an implicit difference method. The method gives the approximations to the solutions of the original differential equations at each grid point. The step size $\Delta\eta$ in η and the position of the edge of the boundary layer η_{∞} have to be adjusted for different values of parameters to maintain accuracy. In this study, the values of $\Delta\eta$ between 0.001 and 0.1 were used, depending on the values of the parameters considered. However, a uniform grid of $\Delta\eta = 0.01$ was found to be satisfactory for a convergence criterion of 10^{-7} , which gives accuracy to six decimal places in nearly all cases. It has quadratic convergence even for non-uniform mesh points and hence very efficient and unconditionally stable. The method gives the approximations to the solutions of the original differential equations at each grid point. This can be achieved by expressing the governing equations in the form of first order system. To get finite difference equations with a second order truncation error, simple centered difference derivatives and average of the midpoints of net rectangles is used. The resulting

finite difference equations are non-linear algebraic equations, which are linearized by Newton's method. The resulting linear algebraic equations can be easily be put in the matrix form $AD=R$ where A is a tridiagonal matrix whose elements are matrices in turn. The Tridiagonal structure can be solved by LU decomposition method.

By using Keller-box technique we have calculated and have compared the values of skin friction $f''(0)$ obtained asymptotically with the same, for different ranges of λ , k and M .

Figure 1:(a) velocity profiles $f'(\eta)$ and (b) gyration profiles $h(\eta)$, with η for $\lambda = -1.3$ obtained by exact method.

Figure 2: (a)velocity profiles $f'(\eta)$ and (b)gyration profiles $h(\eta)$, with η for $M = 2.0$ obtained by exact method.

Table 1: Comparison of the skin friction $f''(0)$ obtained by analytical method and numericalmethod for $M=1.5$ and $m = -\frac{1}{3}$

K = 1.0			K = 2.0		K = 3.0	
λ	Analytical	Numerical	Analytical	Numerical	Analytical	Numerical
-1.1	-0.150231	-0.158041	-0.138766	-0.137121	-0.125633	-0.139835
-1.2	-0.313319	-0.318869	-0.279999	-0.276662	-0.240438	-0.282235
-1.3	-0.489198	-0.482448	-0.423689	-0.428592	-0.378949	-0.427165
-1.4	-0.657888	-0.653645	-0.569762	-0.562882	-0.509585	-0.574589
-1.5	-0.819307	-0.817713	-0.718222	-0.719499	-0.632355	-0.635414

Table 2: Comparison of the skin friction $f''(0)$ obtained by analytical method and numerical method for $m = 1.0$ and $k = 1.0$.

M= 0.0			M = 0.8		M = 1.5	
λ	Analytical	Numerical	Analytical	Numerical	Analytical	Numerical
-1.1	-0.119827	-0.118205	-0.147869	-0.146048	-0.180316	-0.187437
-1.2	-0.275622	-0.272736	-0.300268	-0.306662	-0.364359	-0.368468
-1.3	-0.403246	-0.401346	-0.457132	-0.451295	-0.552071	-0.553036
-1.4	-0.540132	-0.546793	-0.618373	-0.610273	-0.743394	-0.741084
-1.5	-0.681739	-0.680429	-0.783853	-0.783375	-0.938278	-0.932559

4. RESULTS AND DISCUSSIONS

Figure 1 illustrates the effect of magnetic parameter M on the velocity and gyration profiles obtained by exact method, when the other relevant parameters are held as constants. In figure (1a), we observe that the increasing M accelerates the velocity. This results in the decrease in boundary layer thickness which infers that the flow in the boundary layer gets closer to the wedge surface increasing the velocity gradient. The decrease in the boundary layer thickness can be attributed to Lorentz's force induced by more magnetization. In figure (1b), the effect of magnetic parameter M is more dominant near the wall and shows increase in gyration as M increases but reverse happens as we move away from the wall. Figures (1a) and (1b) show that the profiles eventually achieve their end boundary condition asymptotically. Fig 2 shows oscillatory velocity and gyration profiles for negative pressure gradient, similar to that of classical Falkner-Skan equation as discussed by Hastings and Troy[39]. Figure (2a) shows more number of modes of oscillation in the velocity profile for more and more decelerated values of $m(<0)$ when λ, k, M are fixed. The curves show only undershoots ($f'(\eta) < 1$) except for $m = -5/9$ in which neither undershoot nor

overshoot is shown. Similar nature is observed in the gyration profiles for negative values of m in fig (2b) though with more steep oscillations except for the curve pertaining to $m = -5/9$. However the curves achieve their end boundary conditions.

Table 1 exhibits the comparison of the skin friction coefficient values $f''(0)$ obtained by exact solution with that of the numerical solution for the values of k , λ whereas the values of m , M are kept fixed at $m = -\frac{1}{3}$ and $M = 1.5$. The values in the table show that

there is an excellent agreement between the two solutions for all values. Also as the wall stretching parameter increases in absolute sense, skin friction coefficient also increases. As the micropolar parameter k increases horizontally across the table, the skin friction decreases and hence the boundary layer thickness increases which decreases the velocity gradient. Table 2 shows the variation of skin friction with the stretching parameter and the magnetic parameter M when k and m are fixed. The values depict the increase in skin friction when stretching parameter λ increases, both in the absolute sense though. As the value of M increases skin friction increases satisfying the physical phenomenon in the flow which in itself validates the exact method employed. The consistency of the exact method is further ensured by comparison of the values with those obtained by numerical computation and the corresponding values in the table agree well with each other.

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