Inverted Exponentiated Gamma and its bivariate extension

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Abstract

In this paper, a new distribution called Inverted Exponentiated Gamma (IEG) distribution is introduced. Also, the statistical properties of this model are studied, such as, quantiles, moments and moments of order statistics, maximum likelihood estimators of its parameters are discussed. Moreover, an application of IEG distribution to a breaking strengths of glass fibres of length 1.5 cm data set is provided. Also, bivariate Inverted Exponentiated Gamma (BIEG) distribution is introduced as a Marshall-Olkin type. Marginal and conditional distribution functions are studied. Furthermore, joint hazard rate function and maximum likelihood estimates (MLEs) of the parameters are presented. An application of BIEG distribution to an UEFA Champion's League data set is provided and the profiles of the log-likelihood function of parameters of IEGD and BIEGD are plotted.

1 Introduction

Gupta et al. [1] introduced the exponentiated gamma (EG) distribution. This model is flexible enough to accommodate both monotonic as well as nonmonotonic failure rates. The cumulative distribution function (cdf.) of EG distribution is given by

$$G(x) = (1 - e^{-\lambda x} (\lambda x + 1))^\alpha, \quad \alpha, \lambda > 0, \quad x > 0, \quad (1)$$

and the probability density function (pdf) is given by

$$g(x) = \alpha \lambda^2 x e^{-\lambda x} (1 - e^{-\lambda x} (\lambda x + 1))^{\alpha-1}, \quad \alpha, \lambda > 0, \quad x > 0. \quad (2)$$

The aim of this paper is to introduce an Inverted Exponentiated Gamma (IEG) distribution. Also, the bivariate and we call it bivariate Inverted Exponentiated Gamma (BIEG) distribution, whose marginals are IEG distributions is introduced. It is a Marshall-Olkin type. Many authors used this method to introduce a new bivariate distribution, see for example Marshall and Olkin

[3], Kundu and Gupta [2], Sarhan and Balakrishnan [4], Sarhan et al. [7], El-Bassiouny et al. [5, 11] and El-Gohary et al. [6].

This article is organized as follows, the inverted exponentiated gamma (IEG) distribution are proposed in Section 2. Various properties including quantiles, median and moments are investigated in Section 3. Rényi entropy is provided in Section 4. Moments of order statistics are obtained in Section 4. Section 5 is devoted to the maximum likelihood estimates of the parameters and the information matrix of the IEG distribution. An application of IEG distribution to a breaking strengths of glass fibres of length 1.5 cm data set is provided and the profiles of the log-likelihood function of parameters of IEGD are plotted in section 6. In section 7, we introduced the bivariate case and we call it bivariat inverted exponentiated gamma (BIEG) distribution. Also, various properties including the joint survival function, the joint cumulative distribution function, the joint probability density function, marginal probability density functions are investigated in Section 8. Some reliability studies are obtained in Section 9. Section 10 is devoted to the maximum likelihood estimates of the parameters of the BIEG distribution. In Section 11, an application of the BIEG distribution to a UEFA Champion's League data set are provided and the profiles of the log-likelihood function of parameters of BIEGD are plotted. Finally, the results of this paper are concluded in Section 12.

2 Inverted Exponentiated Gamma distribution

In this section, we propose the inverted exponentiated gamma (IEG) distribution.

2.1 IEG Specifications

Theorem 1 *Let a non-negative random variable Y has the exponentiated gamma distribution, symbolically we write $Y \sim EG(\alpha, \lambda)$. Define a new random variable $X = 1/Y$, then the random variable X has the inverted exponentiated gamma distribution, symbolically we write $X \sim IEG(\alpha, \lambda, k)$. The cumulative distribution function and the probability density function of X are respectively given by*

$$F_X(x) = 1 - \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^\alpha, \quad x > 0, \alpha, \lambda > 0 \quad (3)$$

and

$$f_X(x) = \frac{\alpha \lambda^2}{x^3} e^{-\lambda/x} \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^{\alpha-1}, \quad x > 0, \alpha, \lambda > 0 \quad (4)$$

Proof. Since

$$\begin{aligned} F_X(x) &= P[X \leq x] = P\left[\frac{1}{Y} \leq x\right] \\ &= P\left[Y \geq \frac{1}{x}\right] = 1 - G\left(\frac{1}{x}\right) \\ &= 1 - \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^\alpha \end{aligned}$$

By differentiation the cdf of X given in (3) with respect to x , we find the pdf of X given in (4), which complete the proof. ■

Figures 1 and 2 illustrate some of the possible shapes of the pdf and cdf of the IEG distribution.

Fig. 2: The cdf of the IEG distribution at different values of its parameters.

2.2 Hazard rate function

The hazard rate function $h(x)$ and reversed hazard rate function $\tau(x)$ are an important quantity characterizing life phenomena. For the IEG distribution, $h(x)$ and $\tau(x)$ are given by the following theorem:

Theorem 2 *The hazard rate function and reversed hazard rate function for the IEG distribution are given by*

$$h(x) = \frac{\alpha \lambda^2 e^{-\lambda/x}}{x^3 \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)}, \quad (5)$$

$$\tau(x) = \frac{\frac{\alpha \lambda^2}{x^3} e^{-\lambda/x} \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^{\alpha-1}}{1 - \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^\alpha}. \quad (6)$$

Proof. This can be easily deduced by using $h(x) = \frac{f(x)}{1-F(x)}$, $\tau(x) = \frac{f(x)}{F(x)}$. ■

Figure 3 illustrates some of the possible shapes of the hazard rate function of the IEG distribution.

Fig. 3. The $h(x)$ of the IEG distribution at different values of its parameters.

3 Properties of the IEG distribution

In this section, we discuss some properties of the IEG distribution.

3.1 Quantile, Median and Mode

The q th quantile (x_q), can be computed to the IEG distribution by the following relation

$$e^{-\lambda/x_q} \left(\frac{\lambda}{x_q} + 1 \right) = 1 - (1 - q)^{1/\alpha}. \quad (7)$$

The median of the IEG distribution, i.e. $x_{0.5}$, is given by the following relation

$$e^{-\lambda/x_{0.5}} \left(\frac{\lambda}{x_{0.5}} + 1 \right) = 1 - (1 - 0.5)^{1/\alpha}. \quad (8)$$

Moreover, the mode of the IEG distribution can be obtained by deriving its pdf with respect to x and equal it to zero. Thus the mode of the IEG distribution can be obtained as a solution of the following nonlinear equation

$$\frac{(\alpha - 1)\lambda^2}{x^2} e^{-\lambda/x} \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1 \right) \right)^{-1} - \left(\frac{\lambda}{x} - 3 \right) = 0. \quad (9)$$

One can't get an explicit solution of (9) in the general case. Numerical methods should be used to solve it.

3.2 Moments of the IEGD

In this subsection, the r th moment of the IEG distribution is computed.

Theorem 3 *The r th moment of this IEG distribution is as follows*

$$E(X^r) = \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \leq i}}^{\infty} \binom{\alpha-1}{i} \binom{i}{j} (-1)^i \alpha \lambda^{2+j} (\lambda(i+1))^{r-j-2} \Gamma(j-r+2),$$

where $r \leq 2 + j$.

Proof.

$$\begin{aligned} E(X^r) &= \int_{-\infty}^{\infty} x^r f(x) dx \\ &= \int_0^{\infty} x^r \frac{\alpha \lambda^2}{x^3} e^{-\lambda/x} \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1 \right) \right)^{\alpha-1} dx \\ &= \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \leq i}}^{\infty} \binom{\alpha-1}{i} \binom{i}{j} (-1)^i \alpha \lambda^{2+j} \int_0^{\infty} x^{r-j-3} e^{-\frac{\lambda(i+1)}{x}} dx \end{aligned}$$

let $\frac{\lambda(i+1)}{x} = t$, $x = \lambda(i+1)t^{-1}$, $dx = -\lambda(i+1)t^{-2}dt$, $t = \infty \rightarrow 0$

$$\begin{aligned} E(X^r) &= \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \leq i}}^{\infty} \binom{\alpha-1}{i} \binom{i}{j} (-1)^i \alpha \lambda^{2+j} (\lambda(i+1))^{r-j-2} \int_0^{\infty} t^{j-r+1} e^{-t} dt \\ &= \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \leq i}}^{\infty} \binom{\alpha-1}{i} \binom{i}{j} (-1)^i \alpha \lambda^{2+j} (\lambda(i+1))^{r-j-2} \Gamma(j-r+2). \end{aligned}$$

■

Using the appropriate moment expressions, the variance is given as follows:

$$\begin{aligned}
\text{var}(X) &= E(X^2) - (E(X))^2 \\
&= \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \leq i}}^{\infty} \binom{\alpha-1}{i} \binom{i}{j} (-1)^i \alpha \lambda^{2+j} (\lambda(i+1))^{-j} \Gamma(j) \\
&\quad - \left(\sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \leq i}}^{\infty} \binom{\alpha-1}{i} \binom{i}{j} (-1)^i \alpha \lambda^{2+j} (\lambda(i+1))^{-j-1} \Gamma(j+1) \right)^2
\end{aligned} \tag{10}$$

The skewness and kurtosis can now be calculated by using the relations

$$\text{Skewness} = \frac{E(X^3) - 3E(X)E(X^2) + 2E^3(X)}{\text{Var}^{3/2}(X)} \tag{11}$$

$$\text{Kurtosis} = \frac{E(X^4) - 4E(X)E(X^3) + 6E(X^2)E^2(X) - 3E^4(X)}{\text{Var}^2(X)}. \tag{12}$$

4 Rényi entropy of IEGD

In this section, we compute the rényi entropy which is a measure of the uncertain variation

Theorem 4 *The Rényi entropies for the IEG distribution are given by*

$$I_R(\delta) = \frac{1}{1-\delta} \log[I(\delta)], \quad \delta \neq 1 \tag{13}$$

where

$$I(\delta) = \lambda^{2\delta} \alpha^\delta \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \leq i}}^{\infty} \binom{\delta\alpha-\delta}{i} \binom{i}{j} (-1)^i \lambda^j (\lambda(i+\delta))^{-3\delta-j+1} \Gamma(3\delta+j-1). \tag{14}$$

Proof.

$$I_R(\delta) = \frac{1}{1-\delta} \log(I(\delta))$$

$$\begin{aligned}
I(\delta) &= \int_{\mathbb{R}} f^\delta(x) dx = \int_0^\infty \frac{\alpha^\delta \lambda^{2\delta}}{x^{3\delta}} e^{-\lambda\delta/x} \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1 \right) \right)^{\delta\alpha-\delta} dx \\
&= \lambda^{2\delta} \alpha^\delta \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \leq i}}^{\infty} \binom{\delta\alpha-\delta}{i} \binom{i}{j} (-1)^i \lambda^j \int_0^\infty x^{-j-3\delta} e^{\frac{-\lambda(i+\delta)}{x}} dx
\end{aligned}$$

let $\frac{\lambda(i+\delta)}{x} = t$, $x = \lambda(i+\delta)t^{-1}$, $dx = -\lambda(i+\delta)t^{-2}dt$, $t = \infty \rightarrow 0$

$$\begin{aligned} I(\delta) &= \lambda^{2\delta} \alpha^\delta \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \leq i}}^{\infty} \binom{\delta\alpha - \delta}{i} \binom{i}{j} (-1)^i \lambda^j (\lambda(i+\delta))^{-3\delta-j+1} \\ &\quad \times \int_0^\infty e^{-t} t^{3\delta+j-2} dt \\ &= \lambda^{2\delta} \alpha^\delta \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \leq i}}^{\infty} \binom{\delta\alpha - \delta}{i} \binom{i}{j} (-1)^i \lambda^j (\lambda(i+\delta))^{-3\delta-j+1} \Gamma(3\delta+j-1). \end{aligned}$$

■

5 Moments of order statistics of IEGD

Suppose that n random variables $X_1, X_2, \dots, X_n$ are ordered in non-decreasing magnitude and written as $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$. Where the smallest order statistic is denoted by $X_{1:n}$, the second smallest is denoted by $X_{2:n}$, and so on, and the largest order statistic is denoted by $X_{n:n}$ and $X_{r:n}$ is called r^{th} order statistic.

In the definition of order statistics, there is no restriction on whether X_i 's are independent or identically distributed. But many well-known results about order statistics are under the classical assumption that X_i 's are independent and identically distributed (iid). The *pdf* of the r^{th} order statistic is

$$f_{r:n}(x) = \frac{n!}{(n-r)!(r-1)!} f(x) [F(x)]^{r-1} [1-F(x)]^{n-r}, \quad -\infty < x < \infty.$$

where $f(x)$ comes from (4) and $F(x)$ comes from (3).

The k th moment about zero of the r th order statistic is given by the following theorem

Theorem 5 *The k th moment about zero of the r th order statistic for the IEG distribution is given by*

$$\begin{aligned} \mu_{r:n}^{(k)} &= \frac{n! \alpha}{(r-1)!(n-r)!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{r-1}{i} \binom{\alpha(n-r+i+1)-1}{j} \\ &\quad \times \binom{j}{k} (-1)^{i+j} (j+1)^{-k-2} \Gamma(k+2). \end{aligned}$$

Proof.

$$\begin{aligned}
\mu_{r:n}^{(k)} &= \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} f(x) [F(x)]^{r-1} (1-F(x))^{n-r} dx \\
&= \frac{n! \alpha \lambda^2}{(r-1)!(n-r)!} \sum_{i=0}^{\infty} \binom{r-1}{i} (-1)^i \int_0^{\infty} x^{-3} e^{-\lambda/x} \\
&\quad \times \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1 \right) \right)^{\alpha(n-r+i+1)-1} dx \\
&= \frac{n! \alpha \lambda^2}{(r-1)!(n-r)!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{r-1}{i} \binom{\alpha(n-r+i+1)-1}{j} \\
&\quad \times (-1)^{i+j} \int_0^{\infty} x^{-3} e^{-\frac{\lambda(j+1)}{x}} \left(\frac{\lambda}{x} + 1 \right)^j dx \\
&= \frac{n! \alpha \lambda^2}{(r-1)!(n-r)!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{r-1}{i} \binom{\alpha(n-r+i+1)-1}{j} \\
&\quad \times \binom{j}{k} (-1)^{i+j} \lambda^k \int_0^{\infty} x^{-3-k} e^{-\frac{\lambda(j+1)}{x}} dx
\end{aligned}$$

let $\frac{\lambda(i+\delta)}{x} = t$, $x = \lambda(i+\delta)t^{-1}$, $dx = -\lambda(i+\delta)t^{-2}dt$, $t = \infty \rightarrow 0$

$$\begin{aligned}
\mu_{r:n}^{(k)} &= \frac{n! \alpha}{(r-1)!(n-r)!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{r-1}{i} \binom{\alpha(n-r+i+1)-1}{j} \\
&\quad \times \binom{j}{k} (-1)^{i+j} (j+1)^{-k-2} \int_0^{\infty} e^{-t} t^{k+1} dt \\
&= \frac{n! \alpha}{(r-1)!(n-r)!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{r-1}{i} \binom{\alpha(n-r+i+1)-1}{j} \\
&\quad \times \binom{j}{k} (-1)^{i+j} (j+1)^{-k-2} \Gamma(k+2).
\end{aligned}$$

■

6 Estimation and information matrix of IEGD

In this section we discussed the estimation of the IEG parameters by using the method of maximum likelihood based on a complete sample.

6.1 Maximum Likelihood Estimators

Let $x_1, x_2, \dots, x_n$ be a random sample from IEG distribution. Then the log-likelihood function of the IEG may be expressed as

$$L = \log l(x; \alpha, \lambda) = 2n \log \lambda + n \log \alpha - 3 \sum_{i=1}^n \log x_i - \lambda \sum_{i=1}^n \frac{1}{x_i} + (\alpha - 1) \sum_{i=1}^n \log \left(1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1 \right) \right) \quad (15)$$

Differentiating the log-likelihood with respect to α and λ , respectively, and setting the result equal to zero, we have

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log \left(1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1 \right) \right) \quad (16)$$

$$\frac{\partial L}{\partial \lambda} = \frac{2n}{\lambda} - \sum_{i=1}^n \frac{1}{x_i} + (\alpha - 1) \sum_{i=1}^n \frac{e^{-\lambda/x_i} \frac{\lambda}{x_i^2}}{1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1 \right)} \quad (17)$$

The maximum likelihood estimates $\hat{\alpha}$ and $\hat{\lambda}$ of the unknown parameters α and λ respectively, are obtained by setting these above equations (16)- (17) to zero and solving them simultaneously.

6.2 Asymptotic Confidence Bounds

In this subsection, we derive the asymptotic confidence intervals of the unknown parameters α and λ when $\alpha, \lambda > 0$. The simplest large sample approach is to assume that the MLEs(α, λ) are approximately multivariate normal with mean (α, λ) and covariance matrix I_0^{-1} where I_0^{-1} the inverse of the observed information matrix which is defined by

$$I_0^{-1} = - \begin{bmatrix} \frac{\partial^2 L}{\partial \alpha^2} & \frac{\partial^2 L}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 L}{\partial \lambda \partial \alpha} & \frac{\partial^2 L}{\partial \lambda^2} \end{bmatrix}^{-1} = \begin{bmatrix} var(\hat{\alpha}) & cov(\hat{\alpha}, \hat{\lambda}) \\ cov(\hat{\lambda}, \hat{\alpha}) & var(\hat{\lambda}) \end{bmatrix} \quad (18)$$

The second partial derivatives included in I_0^{-1} are given as follows

$$\begin{aligned} \frac{\partial^2 L}{\partial \alpha^2} &= \frac{-n}{\alpha^2} \\ \frac{\partial^2 L}{\partial \lambda \partial \alpha} &= \sum_{i=1}^n \frac{e^{-\lambda/x_i} \frac{\lambda}{x_i^2}}{W_i} \\ \frac{\partial^2 L}{\partial \lambda^2} &= \frac{-2n}{\lambda^2} - (\alpha - 1) \sum_{i=1}^n \frac{W_i e^{-\lambda/x_i} \frac{1}{x_i^2} \left(1 - \frac{\lambda}{x_i} \right) - \left(e^{-\lambda/x_i} \frac{\lambda}{x_i^2} \right)^2}{W_i^2} \end{aligned}$$

where the function W_i is given by

$$W_i = 1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1 \right)$$

The above approach is used to derive the $(1 - \delta)$ 100% confidence intervals for the parameters α and λ as in the following forms

$$\hat{\alpha} \pm Z_{\frac{\delta}{2}} \sqrt{\text{var}(\hat{\alpha})}, \quad \hat{\lambda} \pm Z_{\frac{\delta}{2}} \sqrt{\text{var}(\hat{\lambda})},$$

where $Z_{\frac{\delta}{2}}$ is the upper $(\frac{\delta}{2})$ th percentile of the standard normal distribution.

7 Application of the IEG Distribution

Now we use a real data set to compare the results of fitting the IEG and EG distributions. The data set in Table 1 shows the sixty-three breaking strengths of glass fibres of length 1.5 cm were reported by Smith and Naylor (1987).

Table 1: Sixty-three breaking strengths of glass fibres of length 1.5 cm

0.55	0.74	0.77	0.81	0.84	0.93	1.04	1.11	1.13	1.24
1.25	1.27	1.28	1.29	1.30	1.36	1.39	1.42	1.48	1.48
1.49	1.49	1.5	1.5	1.51	1.52	1.53	1.54	1.55	1.55
1.58	1.59	1.6	1.61	1.61	1.61	1.61	1.62	1.62	1.63
1.64	1.66	1.66	1.66	1.67	1.68	1.68	1.69	1.7	1.7
1.73	1.76	1.76	1.77	1.78	1.81	1.82	1.84	1.84	1.89
2.00	2.01	2.24							

The MLEs of the unknown parameters α and λ and the values of the log-likelihood functions L , the statistics AIC (Akaike Information Criterion), $CAIC$ (Consistent Akaike Information Criteria) and BIC (Bayesian Information Criterion) are given in Table 2. Also, the Kolmogorov-Smirnov (K-S) test statistic with corresponding P-value are given in Table 3 for the distributions in order to verify which distribution fits better this real data set.

Table 2: The MLEs and the values of L , AIC , BIC and $CAIC$

Model	MLEs	L	AIC	BIC	$CAIC$
EG	$\hat{\lambda} = 0.704, \hat{\alpha} = 2.075$	-30.080	64.16	63.7587	64.36
IEG	$\hat{\lambda} = 9.341, \hat{\alpha} = 50.332$	-23.189	50.378	49.977	50.578

Table 3: The K-S test statistic and corresponding P-values

Model	KS-value	p-value
EG	0.211	6.1188×10^{-3}
IEG	0.20386	8.99398×10^{-3}

Since the values of $-L$, AIC , BIC , $CAIC$ and K-S (see Table 2 & Table 3) are smaller for the IEG distribution compared with those values of the other model, then the new distribution seems to be a very competitive model based on this real data set.

Fig. 4. The empirical and fitted survival functions of selected models

Substituting the MLEs of the unknown parameters into (18), we get an estimation of the variance covariance matrix as the following:

$$I_0^{-1} = \begin{bmatrix} 562.31 & 20.63 \\ 20.63 & 0.815 \end{bmatrix}$$

The approximate 95% two sided confidence intervals of the unknown parameters α and λ are, respectively, given as $[10.257, 90.41]$, $[7.8153, 10.867]$.

The profiles of the log-likelihood functions of α and λ of IEG for breaking strengths of glass fibres of length 1.5 cm data are plotted in Fig. 5 and Fig. 6. From the plots of the profiles of the log-likelihood function of α and λ , we observe that the likelihood equations have a unique solution.

Fig.5.The profile of the log-likelihood function of α .

Fig.6.The profile of the log-likelihood function of λ .

8 Bivariate Inverted Exponentiated Gamma distribution

In this section, the BIEG distribution is discussed. Let us start with the joint survival function and derive the corresponding joint probability density function of this distribution

8.1 The Joint Survival Function of BIEG

Suppose that $U_1 \sim IEG(\alpha_1, \lambda)$, $U_2 \sim IEG(\alpha_2, \lambda)$ and $U_3 \sim IEG(\alpha_3, \lambda)$ are independent random variables. Define $X_1 = \min\{U_1, U_3\}$ and $X_2 = \min\{U_2, U_3\}$. Then, the bivariate vector $(X_1, X_2) \sim BIEG(\alpha_1, \alpha_2, \alpha_3, \lambda)$.

In the following lemma, We study the joint survival function of the random variables X_1 and X_2 .

Lemma 6 *The joint survival function of X_1 and X_2 is given by*

$$\begin{aligned} \bar{F}_{BIEG}(x_1, x_2) &= \left(1 - e^{-\lambda/x_1} \left(\frac{\lambda}{x_1} + 1\right)\right)^{\alpha_1} \left(1 - e^{-\lambda/x_2} \left(\frac{\lambda}{x_2} + 1\right)\right)^{\alpha_2} \\ &\quad \times \left(1 - e^{-\lambda/z} \left(\frac{\lambda}{z} + 1\right)\right)^{\alpha_3} \end{aligned} \quad (19)$$

where $z = \max(x_1, x_2)$.

Proof.

$$\begin{aligned} \bar{F}_{BIEG}(x_1, x_2) &= P(X_1 > x_1, X_2 > x_2) \\ &= P(\min\{U_1, U_3\} > x_1, \min\{U_2, U_3\} > x_2) \\ &= P(U_1 > x_1, U_2 > x_2, U_3 > \max(x_1, x_2)). \end{aligned}$$

Where, U_i ($i = 1, 2, 3$) are independent random variables. Then, we obtain

$$\begin{aligned} \bar{F}_{BIEG}(x_1, x_2) &= P(U_1 > x_1) P(U_2 > x_2) P(U_3 > \max(x_1, x_2)) \\ &= \bar{F}_{IEG}(x_1; \alpha_1, \lambda) \bar{F}_{IEG}(x_2; \alpha_2, \lambda) \bar{F}_{IEG}(z; \alpha_3, \lambda) \\ &= \left(1 - e^{-\lambda/x_1} \left(\frac{\lambda}{x_1} + 1\right)\right)^{\alpha_1} \left(1 - e^{-\lambda/x_2} \left(\frac{\lambda}{x_2} + 1\right)\right)^{\alpha_2} \\ &\quad \times \left(1 - e^{-\lambda/z} \left(\frac{\lambda}{z} + 1\right)\right)^{\alpha_3}. \end{aligned}$$

■

8.2 The Joint Probability Density Function of BIEGD

In this subsection, we study the joint probability density function of the random variables X_1 and X_2 in the following theorem.

Theorem 7 *If the joint survival function of (X_1, X_2) is as in (19) then, the joint pdf of (X_1, X_2) is given by*

$$f_{BIEG}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & \text{if } x_2 < x_1 \\ f_2(x_1, x_2) & \text{if } x_1 < x_2 \\ f_3(x) & \text{if } x_1 = x_2 = x \end{cases} \quad (20)$$

where

$$\begin{aligned} f_1(x_1, x_2) &= f_{IEG}(x_1; \alpha_1 + \alpha_3, \lambda) f_{IEG}(x_2; \alpha_2, \lambda) \\ &= \frac{(\alpha_1 + \alpha_3)\lambda^2}{x_1^3} e^{-\lambda/x_1} \left(1 - e^{-\lambda/x_1} \left(\frac{\lambda}{x_1} + 1\right)\right)^{\alpha_1 + \alpha_3 - 1} \\ &\quad \times \frac{\alpha_2 \lambda^2}{x_2^3} e^{-\lambda/x_2} \left(1 - e^{-\lambda/x_2} \left(\frac{\lambda}{x_2} + 1\right)\right)^{\alpha_2 - 1} \end{aligned} \quad (21)$$

$$\begin{aligned} f_2(x_1, x_2) &= f_{IEG}(x_1; \alpha_1, \lambda) f_{IEG}(x_2; \alpha_2 + \alpha_3, \lambda) \\ &= \frac{\alpha_1 \lambda^2}{x_1^3} e^{-\lambda/x_1} \left(1 - e^{-\lambda/x_1} \left(\frac{\lambda}{x_1} + 1\right)\right)^{\alpha_1 - 1} \\ &\quad \times \frac{(\alpha_2 + \alpha_3)\lambda^2}{x_2^3} e^{-\lambda/x_2} \left(1 - e^{-\lambda/x_2} \left(\frac{\lambda}{x_2} + 1\right)\right)^{\alpha_2 + \alpha_3 - 1} \end{aligned} \quad (22)$$

$$\begin{aligned} f_3(x) &= \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} f_{IEG}(x; \alpha_1 + \alpha_2 + \alpha_3, \lambda) \\ &= \frac{\alpha_3 \lambda^2}{x^3} e^{-\lambda/x} \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} \end{aligned} \quad (23)$$

Proof. Let us first assume that $x_2 < x_1$. Then, $\bar{F}_{BIEG}(x_1, x_2)$ in (19) becomes

$$\bar{F}_{BIEG}(x_1, x_2) = \left(1 - e^{-\lambda/x_1} \left(\frac{\lambda}{x_1} + 1\right)\right)^{\alpha_1 + \alpha_3} \left(1 - e^{-\lambda/x_2} \left(\frac{\lambda}{x_2} + 1\right)\right)^{\alpha_2}$$

Then, upon differentiating this function w.r.t. x_1 and x_2 we obtain the expression of $f_1(x_1, x_2)$ given in (21). By the same way we obtain $f_2(x_1, x_2)$ when $x_1 < x_2$. But $f_3(x)$ cannot be derived in a similar way. For this reason, we use the following identity to derive $f_3(x)$

$$\int_0^\infty \int_{x_2}^\infty f_1(x_1, x_2) dx_1 dx_2 + \int_0^\infty \int_{x_1}^\infty f_2(x_1, x_2) dx_2 dx_1 + \int_0^\infty f_3(x) dx = 1$$

let

$$I_1 = \int_0^\infty \int_{x_2}^\infty f_1(x_1, x_2) dx_1 dx_2 \text{ and } I_2 = \int_0^\infty \int_{x_1}^\infty f_2(x_1, x_2) dx_2 dx_1$$

then

$$\begin{aligned} I_1 &= \int_0^\infty \int_{x_2}^\infty \frac{(\alpha_1 + \alpha_3)\lambda^2}{x_1^3} e^{-\lambda/x_1} \left(1 - e^{-\lambda/x_1} \left(\frac{\lambda}{x_1} + 1\right)\right)^{\alpha_1 + \alpha_3 - 1} \\ &\quad \times \frac{\alpha_2 \lambda^2}{x_2^3} e^{-\lambda/x_2} \left(1 - e^{-\lambda/x_2} \left(\frac{\lambda}{x_2} + 1\right)\right)^{\alpha_2 - 1} dx_1 dx_2 \\ &= \int_0^\infty \frac{\alpha_2 \lambda^2}{x_2^3} e^{-\lambda/x_2} \left(1 - e^{-\lambda/x_2} \left(\frac{\lambda}{x_2} + 1\right)\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} dx_2. \end{aligned} \quad (24)$$

Similarly

$$I_2 = \int_0^\infty \frac{\alpha_1 \lambda^2}{x_1^3} e^{-\lambda/x_1} \left(1 - e^{-\lambda/x_1} \left(\frac{\lambda}{x_1} + 1\right)\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} dx_1. \quad (25)$$

From (24) and (25), we get

$$\begin{aligned} \int_0^\infty f_3(x) dx &= 1 - I_1 - I_2 \\ &= \int_0^\infty \frac{(\alpha_1 + \alpha_2 + \alpha_3)\lambda^2}{x^3} e^{-\lambda/x} \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} dx \\ &\quad - \int_0^\infty \frac{\alpha_2 \lambda^2}{x^3} e^{-\lambda/x} \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} dx \\ &\quad - \int_0^\infty \frac{\alpha_1 \lambda^2}{x^3} e^{-\lambda/x} \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} dx \end{aligned}$$

Then

$$\begin{aligned} f_3(x) &= \frac{\alpha_3 \lambda^2}{x^3} e^{-\lambda/x} \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} \\ &= \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} f_{IEG}(x; \alpha_1 + \alpha_2 + \alpha_3, \lambda). \end{aligned}$$

This completes the proof of the theorem. ■

8.3 Marginal Probability Density Functions of BIEGD

The following theorem gives the marginal probability density functions of X_1 and X_2 .

Theorem 8 *The marginal probability density functions of X_i ($i = 1, 2$) is given by*

$$\begin{aligned} f_{X_i}(x_i) &= \frac{(\alpha_i + \alpha_3)\lambda^2}{x_i^3} e^{-\lambda/x_i} \left(1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1\right)\right)^{\alpha_i + \alpha_3 - 1} \\ &= f_{IEG}(x_i; \alpha_i + \alpha_3, \lambda), \quad x_i > 0, \quad i = 1, 2. \end{aligned} \quad (26)$$

Proof. The marginal survival function of X_i , say $\bar{F}(x_i)$, as follows:

$$\begin{aligned} \bar{F}(x_i) &= P(X_i > x_i) \\ &= P(\min\{U_i, U_3\} > x_i) \\ &= P(U_i > x_i, U_3 > x_i) \end{aligned}$$

since, the random variables U_i ($i = 1, 2$) and U_3 are mutually independent, then

$$\begin{aligned} \bar{F}(x_i) &= P(U_i > x_i) P(U_3 > x_i) \\ &= \bar{F}_{IEG}(x_i; \alpha_i + \alpha_3, \lambda) \\ &= \left(1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1\right)\right)^{\alpha_i + \alpha_3} \end{aligned} \quad (27)$$

Since, $f_{X_i}(x_i) = \frac{-d\bar{F}(x_i)}{dx_i}$ then, we obtain the formula given in (26). ■

8.4 The Joint Cumulative Distribution Function of BIEGD

Theorem 9 *The joint cumulative distribution function of (X_1, X_2) is given by*

$$F_{X_1, X_2}(x_1, x_2) = \begin{cases} F_1(x_1, x_2) & \text{if } x_2 < x_1 \\ F_2(x_1, x_2) & \text{if } x_1 < x_2 \\ F_3(x) & \text{if } x_1 = x_2 = x \end{cases} \quad (28)$$

where

$$\begin{aligned} F_1(x_1, x_2) &= 1 - \left(1 - e^{-\lambda/x_2} \left(\frac{\lambda}{x_2} + 1\right)\right)^{\alpha_2 + \alpha_3} - \left(1 - e^{-\lambda/x_1} \left(\frac{\lambda}{x_1} + 1\right)\right)^{\alpha_1 + \alpha_3} \\ &\quad \times \left[1 - \left(1 - e^{-\lambda/x_2} \left(\frac{\lambda}{x_2} + 1\right)\right)^{\alpha_2}\right] \\ F_2(x_1, x_2) &= 1 - \left(1 - e^{-\lambda/x_1} \left(\frac{\lambda}{x_1} + 1\right)\right)^{\alpha_1 + \alpha_3} - \left(1 - e^{-\lambda/x_2} \left(\frac{\lambda}{x_2} + 1\right)\right)^{\alpha_2 + \alpha_3} \\ &\quad \times \left[1 - \left(1 - e^{-\lambda/x_1} \left(\frac{\lambda}{x_1} + 1\right)\right)^{\alpha_1}\right] \end{aligned}$$

and

$$F_3(x) = 1 - \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^{\alpha_3} \left[\left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^{\alpha_1} - \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^{\alpha_2} - \left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)^{\alpha_1 + \alpha_2} \right]$$

Proof. This can be easily deduced by using the following relation:

$$F_{X_1, X_2}(x_1, x_2) = 1 - \bar{F}_{X_1}(x_1) - \bar{F}_{X_2}(x_2) + \bar{F}_{X_1, X_2}(x_1, x_2). \blacksquare$$

8.5 Conditional Probability Density Functions

Given the marginal probability density functions of X_1 and X_2 , one can derive the conditional probability density functions as presented in the following theorem

Theorem 10 *The conditional probability density functions of X_i , given $X_j = x_j$, $f(x_i|x_j)$, $i, j = 1, 2$; $i \neq j$, is given by*

$$f_{X_i|X_j}(x_i|x_j) = \begin{cases} f_{X_i|X_j}^{(1)}(x_i|x_j) & \text{if } x_j < x_i, \\ f_{X_i|X_j}^{(2)}(x_i|x_j) & \text{if } x_i < x_j, \\ f_{X_i|X_j}^{(3)}(x_i|x_j) & \text{if } x_i = x_j = x, \end{cases}$$

where

$$\begin{aligned} f_{X_i|X_j}^{(1)}(x_i|x_j) &= \frac{(\alpha_i + \alpha_3)\lambda^2 e^{-\lambda/x_i} \left(1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1\right)\right)^{\alpha_i + \alpha_3 - 1}}{x_i^3(\alpha_j + \alpha_3) \left(1 - e^{-\lambda/x_j} \left(\frac{\lambda}{x_j} + 1\right)\right)^{\alpha_3}}, \\ f_{X_i|X_j}^{(2)}(x_i|x_j) &= \frac{\alpha_i \lambda^2}{x_i^3} e^{-\lambda/x_i} \left(1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1\right)\right)^{\alpha_i - 1}, \\ f_{X_i|X_j}^{(3)}(x_i|x_j) &= \frac{\alpha_3}{\alpha_i + \alpha_3} \left(1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1\right)\right)^{\alpha_i}. \end{aligned}$$

Proof. The proof follows immediately by substituting the joint probability density function of (X_1, X_2) given in (21), (22) and (23) and the marginal probability density function of X_i ($i = 1, 2$) given in (26), using the relation

$$f_{X_i|X_j}(x_i|x_j) = \frac{f_{X_i, X_j}(x_i, x_j)}{f_{X_j}(x_j)}, \quad i = 1, 2. \blacksquare$$

9 Reliability Studies of BIEGD

In this section, the joint hazard rate function of (X_1, X_2) , the cdf of the random variable $U = \max\{X_1, X_2\}$ and the cdf of the random variable $V = \min\{X_1, X_2\}$ are addressed.

9.1 Joint Hazard Rate Function of BIEGD

Theorem 11 The joint hazard rate function of (X_1, X_2) is given by

$$h_{X_1, X_2}(x_1, x_2) = \begin{cases} h_1(x_1, x_2) & \text{if } x_2 < x_1 \\ h_2(x_1, x_2) & \text{if } x_1 < x_2 \\ h_3(x_1, x_2) & \text{if } x_1 = x_2 = x \end{cases}$$

where

$$h_1(x_1, x_2) = \frac{\frac{(\alpha_1 + \alpha_3)\lambda^4}{x_1^3} e^{-\lambda/x_1} \frac{\alpha_2}{x_2^3} e^{-\lambda/x_2}}{\left(1 - e^{-\lambda/x_1} \left(\frac{\lambda}{x_1} + 1\right)\right) \left(1 - e^{-\lambda/x_2} \left(\frac{\lambda}{x_2} + 1\right)\right)}$$

$$h_2(x_1, x_2) = \frac{\frac{\alpha_1\lambda^4}{x_1^3} e^{-\lambda/x_1} \frac{(\alpha_2 + \alpha_3)}{x_2^3} e^{-\lambda/x_2}}{\left(1 - e^{-\lambda/x_1} \left(\frac{\lambda}{x_1} + 1\right)\right) \left(1 - e^{-\lambda/x_2} \left(\frac{\lambda}{x_2} + 1\right)\right)}$$

and

$$h_3(x, x) = \frac{\frac{\alpha_3\lambda^2}{x^3} e^{-\lambda/x}}{\left(1 - e^{-\lambda/x} \left(\frac{\lambda}{x} + 1\right)\right)}.$$

Proof. This can be easily deduced by using $h_{X_1, X_2}(x_1, x_2) = \frac{f_{X_1, X_2}(x_1, x_2)}{F_{X_1, X_2}(x_1, x_2)}$. ■

Lemma 12 The cdfs of the random variables $U = \max\{X_1, X_2\}$ and $V = \min\{X_1, X_2\}$ are given by

$$F_U(u) = 1 - \left(1 - e^{-\lambda/u} \left(\frac{\lambda}{u} + 1\right)\right)^{\alpha_3} \left[\left(1 - e^{-\lambda/u} \left(\frac{\lambda}{u} + 1\right)\right)^{\alpha_1} - \left(1 - e^{-\lambda/u} \left(\frac{\lambda}{u} + 1\right)\right)^{\alpha_2} - \left(1 - e^{-\lambda/u} \left(\frac{\lambda}{u} + 1\right)\right)^{\alpha_1 + \alpha_2} \right],$$

$$F_V(v) = 1 - \left(1 - e^{-\lambda/v} \left(\frac{\lambda}{v} + 1\right)\right)^{\alpha_1 + \alpha_2 + \alpha_3}.$$

Proof. The cdf of the random variable $U = \max\{X_1, X_2\}$ is

$$\begin{aligned} F_U(u) &= P[U \leq u] \\ &= P[\max\{X_1, X_2\} \leq u] \\ &= P[X_1 \leq u, X_2 \leq u] \\ &= F(u, u) \\ &= 1 - \left(1 - e^{-\lambda/u} \left(\frac{\lambda}{u} + 1\right)\right)^{\alpha_3} \left[\left(1 - e^{-\lambda/u} \left(\frac{\lambda}{u} + 1\right)\right)^{\alpha_1} - \left(1 - e^{-\lambda/u} \left(\frac{\lambda}{u} + 1\right)\right)^{\alpha_2} - \left(1 - e^{-\lambda/u} \left(\frac{\lambda}{u} + 1\right)\right)^{\alpha_1 + \alpha_2} \right] \end{aligned}$$

The *cdf* of the random variables $V = \min \{X_1, X_2\}$ is given by

$$\begin{aligned}
 F_V(v) &= P[V \leq v] \\
 &= P[\min \{X_1, X_2\} \leq v] \\
 &= 1 - P[\min \{X_1, X_2\} > v] \\
 &= 1 - P[X_1 > v, X_2 > v] \\
 &= 1 - \bar{F}(v, v) \\
 &= 1 - \left(1 - e^{-\lambda/v} \left(\frac{\lambda}{v} + 1\right)\right)^{\alpha_1 + \alpha_2 + \alpha_3}.
 \end{aligned}$$

■

10 Maximum Likelihood Estimators of BIEGD

Kundu and Gupta [2] used the method of maximum likelihood to estimate the unknown parameters of the bivariate generalized exponential distribution. In the same maner we use the method of maximum likelihood to estimate the unknown parameters of the BIEG distribution.

Suppose $((x_{11}, x_{21}), \dots, (x_{1n}, x_{2n}))$ is a random sample from BIEG distribution. Consider the following notation

$I_1 = \{i; x_{1i} > x_{2i}\}$, $I_2 = \{i; x_{1i} < x_{2i}\}$, $I_3 = \{i; x_{1i} = x_{2i} = x_i\}$, $I = I_1 \cup I_2 \cup I_3$, $|I_1| = n_1$, $|I_2| = n_2$, $|I_3| = n_3$, and $n_1 + n_2 + n_3 = n$.

The likelihood function of the sample of size n is given by:

$$l(\alpha_1, \alpha_2, \alpha_3, \lambda) = \prod_{i=1}^{n_1} f_1(x_{1i}, x_{2i}) \prod_{i=1}^{n_2} f_2(x_{1i}, x_{2i}) \prod_{i=1}^{n_3} f_3(x_i)$$

The log-likelihood function can be expressed as

$$L(\alpha_1, \alpha_2, \alpha_3, \lambda) = \ln l(\alpha_1, \alpha_2, \alpha_3, \lambda)$$

$$\begin{aligned}
= & 2n_1 \ln(\lambda) + n_1 \ln(\alpha_1 + \alpha_3) - 3 \sum_{i=1}^{n_1} \ln(x_{1i}) - \lambda \sum_{i=1}^{n_1} \frac{1}{x_{1i}} \\
& + (\alpha_1 + \alpha_3 - 1) \sum_{i=1}^{n_1} \ln \left(1 - e^{-\lambda/x_{1i}} \left(\frac{\lambda}{x_{1i}} + 1 \right) \right) + 2n_1 \ln(\lambda) + n_1 \ln(\alpha_2) \\
& - 3 \sum_{i=1}^{n_1} \ln(x_{2i}) - \lambda \sum_{i=1}^{n_1} \frac{1}{x_{2i}} + (\alpha_2 - 1) \sum_{i=1}^{n_1} \ln \left(1 - e^{-\lambda/x_{2i}} \left(\frac{\lambda}{x_{2i}} + 1 \right) \right) \\
& + 2n_2 \ln(\lambda) + n_2 \ln(\alpha_1) - 3 \sum_{i=1}^{n_2} \ln(x_{1i}) - \lambda \sum_{i=1}^{n_2} \frac{1}{x_{1i}} + n_2 \ln(\alpha_2 + \alpha_3) \\
& + (\alpha_1 - 1) \sum_{i=1}^{n_2} \ln \left(1 - e^{-\lambda/x_{1i}} \left(\frac{\lambda}{x_{1i}} + 1 \right) \right) + 2n_2 \ln(\lambda) - 3 \sum_{i=1}^{n_2} \ln(x_{2i}) \\
& - \lambda \sum_{i=1}^{n_2} \frac{1}{x_{2i}} + (\alpha_2 + \alpha_3 - 1) \sum_{i=1}^{n_2} \ln \left(1 - e^{-\lambda/x_{2i}} \left(\frac{\lambda}{x_{2i}} + 1 \right) \right) \\
& + 2n_3 \ln(\lambda) + n_3 \ln(\alpha_3) - 3 \sum_{i=1}^{n_3} \ln(x_i) - \lambda \sum_{i=1}^{n_3} \frac{1}{x_i} \\
& + (\alpha_1 + \alpha_2 + \alpha_3 - 1) \sum_{i=1}^{n_3} \ln \left(1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1 \right) \right).
\end{aligned}$$

Differentiating the log-likelihood with respect to $\alpha_1, \alpha_2, \alpha_3$ and λ respectively, and equating the results to zero, we have

$$\begin{aligned}\frac{\partial L}{\partial \alpha_1} &= \frac{n_1}{\alpha_1 + \alpha_3} + \sum_{i=1}^{n_1} \ln \left(1 - e^{-\lambda/x_{1i}} \left(\frac{\lambda}{x_{1i}} + 1 \right) \right) + \frac{n_2}{\alpha_1} \\ &+ \sum_{i=1}^{n_2} \ln \left(1 - e^{-\lambda/x_{1i}} \left(\frac{\lambda}{x_{1i}} + 1 \right) \right) + \sum_{i=1}^{n_3} \ln \left(1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1 \right) \right)\end{aligned}\quad (29)$$

$$\begin{aligned}\frac{\partial L}{\partial \alpha_2} &= \frac{n_1}{\alpha_2} + \sum_{i=1}^{n_1} \ln \left(1 - e^{-\lambda/x_{2i}} \left(\frac{\lambda}{x_{2i}} + 1 \right) \right) + \frac{n_2}{\alpha_2 + \alpha_3} \\ &+ \sum_{i=1}^{n_2} \ln \left(1 - e^{-\lambda/x_{2i}} \left(\frac{\lambda}{x_{2i}} + 1 \right) \right) + \sum_{i=1}^{n_3} \ln \left(1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1 \right) \right)\end{aligned}\quad (30)$$

$$\begin{aligned}\frac{\partial L}{\partial \alpha_3} &= \frac{n_1}{\alpha_1 + \alpha_3} + \sum_{i=1}^{n_1} \ln \left(1 - e^{-\lambda/x_{1i}} \left(\frac{\lambda}{x_{1i}} + 1 \right) \right) + \frac{n_2}{\alpha_2 + \alpha_3} + \frac{n_3}{\alpha_3} \\ &+ \sum_{i=1}^{n_2} \ln \left(1 - e^{-\lambda/x_{2i}} \left(\frac{\lambda}{x_{2i}} + 1 \right) \right) + \sum_{i=1}^{n_3} \ln \left(1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1 \right) \right)\end{aligned}\quad (31)$$

$$\begin{aligned}\frac{\partial L}{\partial \lambda} &= \frac{2n_1}{\lambda} - \sum_{i=1}^{n_1} \frac{1}{x_{1i}} + (\alpha_1 + \alpha_3 - 1) \sum_{i=1}^{n_1} \frac{\frac{\lambda}{(x_{1i})^2} e^{-\lambda/x_{1i}}}{1 - e^{-\lambda/x_{1i}} \left(\frac{\lambda}{x_{1i}} + 1 \right)} \\ &+ \frac{2n_1}{\lambda} - \sum_{i=1}^{n_1} \frac{1}{x_{2i}} + (\alpha_2 - 1) \sum_{i=1}^{n_1} \frac{\frac{\lambda}{(x_{2i})^2} e^{-\lambda/x_{2i}}}{1 - e^{-\lambda/x_{2i}} \left(\frac{\lambda}{x_{2i}} + 1 \right)} \\ &+ \frac{2n_2}{\lambda} - \sum_{i=1}^{n_2} \frac{1}{x_{1i}} + (\alpha_1 - 1) \sum_{i=1}^{n_2} \frac{\frac{\lambda}{(x_{1i})^2} e^{-\lambda/x_{1i}}}{1 - e^{-\lambda/x_{1i}} \left(\frac{\lambda}{x_{1i}} + 1 \right)} \\ &+ \frac{2n_2}{\lambda} - \sum_{i=1}^{n_2} \frac{1}{x_{2i}} + (\alpha_2 + \alpha_3 - 1) \sum_{i=1}^{n_2} \frac{\frac{\lambda}{(x_{2i})^2} e^{-\lambda/x_{2i}}}{1 - e^{-\lambda/x_{2i}} \left(\frac{\lambda}{x_{2i}} + 1 \right)} \\ &+ \frac{2n_3}{\lambda} - \sum_{i=1}^{n_3} \frac{1}{x_i} + (\alpha_1 + \alpha_2 + \alpha_3 - 1) \sum_{i=1}^{n_3} \frac{\frac{\lambda}{(x_i)^2} e^{-\lambda/x_i}}{1 - e^{-\lambda/x_i} \left(\frac{\lambda}{x_i} + 1 \right)}\end{aligned}\quad (32)$$

The maximum likelihood estimates $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$ and $\hat{\lambda}$ of the unknown parameters α_1 , α_2 , α_3 , and λ respectively, are obtained by solving Equations (29) - (32).

11 Data Analysis of BIEGD

In this section, a real data set is used to compare the fits of the Marshall-Olkin bivariate exponential (MO) distribution and bivariate inverted exponentiated gamma (BIEG) distribution. The data set (see Table 4) was first analyzed by Meintanis [9] and represents the soccer data where at least one goal is scored by the home team and at least one goal is scored directly from a penalty kick, foul kick or any other direct kick (all of them will be called kick goals) by any team that has been considered. It is a bivariate data set, and the variables X_1 and X_2 are as follows: X_1 represents the time in minutes of the first kick goal scored by any team and X_2 represents the first goal of any type scored by the home team. Clearly, the variables X_1 and X_2 have the following structure: (i) $X_1 < X_2$, (ii) $X_1 = X_2$, (iii) $X_1 > X_2$.

Table 4 UEFA Champion's League Data

X_1	X_2	X_1	X_2	X_1	X_2	X_1	X_2
26	20	66	62	51	28	42	3
63	18	25	9	76	64	27	47
19	19	41	3	64	15	28	28
66	85	16	75	26	48	2	2
40	40	18	18	16	16		
49	49	22	14	44	13		
8	8	42	42	25	14		
69	71	36	52	55	11		
39	39	34	34	49	49		
82	48	53	39	24	24		
72	72	54	7	44	30		

The required numerical evaluations are carried out using the Package of Mathcad software. Table 5 provides the MLEs of the model parameters. The model selection is carried out using the *AIC* (Akaike information criterion) and the *BIC* (Bayesian information criterion).

Table 5: The MLEs and the values of L , AIC and BIC .

Model	MLEs	L	AIC	BIC
MO	$\hat{\lambda}_1 = 0.012, \hat{\lambda}_2 = 0.014$ $\hat{\lambda}_3 = 0.022$	-339.006	684.012	-344.423
BIEGD	$\hat{\alpha}_1 = 0.155, \hat{\alpha}_2 = 0.362$ $\hat{\alpha}_3 = 0.389, \hat{\lambda} = 25.663$	-314.279	636.558	-321.501

Since the values of $-L$, AIC and BIC (see Table 5) are smaller for the BIEG distribution than the new distribution seems to be a very competitive model to these data.

The profiles of the log-likelihood function of α_1 , α_2 , α_3 and λ of BIEGD for UEFA Champion's League data are plotted in Fig. 7, Fig. 8, Fig. 9 and Fig. 10.

respectively. From the plots of the profiles of the log-likelihood function of α_1 , α_2 , α_3 and λ , we observe that the likelihood equations have a unique solution.

Fig. 5. The profile of the log-likelihood function of α_1 .

Fig. 8. The profile of the log-likelihood function of α_2 .

Fig. 9. The profile of the log-likelihood function of α_3 .

Fig. 10. The profile of the log-likelihood function of λ .

12 Conclusions

In this paper, we proposed an Inverted Exponentiated Gamma (IEG) distribution. Some statistical properties of this distribution have been studied and discussed such as quantile, median, moments and moments of order statistics. The maximum likelihood estimators of the parameters are derived. A real data set is analyzed using the proposed distribution and Exponentiated Gamma (EG) distribution. Based on the comparisons between all these models, one can conclude that, the introduced model is highly competitive in the sense of fitting this real data set. Also, the bivariate Inverted Exponentiated Gamma (BIEG) distribution is introduced. Marginal and conditional distribution functions are studied. Furthermore, joint hazard rate function and maximum likelihood estimates (MLEs) of the parameters are presented. A real data set is analyzed using the new distribution and Marshall-Olkin bivariate exponential (MO) distribution. It shown that, the introduced model is highly competitive in the sense of fitting this real data set.

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