On the b -chromatic number of T_n and $M(T_n)$

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Abstract

The set of vertices $V(G)$ and set of edges $E(G)$ forms a mathematical structure, which is called graph G . A b -coloring of a graph $G = (V, E)$ is a coloring $f_G: V(G) \rightarrow C$ such that, for each c in C , there exists some vertex v in $V(G)$ which is colored by c and $f_G(N_G(v)) = C \setminus \{c\}$. In this paper, we give the exact value for the b -chromatic number of triangular snake graph and middle graph of triangular snake graph, which is denoted by T_n and $M(T_n)$ respectively.

Keywords: b -coloring, Triangle snake graph, Middle graph.

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1. Introduction

A graph G is a pair of sets $V(G)$ and $E(G)$, where $V(G)$ and $E(G)$ are referred as the vertex set and edges set of a graph G respectively. We consider finite undirected graphs without loops or multiple edges. Definitions not given here may be found in [3]. Proper vertex coloring of a graph G is an assignment of colors C to the vertices of the graph, in such a way that adjacent vertices are colored differently. If $|C| = k$, we say C is k -coloring. Equivalently a proper coloring is a partition of the vertex set V into independent sets $V_1, V_2, \dots, V_k$. The sets V_i are called color classes. The chromatic number of G is the minimum integer $\chi(G)$ such that G has a proper coloring with $\chi(G)$ colors. A graph G is k -chromatic if $\chi(G) = k$. The concept of chromatic number has been extended to almost all types of graph colorings.

A b -coloring of a graph G with k colors is a proper coloring such that each color class contains a vertex with its neighbors in all other $k - 1$ color classes. A vertex of color i that has all other colors in its neighborhood is called color i dominating vertex. The b -chromatic number of a graph G , denoted by $\varphi(G)$, is the maximum k for which G has a b -coloring by k colors.

This concept was introduced in 1999 by Irving and Manlove [7], by considering proper colorings that are minimal with respect to a partial order defined on the set of all the partitions of vertices of G . It is easy to notice that $\varphi(G) \leq m(G)$, where $m(G)$ is the largest integer m such that G has m vertices of degree at least $m - 1$. For a graph G and an integer k , the problem of deciding whether $\varphi(G) \geq k$ is NP -complete [7]. It is NP -hard to determine the b -chromatic number in general but they proved that it is polynomial for trees [7]. The complexity issues of b -colorings were further studied in [4, 9].

The discussion on the b -chromatic number of some power graphs is carried out by Effantin and Kheddouci [5, 6]. The b -coloring of regular graphs is studied by Blidia et al. [2],

while the b -coloring of regular graphs without 4 cycle is studied by Shaebani [10]. Ramin Javadi and Behnaz Omoomi, studied the b -chromatic number for knesar graph [8] and proved that $K(n, 2)$ is b -continuous when $n \geq 17$. The b -chromatic number of some special graph classes has been discussed in [1, 11]. In this paper, we present exact values for the b -chromatic number of triangular snake graph and middle graph of triangular snake graph.

2. Preliminaries

The Triangular cactus is a connected graph all of whose blocks are triangles. It is obtained from a path $P = v_1, v_2, v_3, \dots, v_{n+1}$ by joining v_i and v_{i+1} to a new vertex u_i ; $1 \leq i \leq n$. A triangular snake has $2n + 1$ vertices and $3n$ edges, where n is the number of blocks in the triangular snake. It is denoted by T_n . (See Figure 1)

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph of G , denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G . (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G .

Fig 1: Triangular snake graph T_n

3. b -chromatic number on T_n and $M(T_n)$

Theorem 3.1: If $n \geq 6$, then the b -chromatic number of triangular snake graph is $\varphi\{T_n\} = 5$.

Proof: Let T_n be the triangular snake graph with $2n + 1$ vertices and $3n$ edges. Let $\{v_1, v_2, \dots, v_{n+1}, u_1, u_2, u_3, \dots, u_n\}$ be the vertices of the triangular snake T_n . So $V(T_n) = \{v_1, v_2, v_3, \dots, v_{n+1}\} \cup \{u_1, u_2, u_3, \dots, u_n\}$. (See Figure 1)

For $n \geq 6$, T_n has at least five vertices of degree 4. Thus $\varphi(T_n) \leq m(T_n) = 5$. By assigning proper coloring to the vertices as $c(v_1) = 5$, $c(v_{i+1}) = i$; for $1 \leq i \leq 5$ and $c(v_7) = 1$. Now assign $c(u_i) = i + 2$; for $1 \leq i \leq 3$ and $c(u_{i+3}) = i$; for $1 \leq i \leq 3$. For remaining vertices of T_n , assign $c(v_i) = 3$; $8 \leq i \leq n + 1$ and $c(u_i) = 2$; $7 \leq i \leq n$, this implies that $\varphi(T_n) \geq 5$. Hence the theorem.

□

Theorem 3.2: If $M(T_n)$ is middle graph of triangular snake graph then

$$\varphi\{M(T_n)\} = \begin{cases} 7; & \text{for } 4 \leq n \leq 10 \\ 9; & \text{for } n \geq 11 \end{cases}$$

Proof: Let T_n be the triangular snake graph with $2n + 1$ vertices and $3n$ edges. Let $\{v_1, v_2, \dots, v_{n+1}, u_1, u_2, u_3, \dots, u_n\}$ be the vertices of the triangular snake T_n . Now by definition of middle graph, each edge of graph is subdivided by a new vertex. Therefore assume that each edge (v_i, v_{i+1}) and the line joining v_i and v_{i+1} to a vertex u_i , $i = 1, 2, 3, \dots, n$ are subdivided by the vertices e_i, e'_i , and e''_i , $i = 1, 2, 3, \dots, n$ respectively. (See Figure 2)

Fig 2: Middle graph of Triangular snake graph $M(T_n)$

For $4 \leq n \leq 10$, $M(T_n)$ has at least 7 vertices of degree 6, Then $\varphi\{M(T_n)\} \leq m\{M(T_n)\} = 7$, for $4 \leq n \leq 10$. Now since the 7-coloring $c: V(M(T_n)) \rightarrow \{1, 2, 3, \dots, 7\}$ is defined by $c(e_i) = i$; $1 \leq i \leq 4$, $c(e'_1) = 6, c(e'_2) = 7, c(e'_3) = 1, c(e'_4) = 6, c(e''_i) = i + 4$; $1 \leq i \leq 3, c(e''_4) = 1, c(v_i) = i + 2$; $1 \leq i \leq 3, c(v_4) = 5, c(v_5) = 2, c(u_1) = c(u_2) = 3, c(u_3) = 2, c(u_4) = 2$. So c is b -coloring. Thus for $4 \leq n \leq 10$, $\varphi\{M(T_n)\} \geq 7$. Therefore $\varphi\{M(T_n)\} = 7, 4 \leq n \leq 10$. (See Figure 3)

Fig 3: Example $M(T_4)$

For $n \geq 11$, consider color class $C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Assign b -coloring to $M(T_n)$ as follows. $c(e_1) = 9, c(e_{i+1}) = c(u_i) = i$, $1 \leq i \leq 9, c(e_{11}) = 1, c(u_{10}) = 1, c(u_{11}) = 2, c(v_i) = 3 + i$; $1 \leq i \leq 6, c(v_{6+i}) = i$; $1 \leq i \leq 6$. Now assign $c(e'_i) = i + 1$; $1 \leq i \leq 8, c(e_{8+i}) = i$; $1 \leq i \leq 3, c(e''_i) = 6 + i$; $1 \leq i \leq 3, c(e_{3+i}) = i$; $1 \leq i \leq 8$. For remaining vertices of $M(T_n)$; $n \geq 11$, we can apply any proper coloring. Therefore for $n \geq 11$, $\varphi\{M(T_n)\} \geq 9$. Now for $n \geq 11$, $M(T_n)$ has at least nine vertices of degree 8. It follows that $\varphi\{M(T_n)\} \leq m\{M(T_n)\} = 9$, for $n \geq 11$. Thus $\varphi\{M(T_n)\} = 9; n \geq 11$. (See Figure 4) $\square$

$\square$

Fig 4: Example $M(T_{11})$

4. Conclusion

b-coloring play important role in clustering, automatic reading system and distributed system. We have investigated b-chromatic number of triangular snake graph and middle graph of triangular snake graph. The investigation of similar results for different graphs as well in the context of various graph coloring problems is an open area of research.

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