B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES
ET DES LETTRES DE ŁÓDŹ

SÉRIE:
RECHERCHES SUR LES DÉFORMATIONS

Volume LXVII, no. 1

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DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

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Volume LXVII, no. 1

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”Bulletin de la Société des sciences et des lettres de Łódź, Série: Recherches sur les déformations” – zadanie finansowane w ramach umowy 597/P-DUN/2017 ze środków Ministra Nauki i Szkolnictwa Wyższego przeznaczonych na działalność upowszechniającą naukę

”Bulletin de la Société des sciences et des lettres de Łódź, Série: Recherches sur les déformations” – task financed under the agreement 597/P-DUN/2017 by the Minister of Science and Higher Education from the funds for science dissemination

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PL ISSN 0459-6854; e-ISSN 2450-9329
DOI: 10.26485/0459-6854; DOI: 10.26485/0459-6854/2017/67.1

Wydanie 1
Nakład 120 egz.
Przygotowanie czasopisma do digitalizacji: Justyna Walewska

The journal is indexed in: *Index Copernicus*, *PBN/POL-Index*, *EBSCOhost*, *Zentralblatt MATH*, *MathSciNet (Mathematical Reviews)* and *Referativny Zhurnal*

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Affiliation/Address

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Professor
Waldyr Alves Rodrigues, Jr.
* 14.03.1946 † 03.04.2017

Dear Friends

Fatima,
Adriana, Renata, Fábio,
Maria Paula, Maria Larissa.

It is with great pain and sorrow that we received information about the death of our friend and colleague Professor Waldyr Alves Rodrigues Jr.

He was not only an involved and highly professional researcher but, first of all, we remember Waldyr as a true friend – kind, warm, of rich and empathic personality.

We pray for Waldyr and for you. We hope that you will receive Christian consolation in your pain and loss.

Good Friday, 14th April 2017

Editor's note

Professor W. A. Rodrigues Jr. had joined the Editorial Board of the *Bulletin de la Société des Sciences et des Lettres de Łódź, Série: Recherches sur les Déformations* in 1995. One year after he had published in that journal the paper entitled *Clifford-algebraical approach to particle lifetimes*, vol. **21** (1996), 57–63.

Professor Rodrigues was also, since 2001, a member of the Editorial Board of *Advances in Applied Clifford Algebras* and the Editor-in-Chief of this journal (AACA) from 2011 to 2015. We refer to AACA **27** (2017) for a detailed description of world-wide known Waldyr's results in applied mathematics and mathematical physics, as well as for a description of his academic career concentrated round the Department of Applied Mathematics at the State University of Campinas (Brasil).

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2017

Vol. LXVII

Recherches sur les déformations

no. 1

pp. 11–31

*Marta Dudek***INVARIANTS UNDER CONFORMAL RESCALING
OF THE SPACE-TIME – A STUDY INCLUDING CONSEQUENCES
FOR THE METRIC****Summary**

The conformal transformations play crucial role in the analysis of global structure of the physical space-time. This paper shows some geometrical and physical objects which describe the space-time. There are also given transformation laws for them under conformal rescaling of the metric. The main goal of this article is to check which geometrical and physical objects are invariants under the conformal rescaling of the metric and to present the consequences of the conformal transformation of the metric like creation of the energy and momentum for the gravitational field or creation of the matter.

Keywords and phrases: conformal transformation of the metric, general theory of relativity, space-time, Riemannian manifold (pseudo-Riemannian manifold, Lorentzian manifold), Einstein's equations, Landau-Lifshitz pseudotensor of the energy-momentum

**1. The information about the actual mathematical model
of the physical space-time**

The theory of relativity stated by Albert Einstein concerns the structure of the space-time. It consists of the special theory of relativity (SR) and the general theory of relativity (GR). The former theory is valid for inertial reference frames and does not take into account the gravitation. In the later one there are described events in the non-inertial reference frames, so it means that the gravity is taken into consideration. Both of them give some mathematical model of the physical space-time. This model is called shortly *the space-time*. It is defined as a set of all possible physical events. According to GR the space-time is a 4-dimentional connected manifold M_4 of the class C^∞ , with the Hausdorff topology, orientable, having Lorentzian structure, non-

extendable and time-oriented. The metric tensor of that space-time satisfies the Einstein equations.

Definition 1. *The conformal rescaling or the conformal transformation of the metric g on the Riemannian manifold (pseudoriemannian, Lorentzian) is the following transformation (in the established coordinates)*

$$\widehat{g}_{ab}(x) = \Omega^2(x)g_{ab}(x),$$

where $\Omega(x)$ is a smooth and positive-definite function called the conformal factor.

Fact 1. *Denote by g^{ab} components of the tensor $(2, 0)$ which is inverse to the metric tensor. Then we have*

$$\widehat{g}^{ab} = \Omega^{-2}(x)g^{ab}.$$

Proof. From the definition we have $g^{ab}g_{bd} = \delta_d^a$, where

$$\delta_d^a = \begin{cases} 1 & \text{if } a = d, \\ 0 & \text{if } a \neq d. \end{cases}$$

The symbol δ_d^a is Kronecker's delta or the unit tensor. Because

$$\widehat{g}_{bd} = \Omega^2 g_{bd}$$

and

$$\widehat{g}_{bd}\widehat{g}^{ab} = \Omega^2 g_{bd}\Omega^x g^{ab} = \delta_d^a,$$

so to preserve the equality x must be equal -2 . □

Fact 2. *The length of a vector changes under the conformal rescaling of the metric but the ratio of vectors is preserved.*

Proof. The length of the vector in the metric g_{ab} is given by the formula

$$\|\vec{v}\| = \sqrt{(\vec{v}, \vec{v})} = \sqrt{g_{ik}v^i v^k}.$$

In the new metric $\widehat{g}_{ab}$ it has the following form

$$\|\vec{\widehat{v}}\| = \sqrt{\widehat{g}_{ik}v^i v^k} = \sqrt{\Omega^2 g_{ik}v^i v^k} = \Omega \|\vec{v}\|.$$

Therefore the ratio of the vectors is not changed.

$$\frac{\|\vec{\widehat{u}}\|}{\|\vec{\widehat{v}}\|} = \frac{\|\vec{u}\|}{\|\vec{v}\|}. \quad \square$$

Fact 3. *The angle between two vectors is an invariant of conformal rescaling of the metric.*

Proof. We calculate the angle between two vectors following the formula

$$\cos(\vec{u}, \vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}.$$

Then in the new metric it has the form

$$\cos(\tilde{\vec{u}}, \tilde{\vec{v}}) = \frac{\tilde{\vec{u}} \cdot \tilde{\vec{v}}}{\|\tilde{\vec{u}}\| \|\tilde{\vec{v}}\|} = \frac{\Omega^2 \vec{u} \cdot \vec{v}}{\Omega^2 \|\vec{u}\| \|\vec{v}\|} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}. \quad \square$$

The facts 2 and 3 justify the name “the conformal transformation” or „the conformal rescaling of the metric” from the definition 1.

Fact 4. *The line element changes under conformal rescaling of the metric as follows*

$$d\tilde{s}^2 = \Omega^2(x) ds^2.$$

Proof. The line element is given by the formula

$$ds^2 = g_{ik} dx^i dx^k.$$

After conformal rescaling of the metric the line element has the form

$$d\tilde{s}^2 = \hat{g}_{ik}(x) dx^i dx^k = \Omega^2(x) g_{ik}(x) dx^i dx^k = \Omega^2(x) ds^2. \quad \square$$

2. Selected geometrical aspects of conformal transformations

Definition 2. *Affine connection (in other words Christoffel symbols of the second kind or Christoffel’s connection) is defined as [2, 5] :*

$$\Gamma_{bc}^a = \frac{1}{2} g^{ae} \left(\frac{\partial g_{be}}{\partial x^c} + \frac{\partial g_{ce}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^e} \right) = \frac{1}{2} g^{ae} (g_{be,c} + g_{ce,b} - g_{bc,e}).$$

Affine connection is an geometrical object, which permits us parallel transport of vectors and tensors on a manifold and develop tensor analysis on it.

Fact 5. *Christoffel symbols change under the conformal transformation of the metric as follows*

$$\hat{\Gamma}_{bc}^a = \Gamma_{bc}^a + P_{bc}^a,$$

where $P_{bc}^a = \Omega^{-1} (\delta_b^a \Omega_{,c} + \delta_c^a \Omega_{,b} - g^{ae} g_{bc} \Omega_{,e})$.

Proof. In the proof below we will write the Christoffel symbols in the new gauge and use the transformation law for the metric tensor.

$$\begin{aligned}
\widehat{\Gamma}_{bc}^a &= \frac{1}{2} \widehat{g}^{ae} (\widehat{g}_{be,c} + \widehat{g}_{ce,b} - \widehat{g}_{bc,e}) \\
&= \frac{1}{2} \Omega^{-2} g^{ae} [(\Omega^2 g_{be})_{,c} + (\Omega^2 g_{ce})_{,b} - (\Omega^2 g_{bc})_{,e}] \\
&= \frac{1}{2} \Omega^{-2} g^{ae} (2\Omega\Omega_{,c}g_{be} + \Omega^2 g_{be,c} + 2\Omega\Omega_{,b}g_{ce} + \Omega^2 g_{ce,b} - 2\Omega\Omega_{,e}g_{bc} - \Omega^2 g_{bc,e}) \\
&= \frac{1}{2} g^{ae} (g_{be,c} + g_{ce,b} - g_{bc,e}) + \Omega^{-1} g^{ae} (\Omega_{,c}g_{be} + \Omega_{,b}g_{ce} - \Omega_{,e}g_{bc}) \\
&= \frac{1}{2} g^{ae} (g_{be,c} + g_{ce,b} - g_{bc,e}) + \Omega^{-1} (\delta_b^a \Omega_{,c} + \delta_c^a \Omega_{,b} - g^{ae} g_{bc} \Omega_{,e}) \\
&= \Gamma_{bc}^a + P_{bc}^a.
\end{aligned}$$

If $\Omega(x) = \text{const}$, then $\widehat{\Gamma}_{bc}^a = \Gamma_{bc}^a$. □

Definition 3. Covariant derivative is defined as [4, 1] :

$$\nabla_i v^k := \partial_i v^k + \Gamma_{ic}^k v^c,$$

for the contravariant vector field i.e. for the tensor of type (1, 0) and

$$\nabla_i v_k := \partial_i v_k - \Gamma_{ik}^c v_c$$

for the covariant vector field (covector or the 1-form or the tensor of type (0, 1)).

Fact 6. Covariant derivative changes under the conformal transformation of the metric in the following way:

$$\widehat{\nabla}_i v^k := \nabla_i v^k + P_{ic}^k v^c,$$

and

$$\widehat{\nabla}_i v_k := \nabla_i v_k - P_{ik}^c v_c,$$

where

$$P_{ik}^c = \Omega^{-1} (\delta_i^c \Omega_{,k} + \delta_k^c \Omega_{,i} - g_{ik} g^{ce} \Omega_{,e}).$$

Proof. Writing down the covariant derivative in the new gauge in accordance with the definition 3 and developing it we get

$$\begin{aligned}
\widehat{\nabla}_i v^k &= \partial_i v^k + \widehat{\Gamma}_{ic}^k v^c = \partial_i v^k + \Gamma_{ic}^k v^c + P_{ic}^k v^c = \nabla_i v^k + P_{ic}^k v^c \\
\widehat{\nabla}_i v^k &= \nabla_i v^k \iff \Omega(x) = \text{const},
\end{aligned}$$

$$\begin{aligned}
\widehat{\nabla}_i v_k &= \partial_i v_k - \widehat{\Gamma}_{ik}^c v_c = \partial_i v_k - \Gamma_{ik}^c v_c - P_{ik}^c v_c = \nabla_i v_k - P_{ik}^c v_c \\
\widehat{\nabla}_i v_k &= \nabla_i v_k \iff \Omega(x) = \text{const}.
\end{aligned}$$
□

Fact 7. *Conformal transformation preserves the metricity of the connection.*

Proof.

$$\begin{aligned}\widehat{\nabla}_c \widehat{g}_{ab} &= \partial_c \widehat{g}_{ab} - \widehat{\Gamma}_{ac}^d \widehat{g}_{db} - \widehat{\Gamma}_{bc}^d \widehat{g}_{ad} \\ &= (\Omega^2 g_{ab})_{,c} - (\Gamma_{ac}^d + P_{ac}^d) \Omega^2 g_{db} - (\Gamma_{bc}^d + P_{bc}^d) \Omega^2 g_{ad} \\ &= \Omega_{,c}^2 g_{ab} + \Omega^2 g_{ab,c} - \Gamma_{ac}^d \Omega^2 g_{db} - \Gamma_{bc}^d \Omega^2 g_{ad} - P_{ac}^d \Omega^2 g_{db} - P_{bc}^d \Omega^2 g_{ad}.\end{aligned}$$

Because $g_{ab,c} - \Gamma_{ac}^d g_{db} - \Gamma_{bc}^d g_{ad} = 0$ so we obtain

$$\begin{aligned}\widehat{\nabla}_c \widehat{g}_{ab} &= \Omega_{,c}^2 g_{ab} - P_{ac}^d \Omega^2 g_{db} - P_{bc}^d \Omega^2 g_{ad} \\ &= 2\Omega \Omega_{,c} g_{ab} - \Omega^{-1} (\delta_a^d \Omega_{,c} + \delta_c^d \Omega_{,a} - g_{ac} g^{de} \Omega_{,e}) \Omega^2 g_{db} \\ &\quad - \Omega^{-1} (\delta_b^d \Omega_{,c} + \delta_c^d \Omega_{,b} - g_{bc} g^{de} \Omega_{,e}) \Omega^2 g_{ad} \\ &= 2\Omega \Omega_{,c} g_{ab} - \Omega (g_{ab} \Omega_{,c} + g_{cb} \Omega_{,a} - g_{ac} \Omega_{,b}) \\ &\quad - \Omega (g_{ab} \Omega_{,c} + g_{ac} \Omega_{,b} - g_{bc} \Omega_{,a}) = 0.\end{aligned}$$

Hence the metricity of the connection is preserved. $\square$

Definition 4. *A geodesic is a curve on a Riemannian manifold (pseudoremannian or Lorentzian) which gives the extremum of the distance between any two neighbouring points [1]. Such a curve is a generalization of a straight line and satisfies the equations in so-called affine parameter u*

$$\frac{d^2 x^i}{du^2} + \Gamma_{kl}^i \frac{dx^k}{du} \frac{dx^l}{du} = 0.$$

Fact 8. *Under the conformal rescaling of the metric, the equations of the geodesic develop as follows*

$$\frac{d^2 x^i}{du^2} + \widehat{\Gamma}_{kl}^i \frac{dx^k}{du} \frac{dx^l}{du} = P_{kl}^i \frac{dx^k}{du} \frac{dx^l}{du}.$$

Proof. In the proof below we start with the equation of the geodesic

$$\frac{d^2 x^i}{du^2} + \Gamma_{kl}^i \frac{dx^k}{du} \frac{dx^l}{du} = 0.$$

Then we use the Christoffel symbol after the conformal rescaling of the metric

$$\widehat{\Gamma}_{kl}^i = \Gamma_{kl}^i + P_{kl}^i.$$

Replacing Γ_{kl}^i by the expression $\widehat{\Gamma}_{kl}^i - P_{kl}^i$ in the initial equation we obtain

$$\frac{d^2 x^i}{du^2} + \left(\widehat{\Gamma}_{kl}^i - P_{kl}^i \right) \frac{dx^k}{du} \frac{dx^l}{du} = 0. \quad \square$$

As we can see it is not the equation of the geodesic $\widehat{\Gamma}_{kl}^i$, except the case of $\Omega(x) = \text{const.}$

Generally one gets

$$\begin{aligned} \frac{d^2 x^i}{du^2} + \widehat{\Gamma}_{kl}^i \frac{dx^k}{du} \frac{dx^l}{du} &= \Omega^{-1} (\delta_k^i \Omega_{,l} + \delta_l^i \Omega_{,k} - g_{kl} g^{ie} \Omega_{,e}) \frac{dx^k}{du} \frac{dx^l}{du} \\ &= \Omega^{-1} \left[\Omega_{,k} \frac{dx^i}{du} \frac{dx^k}{du} + \Omega_{,k} \frac{dx^k}{du} \frac{dx^i}{du} - g_{kl} \frac{dx^k}{du} \frac{dx^l}{du} g^{ie} \Omega_{,e} \right]. \end{aligned}$$

If the geodesic was isotropic, then

$$g_{kl} \frac{dx^k}{du} \frac{dx^l}{du} = 0,$$

and hence

$$\frac{d^2 x^i}{du^2} + \widehat{\Gamma}_{kl}^i \frac{dx^k}{du} \frac{dx^l}{du} = \left(2\Omega^{-1} \Omega_{,k} \frac{dx^k}{du} \right) \frac{dx^i}{du} = h(u) \frac{dx^i}{du}.$$

This is an equation of the geodesic line with non-affine parameter u . Then we have

Fact 9. *Under the conformal transformation of the metric only the null geodesics are preserved, but the affine parameter loses its affine character.*

However we can introduce a new parameter $\lambda = \lambda(u)$ on these lines, which will be affine. Namely we have

Fact 10. *The new parameter λ defined by the equation*

$$\frac{d\lambda}{du} = c\Omega^2,$$

where $c = \text{const}$, is an affine parameter.

Proof.

$$\frac{d^2 x^i}{du^2} + \widehat{\Gamma}_{kl}^i \frac{dx^k}{du} \frac{dx^l}{du} = h(u) \frac{dx^i}{du},$$

where

$$h(u) = 2\Omega^{-1} \Omega_{,k} \frac{dx^k}{du}.$$

Lets write $x^i = x^i[\lambda(u)]$.

Then

$$\begin{aligned} \frac{dx^i}{du} &= \frac{dx^i}{d\lambda} \frac{d\lambda}{du}, \\ \frac{d^2 x^i}{du^2} &= \frac{d}{du} \left(\frac{dx^i}{d\lambda} \frac{d\lambda}{du} \right) = \frac{d^2 x^i}{d\lambda^2} \left(\frac{d\lambda}{du} \right)^2 + \frac{dx^i}{d\lambda} \frac{d^2 \lambda}{du^2}. \end{aligned}$$

Replacing $\frac{dx^i}{du^2}$ by the above expression in the initial equation we obtain

$$\frac{d^2 x^i}{d\lambda^2} \left(\frac{d\lambda}{du} \right)^2 + \frac{dx^i}{d\lambda} \frac{d^2 \lambda}{du^2} + \widehat{\Gamma}_{kl}^i \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda} \left(\frac{d\lambda}{du} \right)^2 = h(u) \frac{dx^i}{d\lambda} \frac{d\lambda}{du},$$

or

$$(1) \quad \left(\frac{d^2 x^i}{d\lambda^2} + \hat{\Gamma}_{kl}^i \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda} \right) \left(\frac{d\lambda}{du} \right)^2 = h(u) \frac{dx^i}{d\lambda} \frac{d\lambda}{du} - \frac{d^2 \lambda}{du^2} \frac{dx^i}{d\lambda}.$$

The right side of the equation will be zero iff

$$h(u) \frac{d\lambda}{du} - \frac{d^2 \lambda}{du^2} = 0$$

or

$$(2) \quad 2\Omega^{-1} \Omega_{,k} \frac{dx^k}{du} \frac{d\lambda}{du} - \frac{d^2 \lambda}{du^2} = 0.$$

On the other hand, if we differentiate

$$\frac{d\lambda}{du} = c\Omega^2$$

then we have

$$\frac{d^2 \lambda}{du^2} = 2c\Omega\Omega_{,k} \frac{dx^k}{du}.$$

Putting $\frac{d^2 \lambda}{du^2}$ into the equation (2) we obtain (reminding that $\frac{d\lambda}{du} = c\Omega^2$)

$$2\Omega^{-1} \Omega_{,k} \frac{dx^k}{du} \frac{d\lambda}{du} - 2c\Omega\Omega_{,k} \frac{dx^k}{du} = 0,$$

and the equations (1) take the form

$$\frac{d^2 x^i}{d\lambda^2} + \hat{\Gamma}_{kl}^i \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda} = 0.$$

Therefore the new parameter λ which satisfies the equation

$$\frac{d\lambda}{du} = c\Omega^2,$$

where $c = \text{const}$, is the affine parameter. □

Definition 5. *The curvature tensor (the Riemann-Christoffel tensor or the Riemann tensor) is defined as follows [1]*

$$R_{bcd}^a = \Gamma_{bd,c}^a - \Gamma_{bc,d}^a + \Gamma_{nc}^a \cdot \Gamma_{bd}^n - \Gamma_{nd}^a \cdot \Gamma_{bc}^n.$$

The curvature tensor is the tensor of type (1,3) and its components are defined by the metric tensor and its derivatives. It is a basic tool used in the differential geometry because it is a measure of the local curvature. Lets notice that for a flat manifold the Christoffel's symbols are not already equal zero in the curvilinear coordinates but the curvature tensor R_{bcd}^a is equal zero.

Fact 11. *The Riemann curvature changes under the conformal transformation of the metric as follows*

$$\begin{aligned}
\widehat{R}_{bcd}^a &= R_{bcd}^a + \frac{\Omega^2}{2}(\delta_{[c}^a \Omega_{d]b} - g_{b[c} \Omega_{d]}^a) \\
&= R_{bcd}^a + \frac{\Omega^2}{4}(\delta_c^a \Omega_{db} - \delta_d^a \Omega_{cb} - g_{bc} \Omega_{,d}^a - g_{bd} \Omega_{,c}^a).
\end{aligned}$$

Proof. The idea of our proof is to show that the both sides of the equation are equal. From the definition we have

$$\begin{aligned}
\widehat{R}_{bcd}^a &= \widehat{\Gamma}_{bd,c}^a - \widehat{\Gamma}_{bc,d}^a + \widehat{\Gamma}_{nc}^a \cdot \widehat{\Gamma}_{bd}^n - \widehat{\Gamma}_{nd}^a \cdot \widehat{\Gamma}_{bc}^n \\
&= \Gamma_{bd,c}^a + P_{bd,c}^a - \Gamma_{bc,d}^a - P_{bc,d}^a + (\Gamma_{nc}^a + P_{nc}^a) \cdot (\Gamma_{bd}^n + P_{bd}^n) \\
&\quad - (\Gamma_{nd}^a + P_{nd}^a) \cdot (\Gamma_{bc}^n + P_{bc}^n) \\
&= \Gamma_{bd,c}^a + P_{bd,c}^a - \Gamma_{bc,d}^a - P_{bc,d}^a + \Gamma_{nc}^a \cdot \Gamma_{bd}^n + \Gamma_{nc}^a \cdot P_{bd}^n + P_{nc}^a \cdot \Gamma_{bd}^n \\
&\quad + P_{nc}^a \cdot P_{bd}^n - \Gamma_{nd}^a \cdot \Gamma_{bc}^n - \Gamma_{nd}^a \cdot P_{bc}^n - P_{nd}^a \cdot \Gamma_{bc}^n - P_{nd}^a \cdot P_{bc}^n \\
&= (\Gamma_{bd,c}^a - \Gamma_{bc,d}^a + \Gamma_{nc}^a \cdot \Gamma_{bd}^n - \Gamma_{nd}^a \cdot \Gamma_{bc}^n) + (P_{bd,c}^a - P_{bc,d}^a) \\
&\quad + (\Gamma_{nc}^a \cdot P_{bd}^n + P_{nc}^a \cdot \Gamma_{bd}^n + P_{nc}^a \cdot P_{bd}^n - \Gamma_{nd}^a \cdot P_{bc}^n - P_{nd}^a \cdot \Gamma_{bc}^n - P_{nd}^a \cdot P_{bc}^n) \\
&= R_{bcd}^a + (P_{bd,c}^a - P_{bc,d}^a + \Gamma_{nc}^a \cdot P_{bd}^n + P_{nc}^a \cdot \Gamma_{bd}^n + P_{nc}^a \cdot P_{bd}^n - \Gamma_{nd}^a \cdot P_{bc}^n \\
&\quad - P_{nd}^a \cdot \Gamma_{bc}^n - P_{nd}^a \cdot P_{bc}^n),
\end{aligned}$$

where

$$R_{bcd}^a = \Gamma_{bd,c}^a - \Gamma_{bc,d}^a + \Gamma_{nc}^a \cdot \Gamma_{bd}^n - \Gamma_{nd}^a \cdot \Gamma_{bc}^n.$$

To make easier further calculations we will divide the equation into three parts. Lets denote

$$I_1 = \Gamma_{nc}^a \cdot P_{bd}^n + P_{nc}^a \cdot \Gamma_{bd}^n - \Gamma_{nd}^a \cdot P_{bc}^n - P_{nd}^a \cdot \Gamma_{bc}^n,$$

$$I_2 = P_{nc}^a \cdot P_{bd}^n - P_{nd}^a \cdot P_{bc}^n,$$

$$I_3 = P_{bd,c}^a - P_{bc,d}^a.$$

Then we compute I_1 , I_2 and I_3 .

$$\begin{aligned}
I_1 &= \Omega^{-1}[\Gamma_{nc}^a(\delta_b^n \Omega_{,d} + \delta_d^n \Omega_{,b} - g^{ne} g_{bd} \Omega_{,e}) + \Gamma_{bd}^n(\delta_n^a \Omega_{,c} + \delta_c^a \Omega_{,n} - g^{ae} g_{nc} \Omega_{,e}) \\
&\quad - \Gamma_{nd}^a(\delta_b^n \Omega_{,c} + \delta_c^n \Omega_{,b} - g^{ne} g_{bc} \Omega_{,e}) - \Gamma_{bc}^n(\delta_n^a \Omega_{,d} + \delta_d^a \Omega_{,n} - g^{ae} g_{nd} \Omega_{,e})] \\
&= \Omega^{-1}[\Gamma_{bc}^a \Omega_{,d} + \Gamma_{cd}^a \Omega_{,b} - \Gamma_{nc}^a g^{ne} g_{bd} \Omega_{,e} + \Gamma_{bd}^a \Omega_{,c} + \Gamma_{bd}^n \delta_c^a \Omega_{,n} - \Gamma_{bd}^n g^{ae} g_{nc} \Omega_{,e} \\
&\quad - \Gamma_{bd}^a \Omega_{,c} - \Gamma_{cd}^a \Omega_{,b} + \Gamma_{nd}^a g^{ne} g_{bc} \Omega_{,e} - \Gamma_{bc}^a \Omega_{,d} - \Gamma_{bc}^n \delta_d^a \Omega_{,n} + \Gamma_{bc}^n g^{ae} g_{nd} \Omega_{,e}] \\
&= \Omega^{-1}(\Gamma_{bd}^a \delta_c^n \Omega_{,n} - \Gamma_{nc}^a g^{ne} g_{bd} \Omega_{,e} - \Gamma_{bd}^n g^{ae} g_{nc} \Omega_{,e}) \\
&\quad + \Omega^{-1}(\Gamma_{nd}^a g^{ne} g_{bc} \Omega_{,e} - \Gamma_{bc}^n \delta_d^a \Omega_{,n} + \Gamma_{bc}^n g^{ae} g_{nd} \Omega_{,e}),
\end{aligned}$$

$$\begin{aligned}
I_2 &= \Omega^{-2}(\delta_n^a \Omega_{,c} + \delta_c^a \Omega_{,n} - g^{ae} g_{nc} \Omega_{,e})(\delta_b^n \Omega_{,d} + \delta_d^n \Omega_{,b} - g^{nf} g_{bd} \Omega_{,f}) \\
&\quad - \Omega^{-2}(\delta_n^a \Omega_{,d} + \delta_d^a \Omega_{,n} - g^{ae} g_{nd} \Omega_{,e})(\delta_b^n \Omega_{,c} + \delta_c^n \Omega_{,b} - g^{nf} g_{bc} \Omega_{,f}) \\
&= \Omega^{-2}(\delta_n^a \Omega_{,c} \delta_b^n \Omega_{,d} + \delta_n^a \Omega_{,c} \delta_d^n \Omega_{,b} - \delta_n^a \Omega_{,c} g^{ne} g_{bd} \Omega_{,e} + \delta_c^a \Omega_{,n} \delta_b^n \Omega_{,d} \\
&\quad + \delta_c^a \Omega_{,n} \delta_d^n \Omega_{,b} - \delta_c^a \Omega_{,n} g^{ne} g_{bd} \Omega_{,e} - g^{ae} g_{nc} \Omega_{,e} \delta_b^n \Omega_{,d} - g^{ae} g_{nc} \Omega_{,e} \delta_d^n \Omega_{,b} \\
&\quad + g^{ae} g_{nc} \Omega_{,e} g^{nf} g_{bd} \Omega_{,f} - \delta_n^a \Omega_{,d} \delta_b^n \Omega_{,c} - \delta_n^a \Omega_{,d} \delta_c^n \Omega_{,b} + \delta_n^a \Omega_{,d} g^{ne} g_{bc} \Omega_{,e} \\
&\quad - \delta_d^a \Omega_{,n} \delta_b^n \Omega_{,c} - \delta_d^a \Omega_{,n} \delta_c^n \Omega_{,b} + \delta_d^a \Omega_{,n} g^{ne} g_{bc} \Omega_{,e} + g^{ae} g_{nd} \Omega_{,e} \delta_b^n \Omega_{,c} \\
&\quad + g^{ae} g_{nd} \Omega_{,e} \delta_c^n \Omega_{,b} - g^{ae} g_{nd} \Omega_{,e} g^{nf} g_{bc} \Omega_{,f}) \\
&= \Omega^{-2}(\delta_b^a \Omega_{,c} \Omega_{,d} + \delta_d^a \Omega_{,c} \Omega_{,b} - g^{ae} g_{bd} \Omega_{,c} \Omega_{,e} + \delta_c^a \Omega_{,b} \Omega_{,d} + \delta_c^a \Omega_{,d} \Omega_{,b} \\
&\quad - \delta_c^a \Omega_{,n} g^{ne} g_{bd} \Omega_{,e} - g^{ae} g_{bc} \Omega_{,e} \Omega_{,d} - g^{ae} g_{dc} \Omega_{,e} \Omega_{,b} + g^{ae} \delta_c^f g_{bd} \Omega_{,f} \Omega_{,e} - \delta_b^a \Omega_{,d} \Omega_{,c} \\
&\quad - \delta_c^a \Omega_{,d} \Omega_{,b} + g^{ae} \Omega_{,d} g_{bc} \Omega_{,e} - \delta_d^a \Omega_{,b} \Omega_{,c} - \delta_d^a \Omega_{,c} \Omega_{,b} + \delta_d^a \Omega_{,n} g^{ne} g_{bc} \Omega_{,e} \\
&\quad + g^{ae} g_{bd} \Omega_{,e} \Omega_{,c} + g^{ae} g_{cd} \Omega_{,e} \Omega_{,b} - g^{ae} \delta_d^f \Omega_{,f} \Omega_{,e} g_{bc}) \\
&= \Omega^{-2}(\delta_c^a \Omega_{,b} \Omega_{,d} - \delta_c^a \Omega_{,n} g^{ne} g_{bd} \Omega_{,e} - \delta_d^a \Omega_{,c} \Omega_{,b} + \delta_d^a \Omega_{,n} g^{ne} g_{bc} \Omega_{,e} \\
&\quad + g^{ae} g_{bd} \Omega_{,e} \Omega_{,c} - g^{ae} g_{bc} \Omega_{,d} \Omega_{,e}),
\end{aligned}$$

$$\begin{aligned}
I_3 &= \Omega^{-1}(\delta_b^a \Omega_{,d} + \delta_d^a \Omega_{,b} - g^{ae} g_{bd} \Omega_{,e})_{,c} - \Omega^{-1}(\delta_b^a \Omega_{,c} + \delta_c^a \Omega_{,b} - g^{ae} g_{bc} \Omega_{,e})_{,d} \\
&= \Omega^{-1}(\delta_b^a \Omega_{,dc} + \delta_d^a \Omega_{,bc} - g_{,c}^{ae} g_{bd} \Omega_{,e} - g^{ae} g_{bd,c} \Omega_{,e} - g^{ae} g_{bd} \Omega_{,ec}) \\
&\quad - \Omega^{-1}(\delta_b^a \Omega_{,cd} + \delta_c^a \Omega_{,bd} - g_{,d}^{ae} g_{bc} \Omega_{,e} - g^{ae} g_{bc,d} \Omega_{,e} - g^{ae} g_{bc} \Omega_{,ed}) \\
&\quad + \Omega_{,c}^{-1}(\delta_b^a \Omega_{,d} + \delta_d^a \Omega_{,b} - g^{ae} g_{bd} \Omega_{,e}) - \Omega_{,d}^{-1}(\delta_b^a \Omega_{,c} + \delta_c^a \Omega_{,b} - g^{ae} g_{bc} \Omega_{,e}) \\
&= \Omega^{-1}(\delta_b^a \Omega_{,cd} + \delta_d^a \Omega_{,bc} - g_{,c}^{ae} g_{bd} \Omega_{,e} - g^{ae} g_{bd,c} \Omega_{,e} - g^{ae} g_{bd} \Omega_{,ec}) \\
&\quad - \Omega^{-1}(\delta_b^a \Omega_{,cd} + \delta_c^a \Omega_{,bd} - g_{,d}^{ae} g_{bc} \Omega_{,e} - g^{ae} g_{bc,d} \Omega_{,e} - g^{ae} g_{bc} \Omega_{,ed}) \\
&\quad - \frac{\Omega_{,c}}{\Omega^2}(\delta_b^a \Omega_{,d} + \delta_d^a \Omega_{,b} - g^{ae} g_{bd} \Omega_{,e}) + \frac{\Omega_{,d}}{\Omega^2}(\delta_b^a \Omega_{,c} + \delta_c^a \Omega_{,b} - g^{ae} g_{bc} \Omega_{,e}) \\
&= \Omega^{-1}(\delta_d^a \Omega_{,bc} - g_{,c}^{ae} g_{bd} \Omega_{,e} - g^{ae} g_{bd,c} \Omega_{,e} - g^{ae} g_{bd} \Omega_{,ec} - \delta_c^a \Omega_{,bd} + g_{,d}^{ae} g_{bc} \Omega_{,e} \\
&\quad + g^{ae} g_{bc,d} \Omega_{,e} + g^{ae} g_{bc} \Omega_{,ed}) - \frac{\Omega_{,c}}{\Omega^2}(\delta_b^a \Omega_{,d} + \delta_d^a \Omega_{,b} - g^{ae} g_{bd} \Omega_{,e}) \\
&\quad + \frac{\Omega_{,d}}{\Omega^2}(\delta_b^a \Omega_{,c} + \delta_c^a \Omega_{,b} - g^{ae} g_{bc} \Omega_{,e}).
\end{aligned}$$

Summing I_1 , I_2 and I_3 we obtain

$$\begin{aligned}
&\Omega^{-1}(\Gamma_{bd}^a \delta_c^a \Omega_{,n} - \Gamma_{nc}^a g^{ne} g_{bd} \Omega_{,e} - \Gamma_{bd}^n g^{ae} g_{nc} \Omega_{,e} + \Gamma_{nd}^a g^{ne} g_{bc} \Omega_{,e} - \Gamma_{bc}^n \delta_d^a \Omega_{,n} \\
&\quad + \Gamma_{bc}^n g^{ae} g_{nd} \Omega_{,e} + \delta_d^a \Omega_{,bc} - g_{,c}^{ae} g_{bd} \Omega_{,e} - g^{ae} g_{bd,c} \Omega_{,e} - g^{ae} g_{bd} \Omega_{,ec} - \delta_c^a \Omega_{,bd} \\
&\quad + g_{,d}^{ae} g_{bc} \Omega_{,e} + g^{ae} g_{bc,d} \Omega_{,e} + g^{ae} g_{bc} \Omega_{,ed}) \\
&\quad + \Omega^{-2}(\delta_c^a \Omega_{,b} \Omega_{,d} - \delta_c^a \Omega_{,n} g^{ne} g_{bd} \Omega_{,e} - \delta_d^a \Omega_{,c} \Omega_{,b} + \delta_d^a \Omega_{,n} g^{ne} g_{bc} \Omega_{,e} \\
&\quad + g^{ae} g_{bd} \Omega_{,e} \Omega_{,c} - g^{ae} g_{bc} \Omega_{,d} \Omega_{,e} - \delta_b^a \Omega_{,c} \Omega_{,d} - \delta_d^a \Omega_{,c} \Omega_{,b} + g^{ae} g_{bd} \Omega_{,c} \Omega_{,e} \\
&\quad + \delta_b^a \Omega_{,d} \Omega_{,c} + \delta_c^a \Omega_{,d} \Omega_{,b} - g^{ae} g_{bc} \Omega_{,d} \Omega_{,e}).
\end{aligned}$$

Using the equations $g_{bd,c} = \Gamma_{bc}^n g_{nd} + \Gamma_{dc}^n g_{bn}$ and $g_{bc,d} = \Gamma_{bd}^n g_{nc} + \Gamma_{cd}^n g_{bn}$ we obtain

$$\Omega^{-1}(g_{bc,d} g^{ae} \Omega_{,e} - g_{bd,c} g^{ae} \Omega_{,e} - \Gamma_{bd}^n g^{ae} g_{nc} \Omega_{,e} + \Gamma_{bc}^n g^{ae} g_{nd} \Omega_{,e}) = 0.$$

Lets take the equation

$$g^{ae}{}_{;c} = g^{ae}{}_{,c} + \Gamma_{nc}^a g^{ne} + \Gamma_{nc}^e g^{an},$$

and

$$g^{ae}{}_{;d} = g^{ae}{}_{,d} + \Gamma_{nd}^a g^{ne} + \Gamma_{nd}^e g^{an},$$

then the formula reduces as follows

$$\begin{aligned} & \Omega^{-1} (\Gamma_{bd}^n \delta_c^a \Omega_{,n} + \Gamma_{nc}^e g^{an} g_{bd} \Omega_{,e} - \Gamma_{bc}^n \delta_d^a \Omega_{,n} + \delta_d^a \Omega_{,bc} - \delta_c^a \Omega_{,bd} - g^{ae} g_{bd} \Omega_{,ec} \\ & + g^{ae} g_{bc} \Omega_{,ed} - \Gamma_{nd}^e g^{an} g_{bc} \Omega_{,e}) \\ & + \Omega^{-2} (2\delta_c^a \Omega_{,d} \Omega_{,b} - 2\delta_d^a \Omega_{,b} \Omega_{,c} + 2g^{ae} \Omega_{,e} g_{bd} \Omega_{,c} + \delta_d^a g_{bc} g^{ne} \Omega_{,n} \Omega_{,e} \\ & - 2g^{ae} g_{bc} \Omega_{,e} \Omega_{,d} - \delta_c^a g^{ne} g_{bd} \Omega_{,n} \Omega_{,e}). \end{aligned}$$

After extending the right side of the initial equation we get the following form

$$\begin{aligned} & R_{.bcd}^a + \frac{\Omega^2}{2} \left(\delta_{[c}^a \Omega_{d]b} - g_{b[c} \Omega_{.d]}^a \right) \\ & = R_{.bcd}^a + \frac{\Omega^2}{2} \left[\frac{1}{2} (\delta_c^a \Omega_{db} - \delta_d^a \Omega_{cb}) - \frac{1}{2} (g_{bc} \Omega_{.d}^a - g_{bd} \Omega_{.c}^a) \right] \\ & = R_{.bcd}^a + \frac{\Omega^2}{4} (\delta_c^a \Omega_{db} - \delta_d^a \Omega_{cb} - g_{bc} \Omega_{.d}^a + g_{bd} \Omega_{.c}^a). \end{aligned}$$

Lets denote by I_4 the expression

$$\frac{\Omega^2}{4} (\delta_c^a \Omega_{db} - \delta_d^a \Omega_{cb} - g_{bc} \Omega_{.d}^a + g_{bd} \Omega_{.c}^a).$$

Putting into

$$\begin{aligned} \Omega_{db} &= g_{ad} \Omega_{.b}^a \\ &= g_{ad} \left[4 (\Omega^{-1}) (\Omega^{-1})_{;be} g^{ae} - 2 (\Omega^{-1})_{,p} (\Omega^{-1})_{,t} g^{pt} \delta_b^a \right] \\ &= 4 (\Omega^{-1}) (\Omega^{-1})_{;be} g_{ad} g^{ae} - 2 (\Omega^{-1})_{,p} (\Omega^{-1})_{,t} g_{ad} g^{pt} \delta_b^a \\ &= 4 (\Omega^{-1}) (\Omega^{-1})_{;bd} - 2 (\Omega^{-1})_{,p} (\Omega^{-1})_{,t} g_{bd} g^{pt}, \end{aligned}$$

and developing in a similar way Ω_{cb} , $\Omega_{.d}^a$ and $\Omega_{.c}^a$ we have

$$\begin{aligned} I_4 &= \frac{\Omega^2}{4} \left(\frac{-4\delta_c^a \Omega_{,bd}}{\Omega^3} + \frac{8\delta_c^a \Omega_{,b} \Omega_{,d}}{\Omega^4} + \frac{4\delta_c^a \Gamma_{bd}^e \Omega_{,e}}{\Omega^3} - \frac{4\delta_c^a \Omega_{,p} \Omega_{,t} g^{pt} g_{db}}{\Omega^4} \right) \\ &+ \frac{\Omega^2}{4} \left(\frac{4\delta_d^a \Omega_{,bc}}{\Omega^3} - \frac{8\delta_d^a \Omega_{,b} \Omega_{,c}}{\Omega^4} - \frac{4\delta_d^a \Gamma_{bc}^e \Omega_{,e}}{\Omega^3} + \frac{4\delta_d^a \Omega_{,p} \Omega_{,t} g^{pt} g_{bc}}{\Omega^4} \right) \\ &+ \frac{\Omega^2}{4} \left(\frac{4\Omega_{,de} g^{ae} g_{bc}}{\Omega^3} - \frac{8\Omega_{,d} \Omega_{,e} g^{ae} g_{bc}}{\Omega^4} - \frac{4\Gamma_{de}^p \Omega_{,p} g_{bc} g^{ae}}{\Omega^3} \right) \\ &- \frac{\Omega^2}{4} \left(\frac{4\Omega_{,ce} g^{ae} g_{bd}}{\Omega^3} - \frac{8\Omega_{,c} \Omega_{,e} g^{ae} g_{bd}}{\Omega^4} - \frac{4\Gamma_{ce}^n \Omega_{,n} g_{bd} g^{ae}}{\Omega^3} \right) \end{aligned}$$

$$\begin{aligned}
&= \Omega^{-1}(-\delta_c^a \Omega_{,bd} + \delta_c^a \Gamma_{bd}^e \Omega_{,e} + \delta_d^a \Omega_{,bc} - \delta_d^a \Gamma_{bc}^e \Omega_{,e} + \Omega_{,de} g^{ae} g_{bc} \\
&\quad - \Gamma_{de}^p \Omega_{,p} g_{bc} g^{ae} - \Omega_{,ce} g_{bd} g^{ae} + \Gamma_{ce}^n \Omega_{,n} g_{bd} g^{ae}) \\
&\quad + \Omega^{-2}(2\delta_c^a \Omega_{,b} \Omega_{,d} - \delta_c^a \Omega_{,p} \Omega_{,t} g^{pt} g_{bd} - 2\delta_d^a \Omega_{,b} \Omega_{,c} \\
&\quad + \delta_d^a \Omega_{,p} \Omega_{,t} g^{pt} g_{bc} - 2\Omega_{,d} \Omega_{,e} g^{ae} g_{bc} + 2\Omega_{,c} \Omega_{,e} g_{bd} g^{ae}).
\end{aligned}$$

After changing some of the indexes we finally obtain that the left side of the previous equation equals the right side.

$$\begin{aligned}
&= \Omega^{-1}(-\delta_c^a \Omega_{,bd} + \delta_c^a \Gamma_{bd}^n \Omega_{,n} + \delta_d^a \Omega_{,bc} - \delta_d^a \Gamma_{bc}^n \Omega_{,n} + \Omega_{,de} g^{ae} g_{bc} \\
&\quad - \Gamma_{dn}^e \Omega_{,e} g_{bc} g^{an} - \Omega_{,ce} g_{bd} g^{ae} + \Gamma_{cn}^e \Omega_{,e} g_{bd} g^{an}) \\
&\quad + \Omega^{-2}(2\delta_c^a \Omega_{,b} \Omega_{,d} - \delta_c^a \Omega_{,n} \Omega_{,e} g^{ne} g_{bd} - 2\delta_d^a \Omega_{,b} \Omega_{,c} \\
&\quad + \delta_d^a \Omega_{,n} \Omega_{,e} g^{ne} g_{bc} - 2\Omega_{,d} \Omega_{,e} g^{ae} g_{bc} + 2\Omega_{,c} \Omega_{,e} g_{bd} g^{ae}).
\end{aligned}$$

If $\Omega(x) = \text{const}$, then $\hat{R}_{bcd}^a = R_{bcd}^a$. □

Definition 6. The Ricci tensor is defined as [2]

$$R_{.g}^b = R_g^b = g^{ac} R_{a.cg}^{b..} = R_{..cg}^{cb..}$$

Fact 12. The Ricci tensor changes under the conformal transformation of the metric as follows

$$\hat{R}_g^b = \Omega^{-2} R_g^b + \frac{\Omega_{.g}^b}{2} + \frac{\delta_g^b}{4} \Omega_{.a}^a.$$

Proof. In the proof we are using the transformation laws for the curvature tensor and for the metric tensor.

$$\begin{aligned}
\hat{R}_{.g}^b &= \hat{g}^{al} \hat{R}_{.agl}^b = \Omega^{-2} g^{al} \left[R_{.agl}^b + \frac{\Omega^2}{4} (\delta_g^b \Omega_{la} - g_{ag} \Omega_{.l}^b - \delta_l^b \Omega_{ga} + g_{al} \Omega_{.g}^b) \right] \\
&= \Omega^{-2} R_{.g}^b + \frac{1}{4} (\delta_g^b \Omega_{.a}^a - \delta_g^l \Omega_{.l}^b - g^{ab} \Omega_{ga} + \delta_a^a \Omega_{.g}^b) \\
&= \Omega^{-2} R_{.g}^b + \frac{1}{4} (\delta_g^b \Omega_{.a}^a - \Omega_{.g}^b - \Omega_{.g}^b + 4\Omega_{.g}^b) \\
&= \Omega^{-2} R_{.g}^b + \frac{\Omega_{.g}^b}{2} + \frac{\delta_g^b \Omega_{.a}^a}{4}.
\end{aligned}$$

After lowering the upper index we obtain the form of the Ricci tensor as below

$$\begin{aligned}
\hat{R}_{ag} &= \hat{g}_{ab} \hat{R}_{.g}^b = \Omega^2 g_{ab} \Omega^{-2} R_g^b + \Omega^2 g_{ab} \frac{\Omega_{.g}^b}{2} + \Omega^2 g_{ab} \frac{\delta_g^b \Omega_{.a}^a}{4} \\
&= R_{ag} + \frac{\Omega^2}{2} \Omega_{ag} + \frac{\Omega^2}{4} g_{ab} \delta_g^b \Omega_{.a}^a \\
&= R_{ag} + \frac{\Omega^2}{2} \Omega_{ag} + \frac{\Omega^2}{4} g_{ag} \Omega_{.a}^a.
\end{aligned}$$

If $\Omega(x) = \text{const}$, then $\hat{R}_{ag}^b = \Omega^{-2} R_g^b$ and $\hat{R}_{ag} = R_{ag}$. □

Definition 7. *The curvature scalar we define as follows [2]*

$$R := g^{ab}R_{ab} = R^b_{\cdot b}.$$

Fact 13. *The curvature scalar changes under the conformal transformation of the metric in the following way*

$$\widehat{R} = \Omega^{-2}R + \frac{3}{2}\Omega^a_{\cdot a}.$$

Proof. One has

$$R = R^i_{\cdot i}$$

In our proof we are using the transformation law for the Ricci tensor.

$$\begin{aligned}\widehat{R} &= \widehat{R}^i_{\cdot i} = \Omega^{-2}R^i_{\cdot i} + \frac{\Omega^i_{\cdot i}}{2} + \frac{\delta^i_i}{4}\Omega^a_{\cdot a} \\ &= \Omega^{-2}R^i_{\cdot i} + \frac{\Omega^i_{\cdot i}}{2} + \frac{4}{4}\Omega^a_{\cdot a} \\ &= \Omega^{-2}R^i_{\cdot i} + \frac{3}{2}\Omega^a_{\cdot a}.\end{aligned}$$

If $\Omega(x) = \text{const}$, then $\widehat{R} = \Omega^{-2}R$. □

Definition 8. *The Weyl tensor is defined as [1]*

$$\begin{aligned}C_{abcd} &:= R_{abcd} + \frac{1}{2}(g_{ad}R_{cb} - g_{ac}R_{db} + g_{bc}R_{da} - g_{bd}R_{ca}) \\ &\quad + \frac{1}{6}R(g_{ac}g_{db} - g_{ad}g_{cb}).\end{aligned}$$

In GR the Weyl tensor describes the free gravitational field, i.e. the gravitational field independent of matter.

Fact 14. *The Weyl tensor of the conformal curvature $C^e_{\cdot bcd}$ is an invariant of the conformal transformation*

$$\widehat{C}^a_{\cdot bcd} = C^a_{\cdot bcd}.$$

Proof. Using the formula for C_{abcd} we calculate at first the Weyl tensor $C^a_{\cdot bcd}$.

$$\begin{aligned}C_{abcd} &:= R_{abcd} + \frac{1}{2}(g_{ad}R_{cb} - g_{ac}R_{db} + g_{bc}R_{da} - g_{bd}R_{ca}) \\ &\quad + \frac{1}{6}R(g_{ac}g_{db} - g_{ad}g_{cb}).\end{aligned}$$

$$\begin{aligned}
C_{.bcd}^e &= g^{ae} C_{abcd} \\
&= g^{ae} R_{abcd} + \frac{1}{2} g^{ae} (g_{ad} R_{cb} - g_{ac} R_{db} + g_{bc} R_{da} - g_{bd} R_{ca}) \\
&\quad + \frac{1}{6} g^{ae} R (g_{ac} g_{db} - g_{ad} g_{cb}) \\
&= R_{.bcd}^e + \frac{1}{2} (\delta_d^e R_{cb} - \delta_c^e R_{db} + g_{bc} R_{.d}^e - g_{bd} R_{.c}^e) \\
&\quad + \frac{1}{6} R (\delta_c^e g_{db} - \delta_d^e g_{cb}).
\end{aligned}$$

After the conformal rescaling of the metric one has

$$\begin{aligned}
\widehat{C}_{.bcd}^a &= \widehat{R}_{.bcd}^a + \frac{1}{2} \left(\widehat{g}_d^a \widehat{R}_{cb} - \delta_c^a \widehat{R}_{db} + \widehat{g}_{bc} \widehat{R}_{.d}^a - \widehat{g}_{bd} \widehat{R}_{.c}^a \right) + \frac{1}{6} \widehat{R} (\delta_c^a \widehat{g}_{db} - \delta_d^a \widehat{g}_{cb}) \\
&= R_{.bcd}^a + \frac{\Omega^2}{4} (\delta_c^a \Omega_{db} - \delta_d^a \Omega_{cb} - g_{bc} \Omega_{.d}^a + g_{bd} \Omega_{.c}^a) \\
&\quad + \frac{1}{2} \delta_d^a \left[R_{cb} + \frac{\Omega^2}{2} \left(\Omega_{cb} + \frac{g_{cb}}{2} \Omega_{.e}^e \right) \right] - \frac{1}{2} \delta_c^a \left[R_{db} + \frac{\Omega^2}{2} \left(\Omega_{db} + \frac{g_{db}}{2} \Omega_{.e}^e \right) \right] \\
&\quad + \frac{1}{2} \Omega^2 g_{bc} \left(\Omega^{-2} R_{.d}^a + \frac{\Omega_{.d}^a}{2} + \frac{\delta_d^a \Omega_{.e}^e}{4} \right) - \frac{1}{2} \Omega^2 g_{bd} \left(\Omega^{-2} R_{.c}^a + \frac{\Omega_{.c}^a}{2} + \frac{\delta_c^a \Omega_{.e}^e}{4} \right) \\
&\quad + \frac{1}{6} \left(\Omega^{-2} R + \frac{3}{2} \Omega_{.e}^e \right) (\delta_c^a \Omega^2 g_{bd} - \delta_d^a \Omega^2 g_{cb}) \\
&= R_{.bcd}^a + \frac{\Omega^2}{4} (\delta_c^a \Omega_{db} - \delta_d^a \Omega_{cb} - g_{bc} \Omega_{.d}^a + g_{bd} \Omega_{.c}^a) \\
&\quad + \frac{1}{2} \left(\delta_d^a R_{cb} + \frac{\Omega^2}{2} \delta_d^a \Omega_{cb} + \frac{\Omega^2}{4} g_{cb} \delta_d^a \Omega_{.e}^e - \delta_c^a R_{db} - \frac{\Omega^2}{2} \delta_c^a \Omega_{db} - \frac{\Omega^2}{4} \delta_c^a g_{db} \Omega_{.e}^e \right) \\
&\quad + \frac{1}{2} \left(g_{bc} R_{.d}^a + \frac{\Omega^2}{2} g_{bc} \Omega_{.d}^a + \frac{\Omega^2}{4} g_{bc} \delta_d^a \Omega_{.e}^e - g_{bd} R_{.c}^a - \frac{\Omega^2}{2} g_{bd} \Omega_{.c}^a - \frac{\Omega^2}{4} \delta_c^a g_{bd} \Omega_{.e}^e \right) \\
&\quad + \frac{1}{6} R (\delta_c^a g_{bd} - \delta_d^a g_{cb}) + \frac{1}{4} \delta_c^a \Omega^2 g_{bd} \Omega_{.e}^e - \frac{1}{4} \delta_d^a \Omega^2 g_{cb} \Omega_{.e}^e \\
&= R_{.bcd}^a + \frac{1}{2} (\delta_d^a R_{cb} - \delta_c^a R_{db} + g_{bc} R_{.d}^a - g_{bd} R_{.c}^a) + \frac{1}{6} R (\delta_c^a g_{db} - \delta_d^a g_{cb}) \\
&= C_{.bcd}^a.
\end{aligned}$$

If we change the location of the indexes in the Weyl tensor, then the above invariance is missed, e.g. $\widehat{C}_{.cd}^{ab} = \widehat{g}^{be} \widehat{C}_{.ecd}^a = \Omega^{-2} g^{be} C_{.ecd}^a = \Omega^{-2} C_{cd}^{ab}$. $\square$

Definition 9. *The Einstein equations (in the other words the field equations in GR or the equations of the gravitational field) have the following form [1]*

$$G_{i.}^k = \kappa T_{i.}^k,$$

where $\kappa := \frac{8\pi G}{c^4}$.

If we use the geometrized units $G = c = 1$, then $\kappa = 8\pi$ and the Einstein's equation has the simpler form $G_{i.}^k = 8\pi T_{i.}^k$. The left side of the equation represents

the geometry of the space-time. On the other hand the right side describes matter and energy which fills the space-time. The Einstein's equation can be summarized as:

the distribution of the matter and the energy in the space-time determines its local geometry.

The tensor $G_{i.}^{\cdot k}$ has the following structure

$$G_{i.}^{\cdot k} = R_{i.}^{\cdot k} - \frac{1}{2}\delta_i^k R,$$

and it is called the Einstein tensor. Raising or lowering one of the indexes we obtain the Einstein tensor in the covariant or contravariant form.

$$G_{ik} = R_{ik} - \frac{1}{2}g_{ik}R,$$

$$G^{ik} = R^{ik} - \frac{1}{2}g^{ik}R.$$

Fact 15. *The Einstein tensor changes under the conformal transformation of metric in the following way*

$$\widehat{G}_{i.}^{\cdot k} = \Omega^{-2}G_{i.}^{\cdot k} + \frac{1}{2}(\Omega_{i.}^{\cdot k} - \delta_i^k \Omega_{.a}^a).$$

Proof.

$$\begin{aligned} \widehat{G}_{i.}^{\cdot k} &= \widehat{R}_{i.}^{\cdot k} - \frac{1}{2}\delta_i^k \widehat{R} \\ &= \Omega^{-2}R_{i.}^{\cdot k} + \frac{\Omega_{i.}^{\cdot k}}{2} + \frac{\delta_i^k}{4}\Omega_{.a}^a - \frac{1}{2}\delta_i^k \left(\Omega^{-2}R + \frac{3}{2}\Omega_{.a}^a \right) \\ &= \Omega^{-2}R_{i.}^{\cdot k} - \frac{1}{2}\Omega^{-2}\delta_i^k R + \frac{\Omega_{i.}^{\cdot k}}{2} + \frac{\delta_i^k}{4}\Omega_{.a}^a - \frac{3}{4}\delta_i^k \Omega_{.a}^a \\ &= \Omega^{-2}G_{i.}^{\cdot k} + \frac{\Omega_{i.}^{\cdot k}}{2} - \frac{1}{2}\delta_i^k \Omega_{.a}^a \\ &= \Omega^{-2}G_{i.}^{\cdot k} + \frac{1}{2}(\Omega_{i.}^{\cdot k} - \delta_i^k \Omega_{.a}^a). \end{aligned}$$

As we can see the Einstein tensor is not conformally invariant. □

3. Selected physical aspects of conformal transformations

In the previous chapter we have calculated the formula for the Einstein tensor under the conformal rescaling of the metric. Now we will show the transformation law for the right side of the Einstein equations, the matter tensor $T_{i.}^{\cdot k}$, which is responsible for the curvature of the space-time.

Definition 10. *The matter tensor of the macroscopic matter (the matter energy-momentum tensor or the stress tensor) we define as follows [1]*

$$T_{i.}^{\cdot k} = (\rho + p)u_i u^k - p\delta_i^k,$$

where ρ denotes a density of the matter and p pressure.

Fact 16. *The matter tensor changes under the conformal transformation of the metric in the following way*

$$\widehat{T}_{i.}^{\cdot k} = \Omega^{-4}T_{i.}^{\cdot k}.$$

Proof. In the proof we will use the formula for the conformal rescaling of the metric g_{ik} and also the formula for mass of the matter contained in the volume element $d^4\Omega = \sqrt{|g|}d^4x$ of 4-dimensional space-time, where $g = \det[g_{ik}]$. Hence

$$M = \sqrt{|g|}\rho d^4x,$$

where ρ is proper density of matter. After the conformal rescaling of the metric $d^4\Omega$ transforms into $d^4\widehat{\Omega} = \sqrt{|\widehat{g}|}d^4x$. The conformal rescaling of the metric does not change the mass included in the volume element $d^4\Omega$. It changes only into its density $\rho \rightarrow \widehat{\rho}$. As a consequence one has

$$M = \widehat{\rho}d^4\widehat{\Omega} = \sqrt{|\widehat{g}|}\widehat{\rho}d^4x,$$

where M indicates the mass of the matter contained in the element $\sqrt{|\widehat{g}|}d^4x$.

Because this mass does not change under the conformal rescaling of the metric, we can compare both values and hence we get

$$(3) \quad \sqrt{|\widehat{g}|}\widehat{\rho}d^4x = \sqrt{|g|}\rho d^4x.$$

Using the formula for $\widehat{g} = \Omega^8 g$ and putting the expression $\sqrt{|\widehat{g}|} = \Omega^4 \sqrt{|g|}$ in (3) we obtain the transformation law for the density of the matter under the conformal rescaling of the metric.

$$\begin{aligned} \Omega^4 \sqrt{|g|}\widehat{\rho}d^4x &= \sqrt{|g|}\rho d^4x \\ \widehat{\rho} &= \Omega^{-4}\rho. \end{aligned}$$

Returning to the initial equation which describes the matter tensor and using the transformation formula for the density of the matter we obtain

$$\widehat{T}_{i.}^{\cdot k} = \Omega^{-4}\rho u_i u^k + \widehat{p}\widehat{u}_i \widehat{u}^k - \widehat{p}\delta_i^k.$$

In order $\widehat{T}_{i.}^{\cdot k}$ to be connected with $T_{i.}^{\cdot k}$ by the transformation rule $\widehat{T}_{i.}^{\cdot k} = \Omega^x T_{i.}^{\cdot k}$, the pressure must change under the conformal rescaling of the metric following the formula

$$\widehat{p} = \Omega^{-4}p,$$

because

$$u_i u^k = \widehat{u}_i \widehat{u}^k, \quad \delta_i^k = \widehat{\delta}_i^k.$$

If this condition is satisfied, then the transformation formula has the following form

$$\widehat{T}_{i.}^{\cdot k} = \Omega^{-4} T_{i.}^{\cdot k}$$

Raising or lowering one of the indexes we obtain the other useful transformation rules

$$\widehat{T}^{ik} = \widehat{g}^{il} \widehat{T}_{i.}^{\cdot k} = \Omega^{-6} T^{ik}$$

and

$$\widehat{T}_{ik} = \widehat{g}_{kl} \widehat{T}_{i.}^{\cdot k} = \Omega^{-2} T_{ik}.$$

□

Fact 17. *Einstein's equations are not conformally invariant.*

Proof. In the proof we use the formula for the Einstein tensor after the conformal transformation of the metric, which we have proved in the Fact 15. Using the Einstein's equations in the metric g_{ik} and replacing the Einstein tensor by the matter tensor we get

$$(4) \quad \widehat{G}_{i.}^{\cdot k} = \kappa \Omega^{-2} T_{i.}^{\cdot k} + \frac{1}{2} (\Omega_{i.}^{\cdot k} - \delta_i^k \Omega_{.a}^a).$$

Converting the relation between the matter tensor in the initial gauge and the new one we obtain

$$(5) \quad \widehat{T}_{i.}^{\cdot k} = \Omega^{-4} T_{i.}^{\cdot k} \implies T_{i.}^{\cdot k} = \Omega^4 \widehat{T}_{i.}^{\cdot k}.$$

Replacing $T_{i.}^{\cdot k}$ in (4) by (5) one has

$$\widehat{G}_{i.}^{\cdot k} = \kappa \Omega^2 \widehat{T}_{i.}^{\cdot k} + \frac{1}{2} (\Omega_{i.}^{\cdot k} - \delta_i^k \Omega_{.a}^a)$$

or

$$\widehat{G}_{i.}^{\cdot k} = \kappa \Omega^2 \widehat{T}_{i.}^{\cdot k} + \kappa \widetilde{T}_{i.}^{\cdot k},$$

where $\widetilde{T}_{i.}^{\cdot k} = \frac{1}{\kappa} \frac{1}{2} (\Omega_{i.}^{\cdot k} - \delta_i^k \Omega_{.a}^a)$. □

The final formula shows that Einstein's equations are not conformally invariant, i.e. we do not have the formula of the form $\widehat{G}_{i.}^{\cdot k} = \kappa \widehat{T}_{i.}^{\cdot k}$. As a consequence there is a new additional matter described by the tensor $\widetilde{T}_{i.}^{\cdot k}$, apart from the matter which had already existed before, the conformal rescaling of the metric, and satisfied Einstein's equations $G_{i.}^{\cdot k} = \kappa T_{i.}^{\cdot k}$.

This fact was used in the articles “An interesting property of the Friedman universes” [6], “On Energy of the Friedman Universes in Conformally Flat Coordinates” [7] and “Superenergy, conformal transformations, and Friedman universes” [8] written by J. Garecki. Namely, he used the conformal rescaling of the metric to create the Friedman dust universes from the empty Minkowskian space-time. One should emphasize in this context that Friedman universes are actually the best mathematical models of the real universe.

As a consequence of Einstein's Equivalence Principle the gravitational field does not have an energy-momentum tensor. It has only the energy-momentum pseudotensors. One such pseudotensor is given in the book “Teoria pola” by L. Landau i E. M. Lifshitz. It is described in the definition 11.

Definition 11. The Landau-Lifshitz pseudotensor of the energy-momentum of the gravitational field is defined as [1]

$$\begin{aligned} t^{ik} = & \frac{1}{16\pi} (2\Gamma_{lm}^n \Gamma_{np}^p - \Gamma_{lp}^n \Gamma_{mn}^p - \Gamma_{ln}^p \Gamma_{mp}^p) (g^{il} g^{km} - g^{ik} g^{lm}) \\ & + \frac{1}{16\pi} g^{il} g^{mn} (\Gamma_{lp}^k \Gamma_{mn}^p + \Gamma_{mn}^k \Gamma_{lp}^p - \Gamma_{np}^k \Gamma_{lm}^p - \Gamma_{lm}^k \Gamma_{np}^p) \\ & + \frac{1}{16\pi} g^{kl} g^{mn} (\Gamma_{lp}^i \Gamma_{mn}^p + \Gamma_{mn}^i \Gamma_{lp}^p - \Gamma_{np}^i \Gamma_{lm}^p - \Gamma_{lm}^i \Gamma_{np}^p) \\ & + \frac{1}{16\pi} g^{lm} g^{np} (\Gamma_{ln}^i \Gamma_{mp}^k - \Gamma_{lm}^i \Gamma_{np}^k). \end{aligned}$$

Fact 18. The Landau-Lifshitz pseudotensor of the energy-momentum of the gravitational field changes under the conformal transformation of the metric in the following way

$$\begin{aligned} \hat{t}^{ik} = & \frac{1}{16\pi} (2\hat{\Gamma}_{lm}^n \hat{\Gamma}_{np}^p - \hat{\Gamma}_{lp}^n \hat{\Gamma}_{mn}^p - \hat{\Gamma}_{ln}^p \hat{\Gamma}_{mp}^p) (\hat{g}^{il} \hat{g}^{km} - \hat{g}^{ik} \hat{g}^{lm}) \\ & + \frac{1}{16\pi} \hat{g}^{il} \hat{g}^{mn} (\hat{\Gamma}_{lp}^k \hat{\Gamma}_{mn}^p + \hat{\Gamma}_{mn}^k \hat{\Gamma}_{lp}^p - \hat{\Gamma}_{np}^k \hat{\Gamma}_{lm}^p - \hat{\Gamma}_{lm}^k \hat{\Gamma}_{np}^p) \\ & + \frac{1}{16\pi} \hat{g}^{kl} \hat{g}^{mn} (\hat{\Gamma}_{lp}^i \hat{\Gamma}_{mn}^p + \hat{\Gamma}_{mn}^i \hat{\Gamma}_{lp}^p - \hat{\Gamma}_{np}^i \hat{\Gamma}_{lm}^p - \hat{\Gamma}_{lm}^i \hat{\Gamma}_{np}^p) \\ & + \frac{1}{16\pi} \hat{g}^{lm} \hat{g}^{np} (\hat{\Gamma}_{ln}^i \hat{\Gamma}_{mp}^k - \hat{\Gamma}_{lm}^i \hat{\Gamma}_{np}^k) \\ = & \Omega^{-4} t^{ik} + \Omega^{-5} \frac{1}{16\pi} (I_1^{ik} + I_2^{ik}) + \Omega^{-6} \frac{1}{16\pi} I_3^{ik}. \end{aligned}$$

The analytical structures of I_1^{ik} , I_2^{ik} and I_3^{ik} are given in the proof.

Proof.

$$\begin{aligned} \hat{t}^{ik} = & \frac{1}{16\pi} (2\hat{\Gamma}_{lm}^n \hat{\Gamma}_{np}^p - \hat{\Gamma}_{lp}^n \hat{\Gamma}_{mn}^p - \hat{\Gamma}_{ln}^p \hat{\Gamma}_{mp}^p) (\hat{g}^{il} \hat{g}^{km} - \hat{g}^{ik} \hat{g}^{lm}) \\ & + \frac{1}{16\pi} \hat{g}^{il} \hat{g}^{mn} (\hat{\Gamma}_{lp}^k \hat{\Gamma}_{mn}^p + \hat{\Gamma}_{mn}^k \hat{\Gamma}_{lp}^p - \hat{\Gamma}_{np}^k \hat{\Gamma}_{lm}^p - \hat{\Gamma}_{lm}^k \hat{\Gamma}_{np}^p) \\ & + \frac{1}{16\pi} \hat{g}^{kl} \hat{g}^{mn} (\hat{\Gamma}_{lp}^i \hat{\Gamma}_{mn}^p + \hat{\Gamma}_{mn}^i \hat{\Gamma}_{lp}^p - \hat{\Gamma}_{np}^i \hat{\Gamma}_{lm}^p - \hat{\Gamma}_{lm}^i \hat{\Gamma}_{np}^p) \\ & + \frac{1}{16\pi} \hat{g}^{lm} \hat{g}^{np} (\hat{\Gamma}_{ln}^i \hat{\Gamma}_{mp}^k - \hat{\Gamma}_{lm}^i \hat{\Gamma}_{np}^k). \end{aligned}$$

Putting $\hat{\Gamma}_{lm}^n = \Gamma_{lm}^n + P_{lm}^n$ in the equation, where

$$P_{lm}^n = \Omega^{-1} (\delta_l^n \Omega_{,m} + \delta_m^n \Omega_{,l} - g_{lm} g^{ne} \Omega_{,e})$$

and $\hat{g}^{mn} = \Omega^{-2} g^{mn}$ we obtain

$$\begin{aligned}
&= \Omega^{-4} \frac{1}{16\pi} (2\Gamma_{lm}^n \Gamma_{np}^p - \Gamma_{lp}^n \Gamma_{mn}^p - \Gamma_{ln}^n \Gamma_{mp}^p) (g^{il} g^{km} - g^{ik} g^{lm}) \\
&\quad + \Omega^{-4} \frac{1}{16\pi} g^{il} g^{mn} \left(\Gamma_{lp}^k \Gamma_{mn}^p + \Gamma_{mn}^k \Gamma_{lp}^p - \Gamma_{np}^k \Gamma_{lm}^p - \Gamma_{lm}^k \Gamma_{np}^p \right) \\
&\quad + \Omega^{-4} \frac{1}{16\pi} g^{kl} g^{mn} \left(\Gamma_{lp}^i \Gamma_{mn}^p + \Gamma_{mn}^i \Gamma_{lp}^p - \Gamma_{np}^i \Gamma_{lm}^p - \Gamma_{lm}^i \Gamma_{np}^p \right) \\
&\quad + \Omega^{-4} \frac{1}{16\pi} g^{lm} g^{np} (\Gamma_{ln}^i \Gamma_{mp}^k - \Gamma_{lm}^i \Gamma_{np}^k) \\
&\quad + \Omega^{-4} \frac{1}{16\pi} (2\Gamma_{lm}^n P_{np}^p + 2P_{lm}^n \Gamma_{np}^p - \Gamma_{lp}^n P_{mn}^p) (g^{il} g^{km} - g^{ik} g^{lm}) \\
&\quad - \Omega^{-4} \frac{1}{16\pi} (P_{lp}^n \Gamma_{mn}^p + \Gamma_{ln}^n P_{mp}^p + P_{ln}^n \Gamma_{mp}^p) (g^{il} g^{km} - g^{ik} g^{lm}) \\
&\quad + \Omega^{-4} \frac{1}{16\pi} g^{il} g^{mn} \left(\Gamma_{lp}^k P_{mn}^p + P_{lp}^k \Gamma_{mn}^p + \Gamma_{mn}^k P_{lp}^p + P_{mn}^k \Gamma_{lp}^p \right) \\
&\quad - \Omega^{-4} \frac{1}{16\pi} g^{il} g^{mn} (\Gamma_{np}^k P_{lm}^p + P_{np}^k \Gamma_{lm}^p - \Gamma_{lm}^k P_{np}^p - P_{lm}^k \Gamma_{np}^p) \\
&\quad + \Omega^{-4} \frac{1}{16\pi} g^{kl} g^{mn} \left(\Gamma_{lp}^i P_{mn}^p + P_{lp}^i \Gamma_{mn}^p + \Gamma_{mn}^i P_{lp}^p + P_{mn}^i \Gamma_{lp}^p \right) \\
&\quad - \Omega^{-4} \frac{1}{16\pi} g^{kl} g^{mn} (\Gamma_{np}^i P_{lm}^p + P_{np}^i \Gamma_{lm}^p + \Gamma_{lm}^i P_{np}^p + P_{lm}^i \Gamma_{np}^p) \\
&\quad + \Omega^{-4} \frac{1}{16\pi} g^{lm} g^{np} (\Gamma_{ln}^i P_{mp}^k + P_{ln}^i \Gamma_{mp}^k - \Gamma_{lm}^i P_{np}^k - P_{lm}^i \Gamma_{np}^k) \\
&\quad + \Omega^{-4} \frac{1}{16\pi} (2P_{lm}^n P_{np}^p - P_{lp}^n P_{mn}^p - P_{ln}^n P_{mp}^p) (g^{il} g^{km} - g^{ik} g^{lm}) \\
&\quad + \Omega^{-4} \frac{1}{16\pi} g^{il} g^{mn} \left(P_{lp}^k P_{mn}^p + P_{mn}^k P_{lp}^p - P_{np}^k P_{lm}^p - P_{lm}^k P_{np}^p \right) \\
&\quad + \Omega^{-4} \frac{1}{16\pi} g^{kl} g^{mn} \left(P_{lp}^i P_{mn}^p + P_{mn}^i P_{lp}^p - P_{np}^i P_{lm}^p - P_{lm}^i P_{np}^p \right) \\
&\quad + \Omega^{-4} \frac{1}{16\pi} g^{lm} g^{np} (P_{ln}^i P_{mp}^k - P_{lm}^i P_{np}^k).
\end{aligned}$$

Hence the above equation we can write in the form

$$\hat{t}^{ik} = \Omega^{-4} t^{ik} + \Omega^{-5} \frac{1}{16\pi} (I_1^{ik} + I_2^{ik}) + \Omega^{-6} \frac{1}{16\pi} I_3^{ik},$$

where

$$\begin{aligned}
I_1^{ik} &= \left(2\Gamma_{lm}^n \tilde{P}_{np}^p - \Gamma_{lp}^n \tilde{P}_{mn}^p - \Gamma_{ln}^n \tilde{P}_{mp}^p \right) (g^{il} g^{km} - g^{ik} g^{lm}) \\
&\quad + g^{il} g^{mn} \left(\Gamma_{lp}^k \tilde{P}_{mn}^p + \Gamma_{mn}^k \tilde{P}_{lp}^p - \Gamma_{np}^k \tilde{P}_{lm}^p - \Gamma_{lm}^k \tilde{P}_{np}^p \right) \\
&\quad + g^{kl} g^{mn} \left(\Gamma_{lp}^i \tilde{P}_{mn}^p + \Gamma_{mn}^i \tilde{P}_{lp}^p - \Gamma_{np}^i \tilde{P}_{lm}^p - \Gamma_{lm}^i \tilde{P}_{np}^p \right) \\
&\quad + g^{lm} g^{np} \left(\Gamma_{ln}^i \tilde{P}_{mp}^k - \Gamma_{lm}^i \tilde{P}_{np}^k \right),
\end{aligned}$$

$$\begin{aligned}
I_2^{ik} &= \left(2\tilde{P}_{lm}^n \Gamma_{np}^p - \tilde{P}_{lp}^n \Gamma_{mn}^p - \tilde{P}_{ln}^n \Gamma_{mp}^p \right) (g^{il} g^{km} - g^{ik} g^{lm}) \\
&\quad + g^{il} g^{mn} \left(\tilde{P}_{lp}^k \Gamma_{mn}^p + \tilde{P}_{mn}^k \Gamma_{lp}^p - \tilde{P}_{np}^k \Gamma_{lm}^p - \tilde{P}_{lm}^k \Gamma_{np}^p \right) \\
&\quad + g^{kl} g^{mn} \left(\tilde{P}_{lp}^i \Gamma_{mn}^p + \tilde{P}_{mn}^i \Gamma_{lp}^p - \tilde{P}_{np}^i \Gamma_{lm}^p - \tilde{P}_{lm}^i \Gamma_{np}^p \right) \\
&\quad + g^{lm} g^{np} \left(\tilde{P}_{ln}^i \Gamma_{mp}^k - \tilde{P}_{lm}^i \Gamma_{np}^k \right), \\
I_3^{ik} &= \left(2\tilde{P}_{lm}^n \tilde{P}_{np}^p - \tilde{P}_{lp}^n \tilde{P}_{mn}^p - \tilde{P}_{ln}^n \tilde{P}_{mp}^p \right) (g^{il} g^{km} - g^{ik} g^{lm}) \\
&\quad + g^{il} g^{mn} \left(\tilde{P}_{lp}^k \tilde{P}_{mn}^p + \tilde{P}_{mn}^k \tilde{P}_{lp}^p - \tilde{P}_{np}^k \tilde{P}_{lm}^p - \tilde{P}_{lm}^k \tilde{P}_{np}^p \right) \\
&\quad + g^{kl} g^{mn} \left(\tilde{P}_{lp}^i \tilde{P}_{mn}^p + \tilde{P}_{mn}^i \tilde{P}_{lp}^p - \tilde{P}_{np}^i \tilde{P}_{lm}^p - \tilde{P}_{lm}^i \tilde{P}_{np}^p \right) \\
&\quad + g^{lm} g^{np} \left(\tilde{P}_{ln}^i \tilde{P}_{mp}^k - \tilde{P}_{lm}^i \tilde{P}_{np}^k \right)
\end{aligned}$$

and

$$\tilde{P}_{mn}^p = (\delta_m^p \Omega_{,n} + \delta_n^p \Omega_{,m} - g_{mn} g^{pe} \Omega_{,e}).$$

As we can see, the structures of I_1^{ik} , I_2^{ik} and I_3^{ik} are the same as the structure of t^{ik} with the change $\Gamma\Gamma \rightarrow \Gamma P$ in I_1^{ik} , $\Gamma\Gamma \rightarrow P\Gamma$ in I_2^{ik} and $\Gamma\Gamma \rightarrow PP$ in I_3^{ik} . $\square$

Conformal rescaling of the metric creates an additional energy and momentum for the gravitation. It is represented by expressions with Ω^{-5} and Ω^{-6} in the transformation law of the Landau-Lifshitz pseudotensor $t^{ik} = t^{ki}$.

4. Conclusion

- I have proved several transformation formulas for various geometrical objects under the conformal rescaling of the metric on the Riemannian (or pseudoriemannian) manifold.
- Some of the formulas I have obtained have simpler form than the formulas given in standard books. One can easily check this fact by comparing our formulas with the ones given e.g. in "The Large Structure of Space-Time" by S. Hawking and G. F. R. Ellis.
- The basic equations of the general theory of relativity the Einstein equations are not conformally invariant. This fact means that there is a creation of the new matter and there is also possibility of creation of the Friedman universes from the vacuum.
- A new result of this paper is the transformation law for the Landau-Lifshitz pseudotensor of the energy-momentum of the gravitational field under conformal transformation of the metric. It is easily seen from this law that a conformal rescaling of the metric creates an additional energy and momentum for gravitation.

Acknowledgement

Author would like to thank Professor Julian Ławrynowicz for possibility to deliver lecture at *Hypercomplex Seminar 2015* in Będlewo, Poland.

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Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 16, 2015.

NIEZMIENNIKI KONFOREMNEGO PRZESKALOWANIA CZASOPRZESTRZENI – BADANIE OBEJMUJĄCE KONSEKWENCJE DLA METRYKI

S t r e s z c z e n i e

Transformacje konforemne odgrywają istotną rolę w analizie globalnej struktury czasoprzestrzeni. Głównym celem tego artykułu jest sprawdzenie, które z geometrycznych i fizycznych obiektów są niezmiennikami konforemnego przeskalowania metryki oraz przedstawienie konsekwencji konforemnego przeskalowania metryki takich jak kreacja energii i pędu dla pola grawitacyjnego oraz kreacja materii.

Słowa kluczowe: transformacja konforemna metryki, ogólna teoria względności, rozmaiłość riemannowska pseudoriemmanowska, lorentzowska, czasoprzestrzeń, równania Einsteina, pseudotensor energii-pędu Landaua-Lifszyca

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ
 2017 Vol. LXVII

 Recherches sur les déformations

no. 1

 pp. 33–44
*Agnieszka Palma***COMPLETE SUFFICIENT STATISTICS FOR MARKOV CHAINS****Summary**

This paper presents a complete sufficient statistic for the class of Markov chains with a finite state S for some natural subspaces $\mathcal{P}_{\mathcal{Z}}$ of the set of all transition probabilities $\mathcal{P}$. This statistic is the random transition count F for Markov chains with a fixed length and a fixed initial state. A summary of the recent results gives a brief exposition of completeness of F with some extra restrictions on the sets $\mathcal{Z}$ and S .

Keywords and phrases: Markov chain, random transition count, complete statistic, minimal sufficient statistic

1. Introduction

Complete sufficient statistics are one of the fundamental concepts in mathematical statistics and play a well known role in estimation theory. In particular these play the essential role in the theory of uniformly most powerful unbiased tests and in minimum variance unbiased estimation. Moreover, we found their application in source coding problems [9]. The concept of completeness of a sufficient statistic comes from Lehmann and Scheffe [3], [4].

This work is an attempt to summarize some of the recent results.

Let $(\mathbf{Y}, \mathcal{B}, \mathbb{M})$ be a statistical space, where $\mathbb{M}$ denotes a parametrized family $\{\mu_{\theta} : \theta \in \Theta\}$ of probability measures μ_{θ} on the measurable space $(\mathbf{Y}, \mathcal{B})$.

Definition 1.1. *As usually, we say that a family of distributions $\mathbb{M} = \{\mu_{\theta} : \theta \in \Theta\}$ on $(\mathbf{Y}, \mathcal{B})$ is complete if for any $\mathcal{B}$ measurable real valued function g the condition $\mathbb{E}_{\theta}(g) = 0$ for all $\theta \in \Theta$ implies $g = 0$ a.s. $\mathbb{M}$.*

Let $(X, \mathcal{A}, \mathbb{P})$, $\mathbb{P} = \{P_\theta : \theta \in \Theta\}$ be a statistical space and let $F : (\mathbf{X}, \mathcal{A}) \rightarrow (\mathbf{Y}, \mathcal{B})$ be a statistic. The statistics F is said to be *complete* if the family of its distributions $\mu_\theta^F = P_\theta(F^{-1}(\cdot))$ on $\mathbf{Y}$, $\theta \in \Theta$ is a complete family.

Fix integer $N \geq 2$, and $S = \{1, \dots, n\}$, $n \geq 2$. We shall use the notation $\mathbf{X}_N = \{(x_1, \dots, x_N) : x_i \in S, i = 1, 2, \dots, N\}$ for a Markov chain with a finite state space S and with a fixed length N . The evolution law is described by an unknown initial distribution $q = (q_1, \dots, q_n)$, where $q_i = P(x_1 = i)$, and an unknown stochastic matrix $p = (p_{i,j})$. The space of such distributions we denote by $\mathcal{Q}$ and the space of all such matrices by $\mathcal{P}$,

$$\mathcal{P} = \{p = (p_{i,j}) : \forall_{i,j \in S} \quad p_{i,j} \geq 0, \quad \sum_{j=1}^n p_{i,j} = 1 \text{ for each } i \in S\}.$$

In other words our family of Markov chains is parametrized by $\mathcal{Q} \times \mathcal{P}$.

Now, recall the definition of the random transition count F ,

$$F(\mathbf{x}) = F(x_1, \dots, x_N) = (f_{i,j})_{i,j=1,\dots,n},$$

$$f_{i,j} = \#\{t = 1, \dots, N-1 : x_t = i, x_{t+1} = j\} \text{ for } i, j \in S.$$

Obviously, F is a basic tool in any statistical investigation. It is well known that for fixed x_1 , F is a sufficient statistic. The problems of completeness of the random transition counts still have not been investigated enough. Denny and Wright [1] showed that if the initial state x_1 is fixed then F is a complete sufficient statistic for $\mathcal{P}$, but the proof was delicate. We note that the situation changes when x_1 is not fixed. It turns out that F is not sufficient and the natural statistic $(x_1, F(\mathbf{x}))$ is sufficient but in general is not complete.

Proposition 1.2. *Let $S = \{1, 2\}$, $N = 2$. The statistic $G(x_1, x_2) = (x_1, (f_{i,j}))$ is sufficient, but not complete.*

Proof. Indeed, for a fixed initial distribution (q_1, q_2) ; $q_1, q_2 \in [0, 1]$, $q_1 + q_2 = 1$ and for any transition probability matrix

$$p = \begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{bmatrix} \in \mathcal{P},$$

the statistic G takes the following values with the corresponding probabilities:

$$P(G = G_1) = q_1 \cdot p_{1,1} \quad \text{for } G_1 = \left(1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right),$$

$$P(G = G_2) = q_1 \cdot p_{1,2} \quad \text{for } G_2 = \left(1, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right),$$

$$P(G = G_3) = q_2 \cdot p_{2,1} \quad \text{for } G_3 = \left(2, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right),$$

$$P(G = G_4) = q_2 \cdot p_{2,2} \quad \text{for } G_4 = \left(2, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Then the expectation $\mathbb{E}_{q,p}(g \circ G)$ vanishes whenever the function g satisfies $g(G_1) = g(G_2) = q_2$, $g(G_3) = g(G_4) = -q_1$. $\square$

However, for a fixed N, S : there exists a complete sufficient statistic for the set of all stationary chains if and only if $S = \{1, 2\}$. It is mentioned on p. 338 of Denny and Wright [1]. By definition, the parameter space for the set of all stationary Markov chains with state space S is $\mathcal{P}_S = \{(p_{i,j}) : (p_{i,j}) \in \mathcal{P}, \text{ there exists at least one stationary initial distribution } (p_i) \text{ for } (p_{i,j})\}$. The following function is a complete sufficient statistic for $\mathcal{P}_S$:

$$G(x_1, \dots, x_N) = \begin{cases} (x_1, f_{1,1}, f_{2,2}, 0) & \text{if } f_{1,2} = f_{2,1}, \\ (0, f_{1,1}, f_{2,2}, f_{1,2} + f_{2,1}) & \text{if } f_{1,2} \neq f_{2,1}. \end{cases}$$

Next, it is shown in [10] that a complete sufficient statistic for the class of all stationary n -state Markov chains, $n \geq 3$, does not exist.

Theorem 1.3. *If $n \geq 3$ e.g. $S \neq \{1, 2\}$, then $(x_1, (f_{i,j}))$ is minimal sufficient statistic for $\mathcal{P}_S$ which is not complete. Consequently there does not exist a complete sufficient statistic for $\mathcal{P}_S$.*

2. The statistical assumptions

This paper is devoted to study completeness of the random transition count for largest possible class of Markov chains. The crucial concept is that we use stochastic matrices $\mathcal{P}_{\mathcal{Z}} \subset \mathcal{P}$:

Let $\mathcal{Z} \subset S \times S$ denote a fixed subset satisfying

$$(1) \quad \forall_{i \in S} \quad \exists_{j \in S} \quad (i, j) \notin \mathcal{Z}.$$

We denote by $\mathcal{P}_{\mathcal{Z}}$ the set of stochastic matrices $p \in \mathcal{P}$, such that

$$(2) \quad \forall_{(i,j) \in \mathcal{Z}} \quad p_{i,j} = 0.$$

Now we assume that the initial state x_1 is fixed (e.g. $q_i = \delta_{x_1}(i)$, $i \in S$). Thus probability distribution on the space $\mathbf{X}_{x_1, N} = \{x_1\} \times S^{N-1}$ is given by the formula

$$P_{(p_{i,j})}(\{\mathbf{x}\}) = p_{x_1, x_2} \cdot \dots \cdot p_{x_{N-1}, x_N} \quad \text{for } \mathbf{x} = (x_1, \dots, x_N) \in \mathbf{X}_{x_1, N}.$$

Throughout the paper we are dealing with a statistical space of the form

$$(\mathbf{X}_{x_1, N}, \{P_{(p_{i,j})} : (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}\}).$$

Obviously, the assumption $p \in \mathcal{P}_{\mathcal{Z}}$ means that the transitions in one step from i to j are forbidden for $(i, j) \in \mathcal{Z}$. The space $\mathcal{P}_{\mathcal{Z}}$ remains non empty because of (1). This condition fits well to the characterization of some classical types of Markov chains.

Example 2.1. Assume that $S = \mathbb{Z}$. Then $(x_1, \dots, x_N)$ is a random walk with stationary transition probability, possibly depending on position and direction, if and only if the matrix p is taken from $\mathcal{P}_{\mathcal{Z}}$ with

$$(S \times S) \setminus \mathcal{Z} = \{(i, i + \epsilon); i \in S, \epsilon = \pm 1\}.$$

Example 2.2. Assume now that $S = \{1, \dots, d\}^k$, $x_t = (y_t^1, \dots, y_t^k) \in S$ for $t = 1, \dots, N$, and let the evolution of x_t be given by a matrix $(p_{\mathbf{i}\mathbf{j}})$, $\mathbf{i} = (i^1, \dots, i^k)$, $\mathbf{j} = (j^1, \dots, j^k) \in S$. Then (x_t) is a Markov chain of order k , with a state space $S_1 = \{1, \dots, d\}$, if and only if $(p_{\mathbf{i}\mathbf{j}}) \in \mathcal{P}_{\mathcal{Z}}$ with

$$(S \times S) \setminus \mathcal{Z} = \{(\mathbf{i}, \mathbf{j}) : (i^2, \dots, i^k) = (j^1, \dots, j^{k-1})\}.$$

The random transition count F could be non-complete for some space $\mathcal{P}_{\mathcal{Z}}$, what we may see in an

Example 2.3. Let $S = \{1, 2, 3, 4\}$, $N = 5$ and let $x_1 = 3$ be fixed. For the space $\mathcal{P}_{\mathcal{Z}}$ with

$$(S \times S) \setminus \mathcal{Z} = \{(1, 3), (2, 3), (3, 2)(3, 4), (4, 1)\}$$

the statistic F is not complete.

Proof. For the set $\mathcal{Z}$ any matrix $p \in \mathcal{P}_{\mathcal{Z}}$ can be written as

$$p = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & q & 0 & 1-q \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad q \in [0, 1].$$

The statistic $F(\mathbf{x}) = (f_{i,j})$ takes on the following values with corresponding probabilities

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$P(F(\mathbf{x}) = M_1) = (1 - q)q, \quad P(F(\mathbf{x}) = M_2) = (1 - q)q,$$

Then the expectation $\mathbb{E}_p(g \circ F)$ vanishes for any non-zero function g satisfying $g(M_1) = -g(M_2)$. $\square$

Thus it is necessary to put some extra assumptions about the set $\mathcal{Z}$ and S (cf. condition in Section 4).

3. Auxiliary results and the graph theory

For the convenience of the reader we repeat the relevant material from [5], [6], [7] without proofs, thus making our exposition self-contained.

In what follows, we introduce a specific notation concerning “tables” of numbers. The proofs of lemmas are based on some combinatorial tools.

For any matrix f of dimension $n \times n$, let $f' = (f'_{i,j})$ denote a “table” being the matrix f with deleted elements $f_{i,i+1}$, $i = 1, \dots, n$ ($f_{n,n+1} \equiv f_{n,1}$). More precisely

$$(3) \quad f'_{i,j} = f_{i,j} \quad \text{for } j \neq i+1 \pmod{n}$$

and

$$(4) \quad f' = (f'_{i,j})_{i=1,\dots,n; j=1,\dots,i,i+2,\dots,n}.$$

Lemma 3.1. Fix $i', i'' \in S$ and $N \geq 2$. Let $\mathcal{M}_{N-1}^{i', i''}$ denote a set of matrices f of dimension $n \times n$ satisfying

- (i) $\sum_{i,j} f_{i,j} = N - 1$,
- (ii) $\sum_j f_{i,j} + \delta_{i''}(i) = \sum_j f_{j,i} + \delta_{i'}(i) \quad \text{for } i \in S$,

with δ being the Kronecker delta. The function $f \rightarrow f'$ defined by (3), (4) is one-one on the class $\mathcal{M}_{N-1}^{i', i''}$. There exist functions $\phi_i^{i', i''}$ on tables f' satisfying (4), such that

$$\phi_i^{i', i''}(f') = f_{i,i+1} \quad \text{for } i \in S.$$

Lemma 3.2. Fix $i' \in S$ and $N \geq 2$. Let $\mathcal{M}_{N-1}^{i'}$ denote a set of matrices f of dimension $n \times n$ with integer elements satisfying

- (I) $\sum_{i,j} f_{i,j} = N - 1$,
- (II) there exists $i'' \in S$ such that

$$\sum_j f_{i,j} + \delta_{i''}(i) = \sum_j f_{j,i} + \delta_{i'}(i) \quad \text{for } i \in S.$$

The function $f \rightarrow f'$ defined by (3), (4) is one-one on the class $\mathcal{M}_{N-1}^{i'}$. There exist functions $\phi_i^{i'}$ with integer values, such that

$$\phi_i^{i'}(f') = f_{i,i+1} \quad \text{for } i \in S,$$

for f' satisfying (3) and (4).

Corollary 3.3. The function $f \rightarrow f'$ defined by (3), (4) is one-one on the class $F[\mathbf{x}]$ of the random transition counts F for the trajectories $\mathbf{x} \in \mathbf{X}$ with a fixed initial state $x_1 = i'$ and a fixed length $N \geq 2$.

Fix $i', i'' \in S = \{1, \dots, n\}$. For any matrix f of dimension $n \times n$, let $\bar{f} = (\bar{f}_{i,j})$ denote a "table" being the matrix f with deleted elements $f_{i,i+1}$, $i = 1, \dots, n$, $i \neq i''$, ($f_{n,n+1}$ is another notation for $f_{n,1}$). More precisely we use a set of indices

$$S_2^{i', i''} = \{(i, j) \in S \times S : j \in S \setminus \{i+1\} \text{ for } i \in S \setminus \{i'', n\}, \\ j \in S \text{ for } i = i'', j \in S \setminus \{1\} \text{ for } i = n \text{ in the case } n \neq i''\}$$

and define

$$(5) \quad \bar{f}_{i,j} = f_{i,j} \quad \text{for } (i, j) \in S_2^{i', i''},$$

$$(6) \quad \bar{f} = (\bar{f}_{i,j})_{(i,j) \in S_2^{i', i''}}.$$

Lemma 3.4. *Fix $i', i'' \in S$. Let $\mathcal{M}^{i', i''}$ denote a set of matrices $f = (f_{i,j})$ of dimension $n \times n$ satisfying*

$$\sum_j f_{i,j} + \delta_{i''}(i) = \sum_j f_{j,i} + \delta_{i'}(i) \quad \text{for } i \in S.$$

The function $f \rightarrow \bar{f}$ defined by (5), (6) is one-one on the class $\mathcal{M}^{i', i''}$. There exist functions $\psi_i^{i', i''}$ for $i \neq i''$, defined on tables $(f_{i,j})_{(i,j) \in S_2^{i', i''}}$, with non-negative integer values satisfying

$$\psi_i^{i', i''}(\bar{f}) = f_{i,i+1} \quad \text{for } i \in S \setminus \{i''\} \text{ and any } f \in \mathcal{M}^{i', i''}.$$

Corollary 3.5. *The function $f \rightarrow \bar{f}$ defined by (5), (6) is one-one on the class $\{F(\bar{\mathbf{x}}) : \bar{\mathbf{x}} = (x_1, \dots, x_t), x_1 = i', x_t = i'', 1 \leq t < N\}$ of all values of random transition counts for the trajectories $\bar{\mathbf{x}}$ with a fixed initial state i' and a fixed final state i'' (and any length).*

We will need the following lemma which was used by Denny and Wright. Assume that integers $n \geq 1$, $j(1) \geq 1, \dots, j(n) \geq 1$, $q \geq 0$ and real $c > 0$ are fixed. Denote by $\mathcal{U}$ the set of all systems of positive numbers $u = (u_{i,j})$, $1 \leq i \leq n$, $1 \leq j \leq j(i)$ such that

$$\sum_{j=1}^{j(i)} u_{i,j} \leq c \quad \text{for any } 1 \leq i \leq n.$$

Denote by M the set of systems of non-negative integers $m = (m_{i,j})$, $1 \leq i \leq n$, $1 \leq j \leq j(i)$ satisfying

$$\sum_{i=1}^n \sum_{j=1}^{j(i)} m_{i,j} \leq q.$$

Let $\varphi_i : M \rightarrow \{0, 1, 2, \dots\}$, $i = 1, \dots, n$ be any given functions. For each $m \in M$ let us define a function $W_m : \mathcal{U} \rightarrow \mathbb{R}$, as follows

$$W_m(u) = \prod_{i=1}^n \left[\prod_{j=1}^{j(i)} u_{i,j}^{m_{i,j}} \cdot (c - u_{i,1} - \dots - u_{i,j(i)})^{\varphi_i((m_{i,j}))} \right].$$

Lemma 3.6. *The system of functions $W_m : \mathcal{U} \rightarrow \mathbb{R}$, indexed by $m \in M$, is linearly independent.*

Proof. Cf. Lemma 2 in [1]. □

The following lemmas from graph theory go back to the work by A. Paszkiewicz.

Definition 3.7. *Let $Z \subset S \times S$ be a fixed set, fix $i', i'' \in S$. We say that an oriented graph (Y, U) , $Y \subset S$, with Y being the set of vertices and $U \subset Y \times Y$ being the set of edges, is defined by Z, i', i'' if*

$$Y = \bigcup \{x_1, \dots, x_N\}, \quad U = \bigcup \{(x_1, x_2), \dots, (x_{N-1}, x_N)\}$$

with the unions taken for all sequences (x_t) satisfying $(x_t, x_{t+1}) \notin Z$, $t = 1, \dots, N-1$, and $x_1 = i'$, $x_N = i''$.

As usual, we say that (Y_0, V_0) is a cycle if

$$Y_0 = \{y_1, \dots, y_s\},$$

$$V_0 = \{(y_1, y_2), \dots, (y_{s-1}, y_s), (y_s, y_1)\}.$$

A graph (Y_1, V_1) is a tree with root y if for any $z \in Y_1$ there exists exactly one path (Y_z, V_z) of the form

$$Y_z = \{z_1 = z, z_2, \dots, z_s = y\} \subset Y_1,$$

$$V_z = \{(z_1, z_2), \dots, (z_{s-1}, z_s)\} \subset V_1,$$

($z_1, \dots, z_s$ are mutually different and $s \geq 1$, with $V_z = \emptyset$ for $s = 1$).

Lemma 3.8. *For any graph (Y, U) defined by Z, i', i'' there exists a tree (Y, W) , $W \subset U$, with root i'' .*

Let F be the random transition count and let the evolution be given by transition probabilities p from $\mathcal{P}_{\mathcal{Z}}$ with a fixed set $\mathcal{Z}$. In the following lemma we use the notion of a tree to describe some general properties of F .

Let $Y = \{1, \dots, n\}$ and let a graph (Y, U) be defined by Z, i', i'' . The value $F(\mathbf{x}) = (f_{i,j})$ for any trajectory $\mathbf{x}$ and satisfies $f_{i,j} = 0$ for $(i, j) \notin U$. Thus $F(\mathbf{x})$ can be identified with some function $m : U \rightarrow \{0, 1, \dots\}$, $m = f|_U$, and obviously

$$\sum_{j \in Y, (i,j) \in U} m_{i,j} + \delta_{i''}(i) = \sum_{j \in Y, (j,i) \in U} m_{j,i} + \delta_{i'}(i) \quad \text{for } i \in Y.$$

Denote by M the space of all such function $m : U \rightarrow \{0, 1, 2, \dots\}$.

Lemma 3.9. *Let (Y, U) be defined by Z, i', i'' . For any tree (Y, V) with root i'' and with $V \subset U$, denote by $j(\cdot)$ the uniquely defined function on $Y \setminus \{i''\}$ satisfying $(i, j(i)) \in V$. Then there exist functions Φ_i on $\{m|_{U \setminus V}; m \in M\}$ with non-negative integer values satisfying*

$$m_{i, j(i)} = \Phi_i(m|_{U \setminus V})$$

for any $i \in Y$, $i \neq i''$, $m \in M$.

4. Completeness of the random transition count

In Section 1 we reviewed some of the standard facts of the complete statistic. Now we present some recent results of the completeness of the random transition count for some special classes of transition probabilities.

We know that the random transition count F is complete for Markov chains with a fixed length and a fixed initial state. Moreover, the statistic F is always complete for Markov bridge.

The random walk $\mathbf{x} = (x_1, \dots, x_N)$ is a *Markov bridge*, if $x_1 = i'$, $x_N = i''$ are fixed. Thus in canonical representation

$$(7) \quad P_{(p_{i,j})}(\{(x_1, \dots, x_N)\}) = c \delta_{i'}(x_1) \delta_{i''}(x_N) \prod_{t=1}^{N-1} p_{x_t x_{t+1}}$$

with some positive constant c . **Theorem 4.1.**

Theorem 4.1. *For a Markov bridge and for the evolution laws given by $p \in \mathcal{P}_{\mathcal{Z}}$, cf. (1), (2), (7), the random transition count F is complete.*

Proof. Cf. Theorem 3.1. in [7]. The basic idea of this proof is to apply Lemma 3.6 and graph theory. $\square$

We first present a reduced form of the main result which will be useful in its proving. Let $\mathcal{Z} \in S \times S$ be a fixed set. We always assume that $\mathcal{Z}$ satisfies (1), that is

$$\forall_{i \in S} \quad \exists_{j \in S} \quad (i, j) \notin \mathcal{Z}.$$

Recall that using $\mathcal{Z}$ one can naturally define the class S_0 of inessential states and classes of equivalence $S_1, \dots, S_{\beta_0}$, $\beta_0 \in \mathbb{N}$, in essential states.

To prove the completeness of the statistic F , we need some more special properties of the set $\mathcal{Z}$. Let us remark that, by Example 2.3, such additional assumptions about the set $\mathcal{Z}$ are necessary.

Now we construct the set $\mathcal{Z}$ such that *the state space S will be the whole class of essential states* and we assume that there exists a permutation $(\pi_1, \pi_2, \dots, \pi_n)$ of the set S , satisfying

$$(8) \quad \{(\pi_1, \pi_2), \dots, (\pi_{n-1}, \pi_n), (\pi_n, \pi_1)\} \cap \mathcal{Z} = \emptyset.$$

Theorem 4.2. *Let $\mathcal{Z}$ satisfy (1) and (8). Let $(\mathbf{X}_{x_1, N}, \{P_{(p_{i,j})} : (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}\})$, with $\mathcal{P}_{\mathcal{Z}}$ given by (2), be the statistical space of all trajectories of Markov chains with the state space $S = \{1, \dots, n\}$, a fixed initial state $x_1 = i'$, and the trajectory size $N \geq 2$. Then the random transition count F is complete.*

Proof. To show that the statistic F is complete, it is enough to prove that condition

$$(9) \quad \sum_f d(f) \cdot \xi(x_1, f) \cdot \prod_{i=1}^n \prod_{j=1}^n p_{i,j}^{f_{i,j}} = 0 \quad \text{for each } (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}$$

implies that

$$d(f) = 0 \quad \text{for each } f,$$

where $\xi(x_1, f) = \#\{\mathbf{x} : F(\mathbf{x}) = f\}$ denote the number of corresponding trajectories. Applying assumptions of theorem, notations from Section 3, the equality (9) can be written as

$$\sum_{f'} d(f') \cdot W_{f'}(p) = 0,$$

for any $p = (p_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m(i), j \neq i+1 \pmod n}$,

where

$$W_{f'}(p) = \prod_{i=1}^n \left(\prod_{\substack{j=1 \\ j \neq i+1}}^{m(i)} p_{i,j}^{f'_{i,j}} \cdot \left(1 - \sum_{\substack{j=1 \\ j \neq i+1}}^{m(i)} p_{i,j} \right)^{\phi_i(f')} \right).$$

Thus Lemma 3.6 completes the proof. $\square$

In order to obtain completeness of F in a more general case it is necessary to make some extra assumptions about the set $\mathcal{Z}$ and S . Let $\mathcal{Z} \subset S \times S$ be a fixed set satisfying (1), that is

$$\forall i \in S \quad \exists j \in S \quad (i, j) \notin \mathcal{Z}.$$

Let S_0 denote the class of inessential states and let $S_1, \dots, S_{\beta_0}$, $\beta_0 \in \mathbb{N}$ denote classes of equivalence in essential states.

Now we formulate assumptions about S :

(I) for each β , $1 \leq \beta \leq \beta_0$ there exists a permutation $(i_1^\beta, \dots, i_{n(\beta)}^\beta)$ of set S_β such that

$$\{(i_1^\beta, i_2^\beta), \dots, (i_{n(\beta)-1}^\beta, i_{n(\beta)}^\beta), (i_{n(\beta)}^\beta, i_1^\beta)\} \cap \mathcal{Z} = \emptyset;$$

(II) for each β , $1 \leq \beta \leq \beta_0$ there exists exactly one pair $(i_\beta, j_\beta) \in S_0 \times S_\beta$ such that

$$(i_\beta, j_\beta) \notin \mathcal{Z}.$$

Fix β , $1 \leq \beta \leq \beta_0$. Our statistical space $(\mathbf{X}_{\beta, N}, \{P_{(p_{i,j})}^{\beta, N} : (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}\})$ is defined as follows:

(i) the space $\mathbf{X}_{\beta, N}$ of trajectories $(x_1, \dots, x_N)$ is determined by a fixed length N , a fixed initial state $x_1 = i'$ and a fixed essential state class S_β such that the final state x_N belongs to S_β ;

(ii) the probability distribution

$$P^{\beta, N}_{(p_{i,j})}(\{\mathbf{x}\}) = p_{x_1, x_2} \cdot \dots \cdot p_{x_{N-1}, x_N} \quad \text{for} \quad \mathbf{x} = (x_1, \dots, x_N) \in \mathbf{X}_{\beta, N}.$$

Without loss of generality we adopt $\beta = 1$.

Before we present our main result, let us observe that the random transition count F is a complete statistic if additionally

(III) there exists a permutation $(i_1^0, \dots, i_{n(0)}^0)$ of the class S_0 of inessential states such that

$$\{(i_1^0, i_2^0), \dots, (i_{n(0)-1}^0, i_{n(0)}^0), (i_{n(0)}^0, i_1^0)\} \cap \mathcal{Z} = \emptyset.$$

Theorem 4.3. *Let $\mathcal{Z}$ satisfy (1) and conditions (I)-(III), and let the statistical space $(\mathbf{X}_{1,N}, \{P_{(p_{i,j})}^{1,N} : (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}\})$ as above be given. Then the random transition count F is complete.*

Proof. We only give the main ideas of the proof. Assume first that $x_1 = i' \in S_1$ is an essential state. Then one can assume that the space S is one class of essential states. Theorem 4.2 completes the proof in this case.

Now, let $x_1 = i' \in S_0$ be an inessential state. Let us consider a trajectory $\mathbf{x} \in \mathbf{X}_{1,N}$. Denote $t(\mathbf{x}) < N$ the number of steps on inessential states, so we perform $N-1-t(\mathbf{x})$ steps on S_1 .

According to (9) we have

$$(10) \quad \forall_{p \in \mathcal{P}_{\mathcal{Z}}} \left(\sum_{f=(f_{i,j})} \left(d(f) \cdot \xi(x_1, f) \cdot \prod_{i,j \in S_0} p_{i,j}^{f_{i,j}} \cdot p_{i'', i'''} \cdot \prod_{i,j \in S_1} p_{i,j}^{f_{i,j}} \right) = 0 \right).$$

By condition (III), Lemma 3.4 and Corollary 3.5, the factor

$$\prod_{i,j \in S_0} p_{i,j}^{f_{i,j}} \cdot p_{i'', i'''}$$

can be written as

$$(11) \quad \prod_{\substack{i \in S_0 \\ i \neq i''}} \prod_{\substack{j \in S_0 \\ j \neq i+1 \pmod{n(0)}}} p_{i,j}^{\bar{f}_{i,j}} \cdot \left(1 - \sum_{\substack{j \in S_0 \\ j \neq i+1 \pmod{n(0)}}} p_{i,j} \right)^{\psi_i(\bar{f})} \cdot \prod_{j \in S_0} p_{i'', j}^{\bar{f}_{i'', j}} \cdot \left(1 - \sum_{j \in S_0} p_{i'', j} \right)^{\psi_{i''}(\bar{f})},$$

obviously, $p_{i'', i'''} = 1 - \sum_{j \in S_0} p_{i'', j}$ and we have put $\psi_{i''}(\bar{f}) = 1$.

Observe that by a suitable change of notation the product (11) can be written as some polynomials $W_m(u)$ described in Section 3. Thus Lemma 3.6 and Theorem 4.2 completes the proof, because in the class S_1 , the number of steps on essential states is known and a fixed essential state $i''' = j_1$ (cf. (II)) is the initial state. $\square$

We can now formulate our main result.

Theorem 4.4. *Let $\mathcal{Z} \subset S \times S$ satisfy (1) and conditions (I), (II). Let*

$$(\mathbf{X}_{1,N}, \{P_{(p_{i,j})}^{1,N} : (p_{i,j}) \in \mathcal{P}_{\mathcal{Z}}\})$$

be the statistical space. Then the random transition count F is complete.

Proof. For $x_1 \in S_1$ Theorem 4.2 completes proof.

Now, assume that $x_1 = i' \in S_0$. We want to show that the statistic F is complete, that is that the condition (9) implies that $d(f) = 0$ for each f .

Let us consider a trajectory $\mathbf{x} \in \mathbf{X}_{1,N}$. Let $t(\mathbf{x})$ denote the number of steps on inessential states in the class S_0 . For the first part $(x_1, \dots, x_{t(\mathbf{x})})$ of trajectory $\mathbf{x}$ we will consider an oriented graph $(\bar{S}_0, U)$, $U \subset \bar{S}_0 \times \bar{S}_0$ defined by $\mathcal{Z}$, i' , i'' , that is $\bar{S}_0 = \bigcup \{x_1, \dots, x_{t(\mathbf{x})}\}$; $U = \bigcup \{(x_1, x_2), \dots, (x_{t(\mathbf{x})-1}, x_{t(\mathbf{x})})\}$, with the unions taken for all $\mathbf{x}$ satisfying $x_1 = i'$, $x_{t(\mathbf{x})} = i''$ and $(x_t, x_{t+1}) \notin \mathcal{Z}$, $t = 1, \dots, t(\mathbf{x}) - 1$. By Lemma 3.8 there exists a tree $(\bar{S}_0, W)$, $W \subset U$ with a root i'' . Next applying Lemma 3.9 and putting $p_{i'',j} = 1 - \sum_{j \in \bar{S}_0} p_{i'',j}$, $f_{i'',i''} = \Phi_{i''}(m|_{U \setminus W}) = 1$, where $m = f|_U$, we can write the factor

$$\prod_{i,j \in S_0} p_{i,j}^{f_{i,j}} \cdot p_{i'',i''}$$

in (10) in the form

$$(12) \quad \prod_{\substack{i \in \bar{S}_0 \\ i \neq i''}} \prod_{\substack{j \in \bar{S}_0 \\ (i,j) \in U \setminus W}} p_{i,j}^{m_{i,j}} \cdot \left(1 - \sum_{\substack{j \in \bar{S}_0 \\ (i,j) \in U \setminus W}} p_{i,j}\right)^{\Phi_i(m|_{U \setminus W})} \cdot \prod_{j \in \bar{S}_0} p_{i'',j}^{m_{i'',j}} \cdot \left(1 - \sum_{\substack{j \in \bar{S}_0 \\ (i,j) \in U \setminus W}} p_{i'',j}\right)^{\Phi_{i''}(m|_{U \setminus W})},$$

for a system of functions Φ_i , $i \neq i''$, on the space $\{m|_{U \setminus W}\}$.

The set $U \setminus W$ can be written as

$$U \setminus W = \{(i, j) : i \in \bar{S}_0, j \in \bar{S}_0^i\},$$

where

$$\bar{S}_0^i = \{j \neq i + 1 \pmod{\bar{n}(0)}\} \text{ for } i \in S_0 \text{ and } i \neq i'', \\ \bar{S}_0^{i''} = \bar{S}_0.$$

The rest of the proof is analogical to that of Theorem 4.3. □

Because the sufficiency of F is obvious, so from Bahadur's Theorem [8] we have a

Conclusion. F is the minimal sufficient statistic.

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Presented by Adam Paszkiewicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on January 27, 2016.

ZUPEŁNE STATYSTYKI DOSTATECZNE DLA ŁAŃCUCHÓW MARKOWA

S t r e s z c z e n i e

Praca ta prezentuje zupełność statystyk dostatecznych dla łańcuchów Markowa o skończonej przestrzeni stanów S dla pewnej naturalnej podprzestrzeni przestrzeni wszystkich macierzy prawdopodobieństw przejść. Statystyka będąca macierzą ilości przejść może być zupełną statystyką dostateczną, ale przy pewnych dodatkowych założeniach. Odpowiednie przykłady pokazują, że założenia te są konieczne. Praca jest próbą podsumowania najważniejszych wyników.

Słowa kluczowe: łańcuch Markowa, macierz ilości przejść, statystyka zupełna, statystyka dostateczna

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2017

Vol. LXVII

Recherches sur les déformations

no. 1

pp. 45–48

*Anatoly S. Serdyuk***ON OPTIMAL SUBSPACES FOR KOLMOGOROV WIDTHS OF CLASSES OF 2π -PERIODIC ANALYTIC FUNCTIONS****Summary**

The following problem arises: to find optimal subspaces for Kolmogorov widths of classes of convolutions with generated kernels, which may increase the oscillations.

Keywords and phrases: Kolmogorov width, optimal subspace, Poisson kernel, kernel of analytic functions, CVD-kernels

Let X be a linear normed space and let $\mathfrak{N}$ a centrally symmetric set from X . The quantity

$$(1) \quad d_m(\mathfrak{N}, X) = \inf_{F_m \subset X} \sup_{f \in \mathfrak{N}} \inf_{g \in F_m} \|f - g\|_X, \quad m \in \mathbb{N},$$

is called *the m -dimensional Kolmogorov width* where the external inf is taken over all possible m -dimensional linear subspaces F_m from X [1].

The subspace $F_m^* \subset X$ for which the equality

$$d_m(\mathfrak{N}, X) = E(\mathfrak{N}, F_m^*)_X = \sup_{f \in \mathfrak{N}} \inf_{g \in F_m^*} \|f - g\|_X$$

is satisfied, is called *the extremal subspace* for width $d_m(\mathfrak{N}, X)$.

We consider the space $L = L_1$ functions φ of 2π -periodic summable on $[0, 2\pi)$ with the norm

$$\|\varphi\|_L = \|\varphi\|_1 = \int_0^{2\pi} |\varphi(t)| dt,$$

Let L_∞ be the space of measurable and essentially bounded 2π -periodic functions φ with the norm

$$\|\varphi\|_\infty = \operatorname{ess\,sup}_{t \in \mathbb{R}} |\varphi(t)|$$

and C the space of continuous 2π -periodic functions φ with the norm

$$\|\varphi\|_C = \max_{t \in \mathbb{R}} |\varphi(t)|.$$

Let $C_{\beta,p}^\psi$ denote the class of 2π -periodic functions f , representable as convolutions of the form

$$(2) \quad f(x) = A + (\Psi_\beta * \varphi)(x) = A + \frac{1}{\pi} \int_{-\pi}^{\pi} \Psi_\beta(x-t) \varphi(t) dt, \quad A \in \mathbb{R},$$

where $\varphi \perp 1$, $\|\varphi\|_p \leq 1$, $p = 1, \infty$ and Ψ_β is a fixed kernel of the form

$$\Psi_\beta(t) = \sum_{k=1}^{\infty} \psi(k) \cos\left(kt - \frac{\beta\pi}{2}\right), \quad \psi(k) > 0, \quad \sum_{k=1}^{\infty} \psi(k) < \infty, \quad \beta \in \mathbb{R}.$$

In the case, when $\psi(k) = q^k$, $q \in (0, 1)$, the classes $C_{\beta,p}^\psi$ are classes of convolutions with the Poisson kernels $P_{q,\beta}(t)$ of the form

$$(3) \quad P_{q,\beta}(t) = \sum_{k=1}^{\infty} q^k \cos\left(kt - \frac{\beta\pi}{2}\right), \quad q \in (0, 1), \quad \beta \in \mathbb{R};$$

we denote them by $C_{\beta,p}^q$. These classes consist (see, e.g. [2, Ch. 3, §8]) of functions that admit a regular extension to the strip $|\operatorname{Im} z| < \ln 1/q$ in the complex plane.

If $\psi(k) = \frac{1}{\operatorname{ch} kh}$, $h > 0$, then the classes $C_{\beta,p}^\psi$ are denoted by $C_{\beta,p}^h$ and are classes of convolutions with kernels of analytic functions

$$(4) \quad H_{h,\beta}(t) = \sum_{k=1}^{\infty} \frac{1}{\operatorname{ch} kh} \cos\left(kt - \frac{\beta\pi}{2}\right), \quad h > 0, \quad \beta \in \mathbb{R}.$$

Functions from the classes $C_{\beta,p}^h$ admit a regular extension to the strip $|\operatorname{Im} z| < h$ in the complex plane (see, e.g. [2, Ch. 3, §8]).

In works [3]–[11] the exact values of widths $d_{2n-1}(C_{\beta,\infty}^q, C)$, $d_{2n}(C_{\beta,\infty}^q, C)$, $d_{2n-1}(C_{\beta,1}^q, L)$, $d_{2n-1}(C_{\beta,\infty}^h, C)$, $d_{2n}(C_{\beta,\infty}^h, C)$, $d_{2n-1}(C_{\beta,1}^h, L)$ were found for all numbers n , beginning with some numbers n_q and n_h , respectively. Herewith, it was shown that in the case $\psi(k) = q^k$, $q \in (0, 1)$, or $\psi(k) = \frac{1}{\operatorname{ch} kh}$, $h > 0$, the equalities take place:

$$(5) \quad \begin{aligned} d_{2n-1}(C_{\beta,\infty}^\psi, C) &= d_{2n}(C_{\beta,\infty}^\psi, C) = d_{2n-1}(C_{\beta,1}^\psi, L) = \\ &= E(C_{\beta,\infty}^\psi, \mathcal{T}_{2n-1})_C = E(C_{\beta,1}^\psi, \mathcal{T}_{2n-1})_L = \|\Psi_\beta * \operatorname{sign} \sin(\cdot)\|_C, \end{aligned}$$

where $\mathcal{T}_{2n-1}$ is the subspace of trigonometric polynomials T_{n-1} of order $n-1$:

$$T_{n-1}(t) = a_0 + \sum_{k=1}^{n-1} a_k \cos kt + b_k \sin kt, \quad a_k, b_k \in \mathbb{R}.$$

Equalities (5) imply, that subspaces $\mathcal{T}_{2n-1}$ of trigonometric polynomials T_{n-1} of order $n-1$ are optimal for odd widths $d_{2n-1}(C_{\beta,\infty}^\psi, C)$ and $d_{2n-1}(C_{\beta,1}^\psi, L)$.

At the same time, as it follows from [12, pp. 180, 183] in the case, where kernels Ψ_β of convolutions of the form (2) are CVD-kernels (Cyclic Variation Diminishing Kernels), i.e.

$$\nu(\Psi_\beta * \varphi) \leq \nu(\varphi) \quad \forall \varphi \in C,$$

where $\nu(g)$ is the number of changes of the sign of a function $g \in C$ on $[0, 2\pi)$, the equalities (5) take place for all positive integers n and subspaces

$$(6) \quad S_{2n}(\psi, \beta) = \text{span} \left\{ \Psi_\beta \left(\cdot - \frac{k\pi}{n} \right) \right\}_{k=1}^{2n}$$

are optimal subspaces for widths $d_{2n}(C_{\beta, \infty}^\psi, C)$.

The kernels $H_{h,0}(t) = \sum_{k=1}^{\infty} \frac{1}{\text{ch } kh} \cos kt$ are CVD-kernels (see, e.g. [12, p. 128]), so all said above is true for the classes $C_{0,\infty}^h$ generated by these kernels. In that time in general situation, i.e. for arbitrary $\beta \in \mathbb{R}$, kernels $P_{q,\beta}(t)$ of the form (3) and kernels $H_{h,\beta}(t)$ of the form (4) may not be CVD-kernels. Then the following problem arises: *to find out (to prove or to disprove): are the spaces $S_{2n}(\psi, \beta)$ of the form (6) optimal for widths $d_{2n}(C_{\beta, \infty}^\psi, C)$ in the case where $\psi(k) = q^k$, $q \in (0, 1)$, or $\psi(k) = \frac{1}{\text{ch } kh}$, $h > 0$, for all $\beta \in \mathbb{R}$ and for all positive integers n , for which the equalities (5) are satisfied.*

Acknowledgement

The author acknowledges the Polish-Ukrainian grant no. 39/2014 *Topological-analytical methods in complex and hypercomplex analysis* of the Polish Academy of Sciences and National Academy of Sciences of Ukraine.

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Presented by Zbigniew Jakubowski at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 16, 2016.

O OPTYMALNYCH PODPRZESTRZENIACH DLA SZEROKOŚCI KOŁMOGOROWA KLAS 2π -OKRESOWYCH FUNKCJI ANALITYCZNYCH

S t r e s z c z e n i e

Stawiamy następujący problem: znaleźć optymalne podprzestrzenie dla szerokości Kołmogorowa klas splotów z wygenerowanymi jądrami, które mogą powiększyć oscylację.

Słowa kluczowe: szerokość Kołmogorowa, optymalna podprzestrzeń, jądro Poissona, jądro funkcji analitycznych, jądro zmniejszające cykliczną wariancję

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2017

Vol. LXVII

Recherches sur les déformations

no. 1

pp. 49–70

*Jan Jakóbowski***SOME NONCOMMUTATIVE RINGS CONSTRUCTED ON THE
BASE OF POLYNOMIALS $x^2 + tx + 1$ AND THEIR ZERO DIVISORS****Summary**

For any Galois field $F = GF(p^n)$ we construct some ring extension $\mathfrak{R}(F)$ of order p^{4n} . Such construction may be applied also for any infinite field F with $\text{char } F \neq 0$. Then, for any element of $\mathfrak{R}(F)$ we give necessary and sufficient condition to be a zero divisor. With additional assumptions, we make some variations on this condition. In special cases we are able to calculate easily the number of all zero divisors and of idempotents and nilpotents of degree 2. The method used to construct $\mathfrak{R}(F)$ is the following. First we find $t \in F$, such that the polynomial $x^2 + tx + 1$ does not have roots in F (such t -s exist!). Then we take the 4-dimensional F -vector space with basic elements $1, i, j, k$, where i, j, k not belonging to F are roots of $x^2 + tx + 1$ in the ring extension, and $ji = -k - t$. Thus multiplication of i, j, k (and hence in all ring) is some generalization of multiplication in the real Hamilton quaternions.

In consequence, we have got wide class of noncommutative rings. It is known very much on noncommutative rings of smaller order, e.g. in 1994 J. B. Derr, G. F. Orr, and P. S. Peck classified all noncommutative rings of order p^4 , using radical as a helpful tool. Thus, in the particular case $n = 1$, each of the rings constructed here must be of one kind given by Derr and the others. Our consideration is more general. It turns out that selected properties of $\mathfrak{R}(F)$ depend on $\text{char } F$ and on $t^2 - 2^2$ is a square in F or not. To get these and other results, we use some properties of multiplicative subgroup of nonzero squares in $GF(p^n)$ and of the polynomial $x^2 + tx + 1$. All contents is provided with examples illustrating general situation or special cases.

Keywords and phrases: Galois field, polynomial, noncommutative ring, zero divisor, vector space, skew field of quaternions

1. Introduction

Let F be any field with $\text{char } F = p$, where p is prime (eg. [6, p. 83]). It is obvious that there exists $t \in F$ such that the polynomial $x^2 + tx + 1$ is irreducible in F . Our construction of the ring (denoted by $\mathfrak{R}(F)$) is based on the following foundations:

$$i, j, k \notin F; \quad i^2 + ti + 1 = 0, \quad j^2 + tj + 1 = 0, \quad k^2 + tk + 1 = 0; \quad ji + k + t = 0.$$

Therefore, for $t = 0$, the construction is almost a copy of Hamilton algebra of all real quaternions [6, p. 380]; for $t = 1$ all general formulae become simpler and we are able to determine quickly some combinatorial properties. To vary and apply quaternions construction to a field F with $\text{char } F \neq 0$, some authors assume that $\text{char } F \neq 2$ and put $i^2 = a$, $j^2 = b$, $ij = -ji = k$, where a, b are not squares in F (cf. [1] and [7, p. 1314]). This way they usually obtain either infinite division ring or a ring isomorphic with the ring of matrices $M_{2 \times 2}(F)$ [8, p. 16, 19]. Note that the Hamilton algebra of all real quaternions is isomorphic to some subring of matrices $M_{4 \times 4}(R)$ [5, p. 16].

Our construction is much more general. We regard every nonzero characteristic p and some properties of the ring depends on what this characteristic is.

On the other hand, J. B. Derr and the others in [2] have described all the isomorphism classes of noncommutative rings of order p^4 . Their classification depends only on the radical of a ring. The most of their classes consists of special type of matrices and seemingly has no connection with our construction. However, in the case of order p^4 the rings constructed here must belong to classes given in [2].

More examples of not-division rings based on quaternions construction, are given in [5, §1].

1.1. Preliminaries useful for farther consideration

Remark 1.1. • *For every field F , the multiplicative subgroup of all nonzero squares will be denoted by S .*

- *In this paper a nilpotent (resp. idempotent) z of degree 2 (i.e. $z^2 = 0$ or $z^2 = z$, respectively) will be shortly called a nilpotent (resp. idempotent).*

From Fermat's two squares theorem [3] the following is immediate.

Proposition 1.2. [4, p. 251–252] *Let $GF(p^n)$ be the Galois field with odd p . Then $p^n = 4m + 1$ for some positive integer m , if and only if for every $a \in GF(p^n)$ we have $a \in S \Leftrightarrow -a \in S$. In particular, the element -1 is a square iff $p^n = 4m + 1$.*

We apply properties of polynomial $x^2 + tx + 1$, e.g. the following:

Remark 1.3. *If a is a root of the polynomial $x^2 + tx + 1$ over a field F , then $a^{-1} = -a - t$ is the second root. Moreover, if $t = 1$, then $a^{-1} = a^2$. If a is a root of $x^2 + tx + 1$ then $x^2 - tx + 1$ has the roots $-a$ and $a + t$.*

The following property (proved in further part of the work) is very useful for our consideration: the equation $x^2 + y^2 - txy = 0$ over F , where $\text{char } F > 3$, has a nonzero solution if and only if $t^2 - 2^2$ is a square in F . Using this condition we are able to evaluate, more or less precisely, the numbers of all zero divisors, idempotents and nilpotents.

2. Construction of the ring and its selected properties

Let us take any field F of nonzero characteristic and $t \in F$, such that the polynomial $x^2 + tx + 1$ has no roots in F . Then $-2 \neq t \neq 2$ for $\text{char } F \neq 2$. Let i, j, k are elements not belonging to F , and assume:

$$(2.1) \quad i^2 + ti + 1 = 0, \quad j^2 + tj + 1 = 0, \quad k^2 + tk + 1 = 0, \quad ji + k + t = 0.$$

Then we take the standard vector space $V = \{a + bi + cj + dk; a, b, c, d \in F\}$ over F with the base $1, i, j, k$ and we have to define multiplication so that to get a ring. In particular, we require multiplication to be associative and both-side distributive with respect to addition. Therefore, by (2.1), we obtain subsequently:

$$\begin{aligned} -i &= (j^2 + tj)i = (j + t)(ji) = (j + t)(-k - t) = -jk - tj - tk - t^2, \\ &\quad \text{i.e. } jk = i - t(j + k + t)]; \\ -j &= j(k^2 + tk) = (jk)(k + t) = (jk)k + tjk = \\ &= (i - tj - tk - t^2)k + tjk = ik - tjk - tk^2 - t^2k + tjk = \\ &= ik - t(k^2 + tk) = ik + t \quad \text{i.e. } ik = -j - t. \end{aligned}$$

In the same way we subsequently obtain ij, kj and ki ; thus multiplication on basic elements is defined as follows.

Definition 2.1. *Multiplication of basic elements is given in Table 1, where an element from heading column is taken as the first, from heading row – as the second one.*

Tab. 1: Multiplication of basic elements

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	$-1 - ti$	$k - t(i + j + t)$	$-j - t$
j	j	$-k - t$	$-1 - tj$	$i - t(j + k + t)$
k	k	$j - t(k + i + t)$	$-i - t$	$-1 - tk$

Corollary 2.2. *Multiplication on any two elements of our algebraic structure is defined as*

$$\begin{aligned}
 (a + bi + cj + dk)(p + qi + rj + sk) = & \\
 = [ap - (bq + cr + ds) - t(bs + cq + dr) + & \\
 - t^2(br + cs + dq)] + & \\
 (2.2) \quad + [aq + bp + cs - dr - t(bq + br + dq)]i + & \\
 + [ar - bs + cp + dq - t(br + cr + cs)]j + & \\
 + [as + br - cq + dp - t(cs + dq + ds)]k. &
 \end{aligned}$$

Corollary 2.3. a) *If $\text{char } F \neq 2$, then*

$$\begin{aligned}
 (a + bi + cj + dk)^2 = & \{a^2 - [(b + c + d)^2 + \\
 + (t^2 + t - 2)(bc + cd + db)]\} + & \\
 + [2a - t(b + c + d)](bi + cj + dk); &
 \end{aligned}$$

b) *if $\text{char } F = 2$, then*

$$\begin{aligned}
 (a + bi + cj + dk)^2 = & [a^2 + b^2 + c^2 + d^2] + \\
 + (t^2 + t)(bc + cd + db) + & \\
 + t(b + c + d)(bi + cj + dk); &
 \end{aligned}$$

c) *If $t^2 + t = 2$ or $bc + bd + cd = 0$, then*

$$(a + bi + cj + dk)^2 = 0 \Leftrightarrow a = 0 \wedge b + c + d = 0.$$

If the former assumption is the case, then $t = 1$.

Proposition 2.4. *The structure $(V, +, \cdot)$ with multiplication defined by (2.2) is a noncommutative ring.*

Proof. $(V, +)$ is a group since addition is the same as in the vector space V . We have to verify associativity of multiplication and both-side distributivity. Easy but lengthy calculations, omitted here, give the following final results confirming these properties.

• Multiplication is associative:

$$\begin{aligned}
 & [(a + bi + cj + dk)(p + qi + rj + sk)](u + vi + wj + zk) = \\
 & = \{[ap - (bq + cr + ds) - t(bs + cq + dr) - t^2(br + cs + dq)]u + \\
 & + [-aq - bp - cs + dr + t(-ar + bq + br + bs - cp)] + \\
 & + t^2(-as + cq + cr + cs - dp) + t^3(cs + dq + ds)]v + \\
 & + [-ar + bs - cp - dq + t(-as + cq + cr + cs - dp)] +
 \end{aligned}$$

$$\begin{aligned}
& + t^2(-aq - bp + dq + dr + ds) + t^3(bq + br + dq)]w + \\
& + [-as - br + cq - dp + t(-aq - bp + dq + dr + ds) + \\
& + t^2(-ar + bq + br + bs - cp) + t^3(br + cr + cs)]z + \\
& + \{[aq + bp + cs - dr - t(bq + br + dq)]u + \\
& + [ap - (bq + cr + ds) - t(aq + as + bp + br + bs + cs + dp) + \\
& + t^2(bq + dq + ds)]v + \\
& + [-as - br + cq - dp + t(-aq - bp + dq + dr + ds) + t^2(bq + br + dq)]w + \\
& + [ar - bs + cp + dq - t(br + cr + cs)]z\}i + \\
& + \{[ar - bs + cp + dq - t(br + cr + cs)]u + \\
& + [as + br - cq + dp - t(cs + dq + ds)]v + \\
& + [ap - (bq + cr + ds) - t(aq + ar + bp + cp + cq + cs + dq) + \\
& + t^2(bq + br + cr)]w + \\
& + [-aq - bp - cs + dr + t(-ar + bq + br + bs - cp) + t^2(br + cr + cs)]z\}j + \\
& + \{[as + br - cq + dp - t(cs + dq + ds)]u + \\
& + [-ar + bs - cp - dq + t(-as + cq + cr + cs - dp) + t^2(cs + dq + ds)]v + \\
& + [aq + bp + cs - dr - t(bq + br + dq)]w + [ap - (bq + cr + ds) - \\
& - t(ar + as + br + cp + dp + dq + dr) + t^2(cr + cs + ds)]z\}k = \\
& = (a + bi + cj + dk)[(p + qi + rj + sk)(u + vi + wj + zk)].
\end{aligned}$$

- Law of left distributivity:

$$\begin{aligned}
& (a + bi + cj + dk)[(p + qi + rj + sk) + (u + vi + wj + zk)] = \\
& = \{a(p + u) - [b(q + v) + c(r + w) + d(s + z)] - t[b(s + z) + c(q + v) + \\
& + d(r + w)] - t^2[b(r + w) + c(s + z) + d(q + v)]\} + \{a(q + v) + b(p + u) + \\
& + c(s + z) - d(r + w) - t[b(q + v) + b(r + w) + d(q + v)]\}i + \{a(r + w) - \\
& - b(s + z) + c(p + u) + d(q + v) - t[b(r + w) + c(r + w) + c(s + z)]\}j + \\
& + \{a(s + z) + b(r + w) - c(q + v) + d(p + u) - t[c(s + z) + d(q + v) + \\
& + d(s + z)]\}k = (a + bi + cj + dk)(p + qi + rj + sk) + \\
& + (a + bi + cj + dk)(u + vi + wj + zk).
\end{aligned}$$

- Law of right distributivity:

$$\begin{aligned}
& [(a + bi + cj + dk) + (p + qi + rj + sk)](u + vi + wj + zk) = \\
& = \{(a + p)u - [(b + q)v + (c + r)w + (d + s)z] - t[(b + q)z + (c + r)v + \\
& + (d + s)w] - t^2[(b + q)w + (c + r)z + (d + s)v]\} + \{(a + p)v + (b + q)u + \\
& + (c + r)z - (d + s)w - t[(b + q)v + (b + q)w + (d + s)v]\}i + \{(a + p)w - \\
& - (b + q)z + (c + r)u + (d + s)v - t[(b + q)w + (c + r)w + (c + r)z]\}j +
\end{aligned}$$

$$\begin{aligned}
& + \{(a+p)z + (b+q)w - (c+r)v + (d+s)u - t[(c+r)z + (d+s)v + \\
& + (d+s)z]\}k = (a+bi + cj + dk)(u + vi + wj + zk) + \\
& + (p+qi + rj + sk)(u + vi + wj + zk).
\end{aligned}$$

□

We shall denote the ring constructed above by $\mathfrak{R}(F)$. Since multiplication in $\mathfrak{R}(F)$ is associative, it follows that the left inverse (if exists) of any element is equal to the right one. It is obvious that $\text{char } \mathfrak{R}(F) = \text{char } F$.

Example 2.5. Let us consider possible irreducible polynomials $x^2 + tx + 1$ in small prime fields (cf. Remark 1.3). In $GF(2)$ there is only $x^2 + x + 1$; in $GF(3)$ only $x^2 + 1$; in $GF(5)$ $x^2 + x + 1$, and $x^2 + 4x + 1$; in $GF(7)$, $x^2 + 1$, $x^2 + 3x + 1$ and $x^2 + 4x + 1$; in $GF(11)$ $x^2 + 1$, $x^2 + x + 1$, $x^2 + 5x + 1$, $x^2 + 6x + 1$, and $x^2 + 10x + 1$; in $GF(13)$ $x^2 + 3x + 1$, $x^2 + 5x + 1$, $x^2 + 6x + 1$, $x^2 + 7x + 1$, $x^2 + 8x + 1$, and $x^2 + 10x + 1$.

Tab. 2: Addition and multiplication in $GF(2^3)$

+	0	1	A	B	C	D	E	F
0	0	1	A	B	C	D	E	F
1	1	0	B	A	D	C	F	E
A	A	B	0	1	E	F	C	D
B	B	A	1	0	F	E	D	C
C	C	D	E	F	0	1	A	B
D	D	C	F	E	1	0	B	A
E	E	F	C	D	A	B	0	1
F	F	E	D	C	B	A	1	0

·	0	1	A	B	C	D	E	F
0	0	0	0	0	0	0	0	0
1	0	1	A	B	C	D	E	F
A	0	A	C	E	B	1	F	D
B	0	B	E	D	F	C	1	A
C	0	C	B	F	E	A	D	1
D	0	D	1	C	A	F	B	E
E	0	E	F	1	D	B	A	C
F	0	F	D	A	1	E	C	B

Now, take the fields $GF(2^2)$ with elements 0, 1, A, B where $A + 1 = A^2 = B$, $AB = A + B = 1$, and $GF(2^3)$. In the former case $x^2 + Ax + 1$, and $x^2 + Bx + 1$ are the irreducible polynomials. In the latter case polynomials over $GF(2)$ of degree less than 3, i.e.

$$0, 1, A = x, B = x + 1, C = x^2, D = x^2 + 1, E = x^2 + x, F = x^2 + x + 1,$$

may be used as elements of $GF(2^3)$. Results of multiplication are residuals of division by irreducible polynomial $x^3 + x + 1$. Thus Table 2 presents addition and multiplication. One can easily check that polynomials $x^2 + x + 1$, $x^2 + Ax + 1$, $x^2 + Cx + 1$, and $x^2 + Ex + 1$ are irreducible. The polynomial $x^2 + 1$ has a double root 1, and each of the polynomials $x^2 + Bx + 1$, $x^2 + Dx + 1$, $x^2 + Fx + 1$ has two distinct roots, C and F , B and E , A and D , respectively.

Tab. 3: Addition and multiplication in $GF(3^2)$

+	0	1	2	A	B	C	D	E	F
0	0	1	2	A	B	C	D	E	F
1	1	2	0	B	C	A	E	F	D
2	2	0	1	C	A	B	F	D	E
A	A	B	C	D	E	F	0	1	2
B	B	C	A	E	F	D	1	2	0
C	C	A	B	F	D	E	2	0	1
D	D	E	F	0	1	2	A	B	C
E	E	F	D	1	2	0	B	C	A
F	F	D	E	2	0	1	C	A	B

·	0	1	2	A	B	C	D	E	F
0	0	0	0	0	0	0	0	0	0
1	0	1	2	A	B	C	D	E	F
2	0	2	1	D	F	E	A	C	B
A	0	A	D	2	C	F	1	B	E
B	0	B	F	C	D	1	E	2	A
C	0	C	E	F	1	A	B	D	2
D	0	D	A	1	E	B	2	F	C
E	0	E	C	B	2	D	F	A	1
F	0	F	B	E	A	2	C	1	D

Similarly, consider $GF(3^2)$ with elements 0, 1, 2, $A = \alpha$, $B = \alpha + 1$, $C = \alpha + 2$, $D = 2\alpha$, $E = 2\alpha + 1$, $F = 2\alpha + 2$, where $\alpha \notin GF(3)$, $\alpha^2 = 2$ (see Table 3) Now, polynomials $x^2 + Bx + 1$, $x^2 + Cx + 1$, $x^2 + Ex + 1$, $x^2 + Fx + 1$ are irreducible. The polynomials $x^2 + x + 1$ and $x^2 + 2x + 1$ have double roots 1 and 2, respectively. Each of the polynomials $x^2 + 1$, $x^2 + Ax + 1$, $x^2 + Dx + 1$ has two distinct roots A and D , B and C , E and F , respectively.

Proposition 2.6. *In $\mathfrak{R}(F)$ every left zero divisor is also a right zero divisor and vice versa.*

Proof. An element $a + bi + cj + dk$ is a left zero divisor if there exists nonzero $p + qi + rj + sk \in \mathfrak{R}(F)$ such that $(a + bi + cj + dk)(p + qi + rj + sk) = 0$. Then, by (2.2), the equations

$$(2.3) \quad \begin{cases} ap - (b + tc + t^2d)q - (c + td + t^2b)r - (d + tb + t^2c)s = 0, \\ bp + (a - tb - td)q - (d + tb)r + cs = 0, \\ cp + dq + (a - tb - tc)r - (b + tc)s = 0, \\ dp - (c + td)q + br + (a - tc - td)s = 0 \end{cases}$$

must have a nonzero solution (p, q, r, s) . Therefore the determinant of the main matrix must be equal to 0. Analogously, $a + bi + cj + dk$ is a right zero divisor if there exists nonzero $p + qi + rj + sk \in \mathfrak{R}(F)$ such that $(p + qi + rj + sk)(a + bi + cj + dk) = 0$. Now, the equations

$$(2.4) \quad \begin{cases} pa - q(b + td + t^2c) - r(c + tb + t^2d) - s(d + tc + t^2b) = 0, \\ pb + q(a - tb - tc) + rd - s(c + tb) = 0, \\ pc - q(d + tc) + r(a - tc - td) + sb = 0, \\ pd + qc - r(b + td) + s(a - tb - td) = 0 \end{cases}$$

must have a nonzero solution (p, q, r, s) , i.e. the determinant of their main matrix must be equal to 0.

But the both determinants (of (2.3) and of (2.4)) have the same value:

$$\begin{aligned} & \begin{vmatrix} a & -[b + t(c + td)] & -[c + t(d + tb)] & -[d + t(b + tc)] \\ b & a - t(b + d) & -(d + tb) & c \\ c & d & a - t(b + c) & -(b + tc) \\ d & -(c + td) & b & a - t(c + d) \end{vmatrix} = \\ & = \begin{vmatrix} a & -[b + t(d + tc)] & -[c + t(b + td)] & -[d + t(c + tb)] \\ b & a - t(b + c) & d & -(c + tb) \\ c & -(d + tc) & a - t(c + d) & b \\ d & c & -(b + td) & a - t(b + d) \end{vmatrix}. \end{aligned}$$

□

Theorem 2.7. *An element $a + bi + cj + dk$ is a zero divisor if and only if*

$$a^2 - ta(b + c + d) + (b + c + d)^2 + (t^2 + t - 2)(bc + bd + cd) = 0.$$

The set of solutions $p + qi + rj + sk$ of the equations (2.3) constitute the right annihilator of an element $a + bi + cj + dk$.

Proof. We simply have to calculate the determinant of (2.3) as follows:

$$\begin{vmatrix} a & -[b + t(c + td)] & -[c + t(d + tb)] & -[d + t(b + tc)] \\ b & a - t(b + d) & -(d + tb) & c \\ c & d & a - t(b + c) & -(b + tc) \\ d & -(c + td) & b & a - t(c + d) \end{vmatrix} =$$

$$\begin{aligned}
&= a \cdot \begin{vmatrix} a - t(b + d) & -(d + tb) & c \\ d & a - t(b + c) & -(b + tc) \\ -(c + td) & b & a - t(c + d) \end{vmatrix} + \\
&+ [b + t(c + td)] \cdot \begin{vmatrix} b & -(d + tb) & c \\ c & a - t(b + c) & -(b + tc) \\ d & b & a - t(c + d) \end{vmatrix} + \\
&- [c + t(d + tb)] \cdot \begin{vmatrix} b & a - t(b + d) & c \\ c & d & -(b + tc) \\ d & -(c + td) & a - t(c + d) \end{vmatrix} + \\
&+ [d + t(b + tc)] \cdot \begin{vmatrix} b & a - t(b + d) & -(d + tb) \\ c & d & a - t(b + c) \\ d & -(c + td) & b \end{vmatrix} = \\
&= a \cdot \{[a - t(b + d)][a - t(b + c)][a - t(c + d)] + bcd + \\
&\quad -(c + td)(d + tb)(b + tc) + (c + td)[a - t(b + c)]c + \\
&\quad + b(b + tc)[a - t(b + d)] + [a - t(c + d)](d + tb)d\} + \\
&\quad + [b + t(c + td)] \cdot \{b[a - t(b + c)][a - t(c + d)] + c^2b + \\
&\quad + d(d + tb)(b + tc) - c[a - t(b + c)]d + b^2(b + tc) + \\
&\quad + [a - t(c + d)]c(d + tb)\} - [c + t(d + tb)] \cdot \{bd[a - t(c + d)] + \\
&\quad - c^2(c + td) - d[a - t(b + d)](b + tc) - cd^2 - (b + tc)(c + td)b + \\
&\quad - [a - t(c + d)]c[a - t(b + d)]\} + [d + t(b + tc)] \cdot \{b^2d + \\
&\quad + c(c + td)(d + tb) + d[a - t(b + d)][a - t(b + c)] + \\
&\quad + d^2(d + tb) + [a - t(b + c)](c + td)b - bc[a - t(b + d)]\} = \\
&= a \cdot \{a^3 + ac^2 + ab^2 + ad^2 - t[2a^2(b + c + d) - a(bc + bd + cd) + \\
&\quad + c^2d + b^2c + bd^2 + bc^2 + c^3 + b^3 + b^2d + cd^2 + d^3] + \\
&\quad + t^2[3a(bc + bd + cd) + a(b^2 + c^2 + d^2) - (c^2b + cd^2 + b^2d + c^2d + \\
&\quad + b^2c + bd^2 + 3bcd)] - t^3[b^2c + bc^2 + c^2d + b^2d + bd^2 + cd^2 + 3bcd]\} + \\
&\quad + [b + t(c + td)] \cdot \{a^2b + c^2b + d^2b + b^3 + t[-ab(b + c + d) + \\
&\quad + b^2d + bcd + b^2c] + t^2[b^2c + b^2d + bcd]\} - [c + t(d + tb)] \cdot \{-c^3 - cd^2 + \\
&\quad - b^2c - a^2c + t[ac(b + c + d) - (bcd + c^2d + bc^2)] - t^2[bc^2 + c^2d + bcd]\} + \\
&\quad + [d + t(b + tc)] \cdot \{b^2d + c^2d + a^2d + d^3 + t[-ad(b + c + d) + \\
&\quad + cd^2 + bd^2 + bcd] + t^2[bcd + bd^2 + cd^2]\} =
\end{aligned}$$

$$\begin{aligned}
&= a^4 + a^2(b^2 + c^2 + d^2) - t[2a^3(b + c + d) - a^2(bc + bd + cd) + \\
&+ a(b + c + d)(b^2 + c^2 + d^2)] + t^2[3a^2(bc + bd + cd) + a^2(b^2 + c^2 + d^2) - \\
&- a(bc + bd + cd)(b + c + d)] - t^3a(bc + bd + cd)(b + c + d) + \\
&+ [b + t(c + td)]b \cdot \{a^2 + b^2 + c^2 + d^2 + t[-a(b + c + d) + bc + bd + cd] + \\
&+ t^2(bc + bd + cd)\} - [c + t(d + tb)]c \cdot \{-c^2 - d^2 - \\
&- b^2 - a^2 + t[a(b + c + d) - (bd + cd + bc)] - t^2[bc + cd + bd]\} + \\
&+ [d + t(b + tc)]d \cdot \{b^2 + c^2 + a^2 + d^2 + t[-a(b + c + d) + \\
&+ cd + bd + bc] + t^2[bc + bd + cd]\} = \\
&= a^4 - 2ta^3(b + c + d) + (t + 3t^2)a^2(bc + bd + cd) - \\
&- (t^2 + t^3)a(bc + bd + cd)(b + c + d) + \\
&+ [a^2 - ta(b + c + d) + t^2a^2](b^2 + c^2 + d^2) + \\
&+ [b^2 + c^2 + d^2 + (t + t^2)(bc + bd + cd)][a^2 + b^2 + c^2 + d^2 - ta(b + c + d) + \\
&+ (t + t^2)(bc + bd + cd)] = \\
&= a^4 - 2ta^3(b + c + d) + (t + 3t^2)a^2(bc + bd + cd) - \\
&- (t^2 + t^3)a(bc + bd + cd)(b + c + d) + \\
&+ [a^2 - ta(b + c + d) + t^2a^2][(b + c + d)^2 - 2(bc + bd + cd)] + \\
&+ [(b + c + d)^2 + (t + t^2 - 2)(bc + bd + cd)][a^2 + (b + c + d)^2 - \\
&- ta(b + c + d) + (t^2 + t - 2)(bc + bd + cd)] = \\
&= a^4 - 2ta^3(b + c + d) + (t + 3t^2)a^2(bc + bd + cd) - \\
&- (t^2 + t^3)a(bc + bd + cd)(b + c + d) + \\
&+ a^2(b + c + d)^2 - ta(b + c + d)^3 + t^2a^2(b + c + d)^2 - \\
&- 2a^2(bc + bd + cd) + 2ta(b + c + d)(bc + bd + cd) - \\
&- 2t^2a^2(bc + bd + cd) + (b + c + d)^2a^2 + (b + c + d)^4 - \\
&- ta(b + c + d)^3 + (t^2 + t - 2)(bc + bd + cd)(b + c + d)^2 + \\
&+ (t + t^2 - 2)(bc + bd + cd)a^2 + (t + t^2 - 2)(bc + bd + cd)(b + c + d)^2 - \\
&- (t + t^2 - 2)(bc + bd + cd)ta(b + c + d) + [(t^2 + t - 2)(bc + bd + cd)]^2 = \\
&= a^4 + t^2a^2(b + c + d)^2 + (b + c + d)^4 + [(t^2 + t - 2)(bc + bd + cd)]^2 - \\
&- 2ta^3(b + c + d) + 2a^2(b + c + d)^2 + 2a^2(t^2 + t - 2)(bc + bd + cd) - \\
&- 2ta(b + c + d)^3 - 2ta(b + c + d)(t^2 + t - 2)(bc + bd + cd) + \\
&+ 2(t^2 + t - 2)(bc + bd + cd)(b + c + d)^2 = \\
&= [a^2 - ta(b + c + d) + (b + c + d)^2 + (t^2 + t - 2)(bc + bd + cd)]^2.
\end{aligned}$$

The last part of the Theorem follows from the definition of annihilator. $\square$

Corollary 2.8. *a) Let $t = 1$ or $bc + bd + cd = 0$. Then $a + bi + cj + dk$ is a zero divisor iff $a^2 - ta(b + c + d) + (b + c + d)^2 = 0$. If it is the case then $a = 0 \Leftrightarrow b + c + d = 0$.*

- b) Let $t \neq 1$ and $b + c + d = 0$. Then $a + bi + cj + dk$ is a zero divisor iff $a^2 + (t^2 + t - 2)(bc + bd + cd) = 0$.
 If it is the case then: $a = 0 \Leftrightarrow bc + bd + cd = 0$; for $a \neq 0$ the element $(t^2 + t - 2)(b^2 + c^2 + bc)$ must be a square.
- c) If $b + c + d = 0$ and $bc + bd + cd = 0$, then $bc = d^2$, $bd = c^2$, $cd = b^2$. Hence $b^3 = c^3 = d^3$ and either $b, c, d \in S$ or $b, c, d \notin S$. Every element $z = bi + cj + dk$ satisfying the both assumptions is a nilpotent, i.e. $z^2 = 0$ (see Corollary 2.3).
- d) Let $t = -1$ and $\text{char } F \neq 2$. Then $a + bi + cj + dk$ is a zero divisor iff $a^2 + b^2 + c^2 + d^2 + a(b + c + d) = 0$.
 If it is the case and $a = 0$ then $b + c + d = 0 \Leftrightarrow bc + bd + cd = 0$. Hence every element $z = bi + cj + dk$ satisfying the condition $b + c + d = 0$ is a nilpotent.

Remark 2.9. Note that the possibility $b = c = d \neq 0$ is excluded if $\text{char } F \neq 3$ and $b + c + d = 0 \vee bc + bd + cd = 0$. Otherwise $b + c + d = 3b = 0$ or $bc + bd + cd = 3b^2 = 0$, and zero divisors would exist in the field.

Remark 2.10. If $bc + bd + cd = 0$, then either one of elements b, c, d is arbitrary and the remaining two are equal to 0, or $b + c \neq 0$, $b + d \neq 0$, and $c + d \neq 0$.

Although $\mathfrak{R}(F)$ may be infinite (e.g. if F is the field of rational functions over $GF(2)$), the following condition holds.

Proposition 2.11. An element in $\mathfrak{R}(F)$ is invertible if and only if it is not a zero divisor.

Proof. A nonzero element $a + bi + cj + dk$ is invertible if there exists $p + qi + rj + sk \in \mathfrak{R}(F)$ such that $(a + bi + cj + dk)(p + qi + rj + sk) = 1$. Equivalently, we have obtained the equations like (2.3) with a unique difference: in the 1-st equation on the right side there is 1 instead of 0. But this time we have Cramer equations, so the determinant is different from zero. Summarizing, $a + bi + cj + dk$ is invertible (resp. a zero divisor) if the determinant of the main matrix of (2.3) is different from (resp. equal to) zero. $\square$

The following Remark 2.12 and Lemma 2.13 will be used in further study.

Remark 2.12. In every field F $x^2 - txy + y^2 = 0 \Rightarrow (x = 0 \Leftrightarrow y = 0)$.

Lemma 2.13. Let $\text{char } F > 2$. Then:

- a) There exists a nonzero solution (x, y) of the equation $x^2 - txy + y^2 = 0$ over F if and only if $t^2 - 2^2$ is a square in F .

- b) If $t^2 - 2^2 = \alpha^2$ for some $0 \neq \alpha \in F$, then for every $x \neq 0$ there exist exactly two y -s such that $x^2 - txy + y^2 = 0$: $y_1 = 2^{-1}(\alpha + t)x$ and $y_2 = 2^{-1}(-\alpha + t)x$. Then $y_2 = -y_1 + tx$, $y_1 \neq y_2$, and $y_1 \neq x \neq y_2$.

Proof. Ad. a) Because of Remark 2.12, the values x and y are different from 0 since we seek nonzero solutions (x, y) . Dividing the equation by y^2 , we have

$$\left(\frac{x}{y}\right)^2 - t \cdot \frac{x}{y} + 1 = 0 \quad \text{or equivalently} \quad \left[2\left(\frac{x}{y} - 2^{-1}t\right)\right]^2 = t^2 - 2^2.$$

The left side of the last equation is a square and so must be the right one, if the equation has a solution $\frac{x}{y}$.

Ad. b) The equation $x^2 - txy + y^2 = 0$ may be written as

$$\left(y - \frac{tx}{2}\right)^2 = (t^2 - 2^2) \left(\frac{x}{2}\right)^2. \quad \text{Hence} \quad y_1 = \frac{(\alpha + t)x}{2}, \quad y_2 = \frac{(-\alpha + t)x}{2}.$$

The conditions $y_2 = -y_1 + tx$ and $y_1 \neq y_2$ are obvious. If e.g. $y_1 = x$ then $2x = (\alpha + t)x$, whence $\alpha^2 = t^2 - 2^2 = (2 - t)^2$, $t = 2$ and $\alpha = 0$, a contradiction. $\square$

Corollary 2.14. Assume that $\text{char } F > 2$, $t^2 - 2^2 = \alpha^2$ for some $\alpha \in F$, $b, c, d \in F$, $x = b + c + d \neq 0$, and let $a = \frac{(\alpha+t)x}{2}$ or $a = \frac{(-\alpha+t)x}{2}$. If $t = 1$ or $bc + bd + cd = 0$, then $a + bi + cj + dk$ is a zero divisor which is not a nilpotent.

Proof. It suffices to put $x = b + c + d$ in Lemma 2.13 b) and then use Corollary 2.8 a), where $y = a$. $\square$

The following proposition generalizes Corollary 2.14.

Proposition 2.15. Let $\text{char } F > 2$ and $b, c, d \in F$ are chosen so as

$$(t^2 - 2^2)(b + c + d)^2 - 2^2(t^2 + t - 2)(bc + bd + cd) = \alpha^2 \in S \cup \{0\}$$

for some $\alpha \in F$. Then $a + bi + cj + dk$ is a zero divisor in $\mathfrak{R}(F)$ iff $a = 2^{-1}[t(b + c + d) \pm \alpha]$.

Proof. We use the same method as in Lemma 2.13. Because of Theorem 2.7, $a + bi + cj + dk$ is a zero divisor iff

$$a^2 - ta(b + c + d) + (b + c + d)^2 + (t^2 + t - 2)(bc + bd + cd) = 0.$$

Hence

$$2^2[a - 2^{-1}t(b + c + d)]^2 = (t^2 - 2^2)(b + c + d)^2 - 2^2(t^2 + t - 2)(bc + bd + cd),$$

i.e.

$$2^2[a - 2^{-1}t(b + c + d)]^2 = \alpha^2$$

and we get

$$a = 2^{-1}[t(b + c + d) \pm \alpha].$$

$\square$

Corollary 2.16. *If $\text{char } F > 2$, $b, c, d \in F$ and*

$$(t^2 - 2^2)(b + c + d)^2 - 2^2(t^2 + t - 2)(bc + bd + cd)$$

is not a square in F , then, for every $a \in F$, $a + bi + cj + dk$ is not a zero divisor in $\mathfrak{R}(F)$.

Example 2.17. Let us consider $\mathfrak{R}(GF(7))$ and take $t = 3$ (see Example 2.5). Then $S = \{1, 4, 2\}$ and

$$\begin{aligned} \beta &= (t^2 - 2^2)(b + c + d)^2 - 2^2(t^2 + t - 2)(bc + bd + cd) = \\ &= 5 \cdot [(b + c + d)^2 - (bc + (b + c)d)]. \end{aligned}$$

Let us fix $b = 2$, $c \in \{3, 4, 5\}$, and in accordance with Proposition 2.15 and Corollary 2.16, select all $d, a \in F$, such that $a + bi + cj + dk$ is a zero divisor. If $\beta \in S$ then $\beta = \alpha^2$ and we can determine the two values of a . We have got the results in Table 4.

We see in Table 4 that the set of zero divisors satisfying all conditions listed in Corollary 2.8 c) is not empty ($b = 2$, $c = 4$, $d = 1$). Yet, Corollary 2.8 c) also implies that $z = bi + cj + dk$ is a nilpotent either for $\{b, c, d\} = \{1, 2, 4\}$ or for $\{b, c, d\} = \{6, 5, 3\}$. One can easily find the annihilator of such an element. Taking e.g. $b = 1$, $c = 2$, $d = 4$, we get $(i + 2j + 4k) \cdot [(5r + s) + (5r + 3s)i + rj + sk] = 0$ for every $r, s \in GF(7)$. Note that there exist other nilpotents. It suffices to take $a = 1$, $\{b, c, d\} = \{0, 1, 2\}$, and using Corollary 2.3 a), we obtain

$$\begin{aligned} &(a + bi + cj + dk)^2 = \\ &= \{1^2 - [(0 + 1 + 2)^2 + (3^2 + 3 - 2)(0 \cdot 1 + 0 \cdot 2 + 1 \cdot 2)]\} + \\ &\quad + [2 \cdot 1 - 3(0 + 1 + 2)](bi + cj + dk) = \\ &= \{1 - [2 + 3 \cdot 2]\} + [2 - 3 \cdot 3](bi + cj + dk) = 0. \end{aligned}$$

Example 2.18. Consider $\mathfrak{R}(GF(3^2))$ (cf. Example 2.5 and Remark 2.9). Then the polynomial $x^2 + tx + 1$ is irreducible exactly for $0 \neq t \notin S$. Since $B^2 - 2^2 \notin S$, $C^2 - 2^2 \notin S$, $E^2 - 2^2 \notin S$, and $F^2 - 2^2 \notin S$, every element $a + bi + cj + dk$, such that $bc + bd + cd = 0$ and $b + c + d \neq 0$ is invertible (see Corollary 2.16). Since $\text{char } F = 3$, for every $b \in GF(3^2)$ we have $b + b + b = 0$ and $bb + bb + bb = 0$. Hence every element $bi + bj + bk$ is a zero divisor. There exist many zero divisors $a + bi + cj + dk$ for which $bc + bd + cd \neq 0$. Like in Example 2.17, put e.g. $b = A$, $c = B = t$. Then

$$\begin{aligned} \beta &= (B^2 - 2^2)(A + B + d)^2 - 2^2(B^2 + B - 2)(AB + Ad + Bd) = \\ &= F \cdot (E + d)^2 - (1 - 2) \cdot [C + (A + B)d] = F \cdot (E + d)^2 + C + Ed \end{aligned}$$

and β is a square for $d \in \{0, 2, B, C, D, E\}$ as it has been shown in Table 5. Because of Corollary 2.16, elements $a + Ai + Bj + k$, $a + Ai + Bj + Ak$, $a + Ai + Bj + Fk$

Tab. 4: Zero divisors for $c \in \{3, 4, 5\}$ in Example 2.17

$c = 3$						
d	$(b + c + d)^2$	$bc + (b + c)d$	β	α	a_1	a_2
0	4	6	4	2	5	3
1	1	4	$6 \notin S \cup \{0\}$			
2	0	2	4	2	1	6
3	1	0	$5 \notin S \cup \{0\}$			
4	4	5	2	4	5	1
5	2	3	2	4	3	6
6	2	1	$5 \notin S \cup \{0\}$			

$c = 4$						
d	$(b + c + d)^2$	$bc + (b + c)d$	β	α	a_1	a_2
0	1	1	0	0	2	2
1	0	0	0	0	0	0
2	1	6	$3 \notin S \cup \{0\}$			
3	4	5	2	4	5	1
4	2	4	4	2	2	0
5	2	3	2	4	1	4
6	4	2	$3 \notin S \cup \{0\}$			

$c = 5$						
d	$(b + c + d)^2$	$bc + (b + c)d$	β	α	a_1	a_2
0	0	3	$6 \notin S \cup \{0\}$			
1	1	3	4	2	6	4
2	4	3	$5 \notin S \cup \{0\}$			
3	2	3	2	4	3	6
4	2	3	2	4	1	4
5	4	3	$5 \notin S \cup \{0\}$			
6	1	3	4	2	3	1

are invertible for every $a \in GF(3^2)$.

Tab. 5: β for $b = A$, $c = B = t$ in Example 2.18

d	0	1	2	A	B	C	D	E	F
β	0	$C \notin S$	D	$C \notin S$	A	2	A	A	$E \notin S$
α	0		B		C	A	C	C	
$-\alpha$	0		F		E	A	E	E	

It is obvious that all our consideration becomes simpler if the last component of $a^2 - ta(b + c + d) + (b + c + d)^2 + (t^2 + t - 2)(bc + bd + cd)$ disappears. In particular,

the number of all zero divisors, idempotents and nilpotents may be evaluated more or less precisely. The case $t^2 + t - 2 = 0$ (i.e. $t = 1$) is very simple. We have no additional restriction in the fields where $x^2 + x + 1$ is irreducible, and we take it to construct the ring. Similarly, cases $t = 0$, $\text{char } F = 2$ are easy to investigate. Yet the condition $bc + bd + cd = 0$ implies additional restrictions.

3. Special case $bc + bd + cd = 0$

Theorem 3.1. *Let $\text{char } F > 2$, $|F| = q$ ($|F|$ denotes cardinality of F), and consider zero divisors $a + bi + cj + dk \in \mathfrak{R}(F)$ satisfying the condition $bc + bd + cd = 0$ in F . Then:*

1. *If $\text{char } F = 3$ then there exist at least $q - 1$ proper nilpotents;*
2. *If $\text{char } F > 3$ then there exist exactly $2(q - 1)$ proper nilpotents if $-3 \in S$ and there is no proper nilpotent if $-3 \notin S$;*
3. *If $t^2 - 2^2 \notin S$ then all zero divisors in $\mathfrak{R}(F)$ are nilpotents;*
4. *If $t^2 - 2^2 \in S$ and $\text{char } F > 3$, then there exist at least $4(q - 1)$ zero divisors which are not nilpotents.*

Proof. The expression $a^2 - ta(b + c + d) + (b + c + d)^2$ used in Proposition 2.7 may be written as $x^2 - txy + y^2$, where $x = a$, $y = b + c + d$.

Assume that $x = a = 0$, $y = b + c + d = 0$ (see Remark 2.12). By the assumption $bc + bd + cd = 0$ and Remark 2.10, we obtain $c \neq -b \neq d$ for $b \neq 0$, and

$$d = -(b + c) = -\frac{bc}{b + c}, \quad \text{i.e.} \quad b^2 + bc + c^2 = 0.$$

If $\text{char } F = 3$ then every pair (b, c) with $b = c \neq 0$ satisfies the latter equation, whence every element $bi + bj - 2bk$ is a nilpotent (see Corollary 2.8 c)), which ends the proof of item 1.

If $\text{char } F > 3$, then by Lemma 2.13 a) with $x = c$, $y = b$, $t = -1$, this equation has nonzero solutions if -3 is a square in F . Let $u \in F$ be an element such that $u^2 = -3$. Because of Lemma 2.13 b), for every $c \neq 0$ there exist exactly two distinct values of b and then the values of $d = -(b + c)$ are uniquely determined: $b_1 = 2^{-1}c(u - 1) = d_2$, $b_2 = -2^{-1}c(u + 1) = d_1$. If $c = 0$ then $d = -b$, a contradiction. Therefore we have got exactly $2(q - 1) + 1$ quadruples (a, b, c, d) (together with $(0, 0, 0, 0)$), satisfying the conditions $a = 0$, $b + c + d = 0$. By Corollary 2.3 c), such elements and only such elements are nilpotents. Thus item 2 is proved.

If $t^2 - 2^2$ is not a square in F then the equation $x^2 - txy + y^2 = 0$ has only zero solution (see Lemma 2.13 a)): $x = a = 0$ and $y = b + c + d = 0$. The assumption $bc + bd + cd = 0$ and Corollary 2.8 c) imply that all such elements are nilpotents. So item 3 is proved.

Now let $t^2 - 2^2$ be a square in F . Then the equation $x^2 - txy + y^2 = 0$ has also nonzero solutions. Using Lemma 2.13 b), we know that for every nonzero $a = x \in F$, there exist exactly two nonzero y -s such that $x^2 - txy + y^2 = 0$: for $t^2 - 2^2 = \beta^2$ we obtain $y_1 = 2^{-1}a(t + \beta)$, $y_2 = 2^{-1}a(t - \beta)$. If one of two y -s is fixed, then we put $y = b + c + d$. But we have to recall the assumption $bc + bd + cd = 0$. Hence

$$d = y - b - c = -\frac{bc}{b+c}, \quad \text{i.e.} \quad c^2 + (b-y)(b+c) = 0,$$

$$\text{or, equivalently,} \quad [2 \cdot (c + 2^{-1}(b-y))]^2 = -3b^2 + 2by + y^2.$$

Like before, the latter equation has solutions (b, c) if and only if the right side is a square in F . It depends on the kind of the field for which relations between b and y is $-3b^2 + 2by + y^2 \in S \cup \{0\}$. Yet for every field F , if $y = b \vee y = -3b$; then $-3b^2 + 2by + y^2 = 0$. For $b = y$ Remark 2.10 yields $c = d = 0$. If $b = (-3^{-1})y$ then $0 = c + 2^{-1}(b - y)$, whence $c = -2^{-1}(b - y) = -2^{-1}(-3^{-1}y - y) = 2 \cdot 3^{-1}y$, and $d = y - b - c = y + 3^{-1}y - 2 \cdot 3^{-1}y = 2 \cdot 3^{-1}y$. Summarizing, for every $a \neq 0$ the following elements are zero divisors and they are not nilpotents:

$$\begin{aligned} & a[1 + 2^{-1}(t + \beta)i], \quad a[1 + 2^{-1}(t - \beta)i], \\ & a\{1 + 2^{-1}3^{-1}(t + \beta)(-i + 2j + 2k)\}, \\ & a\{1 + 2^{-1}3^{-1}(t - \beta)(-i + 2j + 2k)\}. \end{aligned}$$

This ends the proof of item 4. □

Corollary 3.2. *If there exist $y, b, c, d \in F$ such that $y = b + c + d$ and $bc + bd + cd = 0$, then $-3b^2 + 2by + y^2 = (c - d)^2$.*

Proposition 3.3. *Let $bc + bd + cd = 0$, where $0 \notin \{b, c, d\}$. Then $a + bi + cj + dk$ is an idempotent if and only if*

$$t^2 - 2^2 \in S, \text{ and } t(b + c)^2 - (2a - 1)(b + c) - tbc = 0.$$

Proof. Let $(a + bi + cj + dk)^2 = a + bi + cj + dk$. By Corollary 2.3 we have

$$(3.1) \quad \begin{cases} a^2 - (b + c + d)^2 = a, \\ 2a - t(b + c + d) = 1. \end{cases}$$

Hence

$$[2(a - 2^{-1})]^2 = t^2(t^2 - 2^2)^{-1}.$$

The left side is a square and so must be the right one if the equation has a solution a . From the latter equation of (3.1) we have the value $b + c + d = \frac{2a-1}{t}$, i.e. $d = -b - c + \frac{2a-1}{t}$.

By the assumptions $bc + bd + cd = 0$, $0 \notin \{b, c, d\}$ and Remark 2.10, we obtain $d = -\frac{bc}{b+c}$. Hence we have got the equality

$$t(b+c)^2 - (2a-1)(b+c) - tbc = 0.$$

Now, assume that $t(b+c)^2 - (2a-1)(b+c) - tbc = 0$. By the assumption $bc + bd + cd = 0$, for $d \neq 0$ we obtain $b+c = -bcd^{-1}$. Then $bc \neq 0$ and

$$\begin{aligned} t(bc)^2 d^{-2} + (2a-1)bcd^{-1} - tbc &= 0; \\ tbc + (2a-1)d - td^2 &= 0; \\ (2a-1)d = t(d^2 - bc) = t(d^2 + bd + cd); \\ 2a-1 &= t(d+b+c). \end{aligned}$$

The last equality coincides with the latter one of (3.1). It remains to verify that $a^2 - (b+c+d)^2 = a$. We already have $b+c+d = (2a-1)t^{-1}$. Thus

$$\begin{aligned} a^2 - (b+c+d)^2 = a &\Leftrightarrow a^2 - [(2a-1)t^{-1}]^2 = a \Leftrightarrow \\ (a^2 - a)(t^2 - 2^2) &= 1 \Leftrightarrow (2a-1)^2 = t^2(t^2 - 2^2)^{-1}. \end{aligned}$$

□

Corollary 3.4. *If $t^2 - 2^2 \in S \subset F$ and $\beta^2 = t^2 - 2^2$, then the following elements are idempotents in $\mathfrak{R}(F)$:*

$$\begin{aligned} \frac{t+\beta}{2\beta} + \beta^{-1}i; & \quad \frac{-t+\beta}{2\beta} - \beta^{-1}i; & \quad \frac{t+\beta}{2\beta} + \beta^{-1}j; \\ \frac{-t+\beta}{2\beta} - \beta^{-1}j; & \quad \frac{t+\beta}{2\beta} + \beta^{-1}k; & \quad \frac{-t+\beta}{2\beta} - \beta^{-1}k. \end{aligned}$$

4. Special cases: $\text{char } F = 2$, $t = 0$, and $t = 1$

Example 4.1. $\mathfrak{R}(GF(2))$ is the smallest ring among $\mathfrak{R}(F)$. Its all invertible elements are presented in Table 6.

One can easily prove that the group of invertible elements contains the following proper subgroups:

3 groups of order 2: $(\{1, 1+i+j\}, \{1, 1+i+k\}, \{1, 1+j+k\})$;

4 groups of order 3: $(\{1, i, 1+i\}, \{1, j, 1+j\}, \{1, k, 1+k\}, \{1, i+j+k, 1+i+j+k\})$.

Tab. 6: Invertible elements in $\mathfrak{R}(GF(2))$

x	1	i	j	k	$1+i+j$	$1+i+k$	$1+j+k$	$i+j+k$
x^{-1}	1	$1+i$	$1+j$	$1+k$	$1+i+j$	$1+i+k$	$1+j+k$	$1+i+j+k$

There are exactly three proper both-side zero divisors $i + j$, $i + k$, $j + k$, multiplication of which is presented in Table 7.

Tab. 7: Multiplication of zero divisors in $\mathfrak{R}(GF(2))$

$\cdot$	$i + j$	$i + k$	$j + k$
$i + j$	0	0	0
$i + k$	0	0	0
$j + k$	0	0	0

This means that there exists a unique annihilator.

From Proposition 2.7 the following is immediate:

- Proposition 4.2.** *a) If $t = 0$ then $a + bi + cj + dk$ is a zero divisor if and only if $a^2 + b^2 + c^2 + d^2 = 0$;*
- b) If $t = 1$ then $a + bi + cj + dk$ is a zero divisor if and only if $a^2 - a(b + c + d) + (b + c + d)^2 = 0$.*

Proposition 4.3. *Let $t = 1$.*

- 1. An element $z = a + bi + cj + dk$ is a nilpotent (i.e. $z^2 = 0$) in $\mathfrak{R}(F)$ if and only if $a = 0$ and $b + c + d = 0$;*
- 2. An element $z = a + bi + cj + dk$ is a nontrivial idempotent (i.e. $z^2 = z$ and $1 \neq z \neq 0$) in $\mathfrak{R}(F)$ if and only if the following conditions hold:*

$$3 \neq \text{char } F \neq 2, \quad 3a^2 - 3a + 1 = 0, \quad b + c + d = 2a - 1.$$

Proof. From Corollary 2.3 **c)** item 1. is immediate. To prove item 2. we have to consider $a + bi + cj + dk$ such that $(a + bi + cj + dk)^2 = a + bi + cj + dk$. Using Corollary 2.3 **a)** and **b)**, we obtain the following equations:

$$[a^2 - (b + c + d)^2] + [2a - (b + c + d)](bi + cj + dk) = a + bi + cj + dk$$

if $\text{char } F \neq 2$;

$$[a^2 + (b + c + d)^2] + (b + c + d)(bi + cj + dk) = a + bi + cj + dk$$

if $\text{char } F = 2$.

If $b = c = d = 0$, then $a^2 = a$, whence $z = 0$ or $z = 1$. Assume that $(a, b, c) \neq (0, 0, 0)$. If $\text{char } F = 2$ then $b + c + d = 1$ and we have $a^2 + 1 = a$, i.e. $a^2 + a + 1 = 0$, which contradicts the assumption that the polynomial $x^2 + x + 1 = 0$ does not have a root. If $\text{char } F \neq 2$ then $2a - (b + c + d) = 1$, whence $a^2 - (2a - 1)^2 = a$, i.e. $1 = 0$ (a contradiction) for $\text{char } F = 3$ and $3a^2 - 3a + 1 = 0$ for $\text{char } F \neq 3$.

Assume that $3 \neq \text{char } F \neq 2$, $3a^2 - 3a + 1 = 0$ and $b + c + d = 2a - 1$. Then the last condition implies:

$$\begin{aligned} a^2 - (b + c + d)^2 = a &\Leftrightarrow a^2 - (2a - 1)^2 = a \Leftrightarrow 3a^2 - 3a + 1 = 0; \\ 2a - (b + c + d) &= 2a - (2a - 1) = 1. \end{aligned}$$

□

Remark 4.4. The condition $3a^2 - 3a + 1 = 0$ yields $-3 \in S$ since the equation $3x^2 - 3x + 1 = 0$, or equivalently $[2 \cdot (a - 2^{-1})]^2 = (-3)^{-1}$, has a solution.

Example 4.5. It follows from Proposition 4.3 that proper idempotents exist neither in $\mathfrak{R}(GF(2))$ nor in $\mathfrak{R}(GF(2^3))$ where we take $t = 1$ (see Example 2.5). We already know from Example 4.1 that in $\mathfrak{R}(GF(2))$ all three zero divisors are nilpotents. One can also easily find the number of nilpotents in $\mathfrak{R}(GF(2^3))$. By Corollary 2.3 c), we have to put $a = 0$ and find all b, c, d , such that $b + c + d = 0$, where $(b, c, d) \neq (0, 0, 0)$. We have 7^3 ordered triples (b, c, d) without 0, $3 \cdot 7^2$ triples with one 0, and $3 \cdot 7$ triples with two 0-s. Together there is 511 such triples. Let us try to find in $\mathfrak{R}(GF(2^3))$ zero divisors $a + bi + cj + dk$ which are not nilpotents (by Proposition 4.3 2., idempotents either). Then, by Theorem 2.7 and the assumptions $t = 1$, $\text{char } F = 2$, the following condition holds

$$a^2 + a(b + c + d) + (b + c + d)^2 = 0, \quad a \neq 0 \neq b + c + d.$$

Denote $b + c + d = y$ and obtain $a^2 + ay + y^2 = 0$, i.e. $(a + y)^2 = ay$. Yet, this equation has no solutions (a, y) in $F(2^3)$. Therefore all zero divisors in $\mathfrak{R}(GF(2^3))$ are nilpotents. If we change the assumption $t = 1$ by $bc + bd + cd = 0$ and $t \in \{A, C, E\}$, then we have to solve the equation $(a + y)^2 = tay$. Yet, for $t \in \{A, C, E\}$ such equation has no solution (a, y) again.

Theorem 4.6. Let $t = 1$, $\text{char } F > 3$ and $|F| = q$. Then:

1. The number of all proper zero divisors in $\mathfrak{R}(F)$ is equal to:
 - a) $q^2 - 1$ if -3 is not a square in F . Then all zero divisors are nilpotents;
 - b) $2q^3 - q^2 - 1$ (and $q^2 - 1$ nilpotents among them) if -3 is a square in F .
2. a) 0 and 1 are the only idempotents in $\mathfrak{R}(F)$ if -3 is not a square in F ;
- b) The number of all idempotents in $\mathfrak{R}(F)$ is equal to $2 \cdot q^2 + 2$ if -3 is a square in F .

Proof. Ad. 1. We have to determine how many quadruples (a, b, c, d) are solutions of the equation $a^2 - a(b + c + d) + (b + c + d)^2 = 0$. As before, we use Lemma 2.13 and Remark 2.12, where $x = a$, $y = b + c + d$.

Assume that $a = 0$, $b + c + d = 0$. The value d is uniquely determined for every fixed pair b, c . Therefore we have got exactly q^2 quadruples (a, b, c, d) (together with $(0, 0, 0, 0)$) satisfying the conditions $a = 0$, $b + c + d = 0$. By Corollary 2.3 c), such elements and only such elements are nilpotents.

If -3 is not a square in F then the equation $x^2 - xy + y^2 = 0$ has only the solution $(0, 0)$ and there is no other zero divisor.

Now let $-3 \in S$. By Lemma 2.13 b), for every nonzero $a = x \in F$ there exist exactly two nonzero y -s such that $x^2 - xy + y^2 = 0$. If one of two y -s is fixed, then for every pair $(b, c) \in F^2$, the value of d is uniquely determined from the equality $y = b + c + d$. Hence we obtain $(q - 1) \cdot 2 \cdot q^2$ quadruples (a, b, c, d) with $a \neq 0$, $b + c + d \neq 0$. The total number of proper zero divisors in this case is equal to $q^2 - 1 + (q - 1) \cdot 2 \cdot q^2 = 2q^3 - q^2 - 1$.

Ad. 2. Using Corollary 2.3, the condition $(a + bi + cj + dk)^2 = a + bi + cj + dk$ may be written as the following equations in F :

$$(4.1) \quad \begin{cases} a^2 & -(b + c + d)^2 & = & a, \\ b & (2a - b - d - c - 1) & = & 0, \\ c & (2a - c - b - d - 1) & = & 0, \\ d & (2a - d - b - c - 1) & = & 0. \end{cases}$$

If $b + c + d \neq 2a - 1$ then $b = c = d = 0$, whence $a^2 = a$, i.e. $a = 0$ or $a = 1$. Now we use Proposition 4.3. If $b + c + d = 2a - 1$ and $-3 \in S$ then the equation $3a^2 - 3a + 1 = 0$ (or equivalently $[2 \cdot (a - 2^{-1})]^2 = (-3)^{-1}$) has two distinct solutions. Let $(a - 2^{-1})^2 = v^2 = (-3)^{-1} \cdot 2^{-2}$ for some $v \in F$. Thus $a = 2^{-1} + v$ or $a = 2^{-1} - v$. For each such a and every pair $(b, c) \in F \times F$, the value of d is uniquely determined. Therefore we have got exactly $2q^2$ idempotents different from 0 and 1. $\square$

From Proposition 2.11 and Theorem 4.6 1. the following is immediate.

Corollary 4.7. *Let $t = 1$ and $|F| = q$. Then the group of all invertible elements in $\mathfrak{R}(F)$ consists of $q^2(q^2 - 1)$ elements if -3 is not a square in F , and of $q^2(q - 1)^2$ elements if -3 is a square.*

Example 4.8. Assume that $t = 1$. We know from Example 2.5 that $x^2 + x + 1$ is irreducible in $GF(2)$, $GF(5)$ and $GF(11)$. This polynomial is also irreducible in $GF(17)$, $GF(19)$, $GF(23)$, $GF(29)$, ... In $GF(5)$, $GF(11)$, $GF(17)$, $GF(23)$ and $GF(29)$, -3 is not a square (Proposition 1.2 is helpful here). So, in $\mathfrak{R}(GF(5))$, $\mathfrak{R}(GF(11))$, $\mathfrak{R}(GF(17))$, $\mathfrak{R}(GF(23))$, $\mathfrak{R}(GF(29))$ only 1 and 0 are idempotents and total number of zero divisors is equal to 5^2 , 11^2 , 17^2 , 23^2 , 29^2 , respectively. Now, consider $GF(19)$. This time the subgroup of nonzero squares consists of: 1, 4, 9, 16, 6, 17, 11, 7, 5. Note that $-3 = 16$ is a square. Since $[2(a - 2^{-1})]^2 = (4^{-1})^2$, we obtain $2(a - 10) = 5$ or $2(a - 10) = -5 = 14$, i.e. $a = 3$ or $a = 17$. There exist exactly $2 \cdot 19^2 + 2 = 724$ idempotents, e.g. $3 + 2i + 4j + 18k$ and $17 + 5i + 7j + 2k$ are idempotents. Total number of zero divisors is equal to $2 \cdot 19^3 - 19^2 = 13357$. There

exist exactly 361 nilpotents among them. In the same way, one can study $GF(37)$, where $-3 = 34 = 16^2$ is a square.

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Presented by Adam Paszkiewicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on March 23, 2017.

PEWNE PIERŚCIEŃ NIEPRZEMIENNE KONSTRUOWANE NA PODSTAWIE WIELOMIANÓW $x^2 + tx + 1$ ORAZ ICH DZIELNIKI ZERA

Streszczenie

W pracy konstruujemy pewne rozszerzenie pierścieniowe dla dowolnego ciała Galois $F = GF(p^n)$. Najpierw znajdujemy $t \in F$, takie że wielomian $x^2 + tx + 1$ nie ma pierwiastków w F (takie t zawsze istnieją!). Następnie bierzemy 4-wymiarową przestrzeń wektorową nad F z elementami bazowymi $1, i, j, k$, gdzie i, j, k nie należą do F i zakładamy, że są one pierwiastkami wielomianu $x^2 + tx + 1$ w konstruowanym rozszerzeniu. Ponadto przyjmujemy $ji = -k - t$. Te warunki wystarczają do zdefiniowania mnożenia na elementach bazowych i w konsekwencji w całej algebraicznej strukturze tak, by otrzymać pierścień. Taka konstrukcja jest podobna do konstrukcji Hamiltona kwaternionów rzeczywistych, ale jest ona dużo bardziej ogólna i zastosowana do ciał skończonych. Najpierw w celu samej konstrukcji, a potem do badania własności pierścieni $\mathfrak{R}(F)$, używamy wybranych własności ciał oraz wielomianów $x^2 + tx + 1$.

Wśród otrzymanych wyników, w szczególności podajemy warunek konieczny i dostateczny dla dowolnego elementu z $\mathfrak{A}(F)$, aby był on dzielnikiem zera, w tym nilpotentem lub idempotentem rzędu 2. Rozważono wiele wersji tego warunku w szczególnych przypadkach i opisano przykłady. Zbadano także własności kombinatoryczne. Pewne ważne własności badanych w pracy pierścieni zależą od charakterystyki ciała i od tego, czy pewne elementy są kwadratami w danym ciele. Podsumowując, w pracy skonstruowano szeroką klasę pierścieni nieprzemiennych i zbadano ich wybrane własności. Ta konstrukcja może być zastosowana dla dowolnych ciał (także nieskończonych) o charakterystyce różnej od zera.

Słowa kluczowe: ciało Galois, wielomian, pierścień nieprzemienny, dzielnik zera, przestrzeń wektorowa, skośne ciało kwaternionów

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2017

Vol. LXVII

Recherches sur les déformations

no. 1

pp. 71–78

*Ewa Krot-Sieniawska***REDUCED INCIDENCE ALGEBRAS DESCRIPTION OF COBWEB POSETS AND KoDAGs****Summary**

The notion of reduced incidence algebra of an arbitrary cobweb poset is delivered. The incidence algebra combinatorial characterization for an arbitrary positive integers valued sequence is given.

Keywords and phrases: cobweb poset, incidence algebra of locally finite poset, order compatible equivalence relation, reduced incidence algebra.

1. Cobweb posets

The family of the so called cobweb posets Π has been invented by A. K. Kwaśniewski few years ago (for references see: [6, 7]). These structures are such a generalization of the Fibonacci tree growth that allows joint combinatorial interpretation for all of them under the admissibility condition (see [8, 9]).

Let $\{F_n\}_{n \geq 0}$ be a positive integers valued sequence with $F_0 = 1$ (with $F_0 = 0$ being exceptional as in case of Fibonacci numbers). Any sequence satisfying this property uniquely designates cobweb poset defined as follows.

For $s \in \mathbf{N}_0 = \mathbf{N} \cup \{\mathbf{0}\}$ let us to define levels of Π :

$$\Phi_s = \{\langle j, s \rangle, \ 1 \leq j \leq F_s\},$$

(in case of $F_0 = 0$ level Φ_0 corresponds to the empty root $\{\emptyset\}$). Then

Definition 1. *Corresponding cobweb poset is an infinite partially ordered set $\Pi = (V, \leq)$, where*

$$V = \bigcup_{0 \leq s} \Phi_s$$

are the elements (vertices) of Π and the partial order relation $\leq$ on V for $x = \langle s, t \rangle, y = \langle u, v \rangle$ being elements of cobweb poset Π defined by formula

$$(x \leq_P y) \iff [(t < v) \vee (t = v \wedge s = u)].$$

Obviously any cobweb poset can be represented, via its Hasse diagram, as infinite directed graph $\Pi = (V, E)$, where set V of its vertices is defined as above and

$$E = \{(\langle j, p \rangle, \langle q, (p+1) \rangle)\} \cup \{(\langle 1, 0 \rangle, \langle 1, 1 \rangle)\},$$

where $1 \leq j \leq F_p$ and $1 \leq q \leq F_{(p+1)}$ stay for set of (directed) edges.

The Kwaśniewski cobweb posets under consideration represented by graphs are examples of orderable directed acyclic graphs (oDAG) which we start to call from now in brief: KoDAGs. These are structures of universal importance for the whole of mathematics and computer sciences in general (quotation from [8, 9]):

For any given natural numbers valued sequence the graded (layered) cobweb posets' DAGs are equivalently representations of a chain of binary relations. Every relation of the cobweb poset chain is biunivocally represented by the uniquely designated **complete** bipartite digraph-a digraph which is a di-biclique designated by the very given sequence. The cobweb poset is then to be identified with a chain of di-bicliques i.e. by definition-a chain of complete bipartite one direction digraphs. Any chain of relations is therefore obtainable from the cobweb poset chain of complete relations via deleting arcs (arrows) in di-bicliques.

According to the definition above arbitrary cobweb poset $\Pi = (V, \leq)$ is a graded poset (ranked poset) and for $s \in \mathbf{N}_0$:

$$x \in \Phi_s \longrightarrow r(x) = s,$$

where $r : \Pi \rightarrow \mathbf{N}_0$ is a rank function on Π .

Let us then define Kwaśniewski finite cobweb sub-posets as follows

Definition 2. Let $P_n = (V_n, \leq)$, ($n \geq 0$), for $V_n = \bigcup_{0 \leq s \leq n} \Phi_s$ and $\leq$ being the induced partial order relation on Π .

It is easy to see that P_n is ranked poset with rank function r as above. P_n has a unique minimal element $0 = \langle 1, 0 \rangle$ (with $r(0) = 0$). Moreover Π and all P_n -s are locally finite, i.e. for any pair $x, y \in \Pi$, the segment $[x, y] = \{z \in \Pi : x \leq z \leq y\}$ is finite.

Let us recall that one defines the incidence algebra of a locally finite partially ordered set P as follows (see [10–12]):

$$I(P) = I(P, R) = \{f : P \times P \longrightarrow R; \quad f(x, y) = 0 \text{ unless } x \leq y\}.$$

The sum of two such functions f and g and multiplication by scalar are defined as usual. The product $h = f * g$ is defined as follows:

$$h(x, y) = (f * g)(x, y) = \sum_{z \in \mathbf{P}: x \leq z \leq y} f(x, z) \cdot g(z, y).$$

It is immediately verified that this is an associative algebra (with an identity element $\delta(x, y)$, the Kronecker delta), over any associative ring R .

In [5] the incidence algebra of an arbitrary cobweb poset Π (or its subposets P_n) uniquely designated by the natural numbers valued sequence $\{F_n\}_{n \geq 0}$, was considered by the present author. The explicit formulas for some typical elements of incidence algebra $I(\Pi)$ of Π where delivered there.

So for x, y being some arbitrary elements of Π such that $x = \langle s, t \rangle$, $y = \langle u, v \rangle$, ($s, u \in \mathbf{N}$, $t, v \in \mathbf{N}_0$), $1 \leq s \leq F_t$ and $1 \leq u \leq F_v$ one has:

(1) ζ function of Π being a characteristic function of partial order in Π

$$(1) \quad \zeta(x, y) = \zeta(\langle s, t \rangle, \langle u, v \rangle) = \delta(s, u) \delta(t, v) + \sum_{k=1}^{\infty} \delta(t + k, v),$$

one can also verify, that ζ^k enumerates all multichains of length k ,

(2) Möbius function of Π being a inverse of ζ

$$(2) \quad \mu(x, y) = \delta(t, v) \delta(s, u) - \delta(t + 1, v) + \sum_{k=2}^{\infty} \delta(t + k, v) (-1)^k \prod_{i=t+1}^{v-1} (F_i - 1),$$

(3) function $\zeta^2 = \zeta * \zeta$ counting the number of elements in the segment $[x, y]$

$$(3) \quad \zeta^2(x, y) = \text{card}[x, y] = \left(\sum_{i=t+1}^{v-1} F_i \right) + 2,$$

(4) function η

$$(4) \quad \eta(x, y) = \sum_{k=1}^{\infty} \delta(t + k, v) = \begin{cases} 1 & t < v \\ 0 & w.p.p. \end{cases},$$

(5) function $\eta^k(x, y)$, ($k \in \mathbf{N}$) counting the number of chains of length k , (with $(k + 1)$ elements) from x to y

$$(5) \quad \begin{aligned} \eta^k(x, y) &= \sum_{x < z_1 < z_2 < \dots < z_{k-1} < y} 1 \\ &= \sum_{t < i_1 < i_2 < \dots < i_{k-1} < v} F_{i_1} F_{i_2} \dots F_{i_{k-1}}, \end{aligned}$$

(6) function $\mathcal{C}$

$$(6) \quad \mathcal{C}(\langle s, t \rangle, \langle u, v \rangle) = \delta(t, v)\delta(s, u) - \sum_{k=1}^{\infty} \delta(t + k, v),$$

such that its inverse function $\mathcal{C}^{-1}(x, y)$ counts the number of all chains from x to y

(7) function χ

$$(7) \quad \chi(x, y) = \delta(t + 1, v),$$

(8) function $\chi^k(x, y)$, ($k \in \mathbf{N}$) counting the number of maximal chains of length k , (with $(k + 1)$ elements) from x to y

$$(8) \quad \chi^k(x, y) = \sum_{x \leq z_1 \leq \dots \leq z_{k-1} \leq y} 1 = \delta(t + k, v) F_{t+1} F_{t+2} \dots F_{v-1}.$$

(9) function $\mathcal{M}$

$$(9) \quad \mathcal{M}(\langle s, t \rangle, \langle u, v \rangle) = \delta(t, v)\delta(s, u) - \delta(t + 1, v),$$

such that its inverse function $\mathcal{M}^{-1}$ counts the number of all maximal chains from x to y .

In this paper the notion of the standard reduced incidence algebra [1, 11, 12] of an arbitrary cobweb poset Π will be delivered. As we shall see, it enables us for example to facilitate the formulas presented above. The results presented below stay true when considering finite subposets P_n defined above.

2. The Standard Reduced Incidence Algebra of an arbitrary cobweb poset

Let $S(\Pi)$ be the set of all segments in Π and let $\sim$ be the equivalence relation $\sim \subseteq S(\Pi) \times S(\Pi)$.

Let us recall that $\sim$ is compatible ([1]), i.e. it satisfies the following condition: if f and g belong to the incidence algebra $I(P)$ and $f(x, y) = f(u, v)$ as well as $g(x, y) = g(u, v)$ for all pairs of segments such that $[x, y] \sim [u, v]$, then $(f * g)(x, y) = (f * g)(u, v)$.

The equivalence classes of segments of Π relative to $\sim$ are called types. The set of all functions defined on types (i.e. all functions taking the same value on equivalent segments) forms an associative algebra with identity. One calls it the reduced incidence algebra $R(\Pi, \sim)$ (modulo of equivalence relation $\sim$). Let us note that $R(\Pi, \sim)$ is isomorphic to a subalgebra of $I(\Pi)$, ([1]).

Now let $\sim$ be defined as follows

$$(10) \quad [x, y] \sim [u, v] \iff \text{segments } [x, y], [u, v] \text{ are isomorphic.}$$

One can show that it is order compatible. Then one calls $R(\Pi, \sim) = R(\Pi)$ the standard reduced incidence algebra of Π . Also from the definitions of $\sim$ and partial order on Π one infers that

$$[x, y] \sim [u, v] \iff [r(x) = r(u) \wedge r(y) = r(v)].$$

So let T be the set of types of relation $\sim$ defined above. Then

$$T = \{(k, n) : k, n \in \mathbf{N}_0\}$$

and for $k \leq n$

$$(11) \quad (k, n) = \{[x, y] \in S(\Pi) : r(x) = k \wedge r(y) = n\},$$

or equivalently

$$(k, n) = \{[x, y] \in S(\Pi) : x \in \Phi_k \wedge y \in \Phi_n\}.$$

Also let $(k, n) = \emptyset$ for $k > n$.

Definition 3. Let $[x, y] \in \alpha_{k, n}$. For $l \in \mathbb{N}_0$ one can define the incidence coefficients in $R(\Pi)$ as follows

$$(12) \quad \left\langle \begin{smallmatrix} k, n \\ l \end{smallmatrix} \right\rangle = |\{z \in [x, y] : [x, z] \in (k, l) \wedge [z, y] \in (l, n)\}|.$$

The following formula holds.

Proposition 1.

$$(13) \quad \left\langle \begin{smallmatrix} k, n \\ l \end{smallmatrix} \right\rangle = \begin{cases} F_l & k \leq l \leq n \\ 0 & \text{otherwise} \end{cases}.$$

Now one can define the product $*$ in $R(\Pi)$ as follows.

Proposition 2. Let $[x, y] \in (k, n)$, $(k \leq n)$ and $f, g \in R(\Pi)$. Then

$$(14) \quad (f * g)(k, n) = \sum_{l \geq 0} F_l f(k, l) g(l, n)$$

with the assumption that $(k, n) = \emptyset$, for $l < k$ or $l > n$.

$$(f * g)(k, n) = (f * g)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$$

$$= \sum_{k \leq l \leq n} \sum_{\{z: [x, z] \in (k, l), [z, y] \in (l, n)\}} f(x, z)g(z, y)$$

Proof.

$$= \sum_{k \leq l \leq n} F_l f(k, l)g(l, n)$$

$$= \sum_{l \geq 0} F_l f(k, l)g(l, n).$$

□

So we have proved

Theorem 1. *Let $F = \{F_n\}_{n \geq 0}$ $\sum F_0 = 1$, be an arbitrary natural numbers valued sequence with $F_0 = 1$ (with $F_0 = 0$ being exceptional). The the numbers F_n ($n \geq 0$) are the incidence coefficients in the standard reduced algebra $R(\Pi)$ of cobweb poset Π uniquely designated by the sequence $F = \{F_n\}_{n \geq 0}$.*

One can also show the following

Theorem 2. *Let $f \in R(\Pi)$. Then for $x, y \in \Pi$ such that $[x, y] \in (k, n)$ the value $f(x, y)$ depends on $r(x) = k$ and $r(y) = n$ only, i.e.*

$$(15) \quad f(x, y) = f(k, n) = f(r(x), r(y)).$$

From the definition of partial order on Π one can also infer that for $x, y \in \Pi$ satisfying $r(x) = r(y)$ and for $f \in I(\Pi)$, one has

$$f(x, y) = \delta(x, y).$$

It is known that all elements of $I(\Pi)$ mentioned above, i.e. functions: $\zeta, \mu, \zeta^2, \zeta^k, \eta, \eta^k, \mathcal{C}, \mathcal{C}^{-1}, \chi, \chi^k, \mathcal{M}, \mathcal{M}^{-1}$ are the elements of an arbitrary reduced incidence algebra $R(\Pi, \sim)$, (i.e. modulo an arbitrary order compatible equivalence relation $\sim$ on $S(\Pi)$). Then the next results follow immediately from this fact and above theorems.

Corollary 1. *Let $(k, n) \in T$. Then:*

$$(16) \quad \zeta(k, n) = \begin{cases} 1 & k \leq n \\ 0 & k > n \end{cases};$$

$$(17) \quad \zeta^2(k, n) = \begin{cases} \left(\sum_{i=k}^{n-1} F_i \right) + 2 & k \leq n \\ 0 & k > n \end{cases};$$

$$(18) \quad \eta(k, n) = \delta_{k < n} = \begin{cases} 1 & k < n \\ 0 & k \geq n \end{cases};$$

$$(19) \quad \eta^2(k, n) = \begin{cases} \sum_{i=k+1}^{n-1} F_i & k \leq n \\ 0 & k > n \end{cases};$$

$$(20) \quad \eta^s(k, n) = \begin{cases} \sum_{k < i_1 < \dots < i_{s-1} < n} F_{i_1} F_{i_2} \dots F_{i_{s-1}} & k \leq n \\ 0 & k > n \end{cases};$$

$$(21) \quad \mathcal{C}(k, n) = \begin{cases} 1 & k = n \\ -1 & k < n \\ 0 & k > n \end{cases};$$

$$(22) \quad \chi(k, n) = \delta(k + 1, n);$$

$$(23) \quad \chi^s(k, n) = \delta(k + s, n) \cdot F_{k+1} F_{k+2} \dots F_{n-1};$$

$$(24) \quad \mu(k, n) = \begin{cases} (-1)^{n-k} \prod_{i=k+1}^{n-1} (F_i - 1) & k \leq n \\ 0 & k > n \end{cases}.$$

Corollary 2. *The standard reduced incidence algebra $R(\Pi)$ is the maximally reduced incidence algebra $\bar{R}(\Pi)$, i.e. the smallest reduced incidence algebra on Π . Equivalently the equivalence relation defined by (10) is the maximal element in the lattice of all order compatible equivalence relations on $S(\Pi)$.*

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Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on January 27, 2016.

OPIS COBWEB POSETÓW W JĘZYKU ZREDUKOWANYCH ALGEBR INCYDENCJI

S t r e s z c z e n i e

W pracy opisano strukturę zredukowanej algebry incydencji dowolnego cobweb posetu. Na tej podstawie podano kombinatoryczną interpretację dowolnego ciągu liczbowego o wartościach ze zbioru liczb naturalnych.

Słowa kluczowe: zbiór częściowo uporządkowany (poset), cobweb poset, algebra incydencji zbioru częściowo uporządkowanego lokalnie skończonego, zredukowana algebra incydencji

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2017

Vol. LXVII

Recherches sur les déformations

no. 1

pp. 79–90

*Bogumiła Kowalczyk, Adam Lecko, and Barbara Śmiarowska***ON SOME COEFFICIENT INEQUALITY IN THE SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS****Summary**

A coefficient inequality related to the Fekete-Szegő-Goluzin problem in some subclass of close-to-convex functions is shown.

Keywords and phrases: Coefficient inequality, close-to-convex functions, Fekete-Szegő-Goluzin problem.

1. Introduction

To find for each $\lambda \in [0, 1]$ the maximum value of the coefficient functional

$$\Phi_\lambda(f) := |a_3 - \lambda a_2^2|$$

over the class $\mathcal{S}$ of univalent functions f in the unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbf{D},$$

is a well known problem having the source in the paper [5] by Fekete and Szegő. They considered the case $\lambda := (k-1)/(2k)$, $k = 2, 3, \dots$, however the case $\lambda \in (0, 1)$ was first discussed and solved by Goluzin [6]. Particularly, recall that

$$\max_{f \in \mathcal{S}} \Phi_\lambda(f) = \begin{cases} 1 + 2 \exp(-2\lambda/(1-\lambda)), & \lambda \in [0, 1), \\ 1, & \lambda := 1. \end{cases}$$

The problem to find $\max_{f \in \mathcal{F}} \Phi_\lambda(f)$ over compact subclasses $\mathcal{F}$ of the class $\mathcal{A}$ of all analytic functions f in $\mathbf{D}$ of the form (1.1), as well as for λ being an arbitrary real or complex number, was studied by many authors (see e.g., [8], [12], [9], [10], [15], [32], [28], [17], [13], [11], [16], [2]).

Let $\mathcal{S}^*$ denote the class of functions $f \in \mathcal{A}$ such that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbf{D},$$

called *starlike*, and let $\mathcal{S}^c$ denote the class of functions $f \in \mathcal{A}$ such that

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad z \in \mathbf{D},$$

called *convex*. Clearly, $\mathcal{S}^c \subsetneq \mathcal{S}^*$.

Given $\delta \in (-\pi/2, \pi/2)$ and $g \in \mathcal{S}^*$, let $\mathcal{C}_\delta(g)$ denote the class of functions $f \in \mathcal{A}$ such that

$$\operatorname{Re} \left\{ e^{i\delta} \frac{zf'(z)}{g(z)} \right\} > 0, \quad z \in \mathbf{D},$$

called *close-to-convex with argument δ with respect to g* . For $g \in \mathcal{S}^*$ let

$$\mathcal{C}(g) := \bigcup_{\delta \in (-\pi/2, \pi/2)} \mathcal{C}_\delta(g)$$

be the class of functions called *close-to-convex with respect to g* . For $\delta \in (-\pi/2, \pi/2)$ let

$$\mathcal{C}_\delta := \bigcup_{g \in \mathcal{S}^*} \mathcal{C}_\delta(g), \quad \mathcal{C}_\delta^c := \bigcup_{h \in \mathcal{S}^c} \mathcal{C}_\delta(h).$$

Let

$$\mathcal{C} := \bigcup_{\delta \in (-\pi/2, \pi/2)} \bigcup_{g \in \mathcal{S}^*} \mathcal{C}_\delta(g)$$

denote the class of *close-to-convex* functions (see [27, pp. 184-185], [14]), and let

$$\mathcal{C}^c := \bigcup_{\delta \in (-\pi/2, \pi/2)} \bigcup_{h \in \mathcal{S}^c} \mathcal{C}_\delta(h).$$

In [15] Keogh and Merkes proved that

$$\max_{f \in \mathcal{C}_0} \Phi_\lambda(f) = \begin{cases} |3 - 4\lambda|, & \lambda \in \mathbf{R} \setminus (1/3, 1), \\ 1/3 + 4/(9\lambda), & \lambda \in [1/3, 2/3], \\ 1, & \lambda \in [2/3, 1]. \end{cases}$$

For $\lambda \in [0, 1]$ Koepf [17] extended the above result for the whole class $\mathcal{C}$ showing that

$$\max_{f \in \mathcal{C}} \Phi_\lambda(f) = \max_{f \in \mathcal{C}_0} \Phi_\lambda(f).$$

In fact, the last result holds for all real λ .

For the class $\mathcal{C}_0^c$ the Fekete-Szegő problem was considered by Abdel-Gawad and Thomas [1]. For the whole class $\mathcal{C}^c$ the computing was done by Srivastava, Mishra

and Das [31]. Their results together with some remark of [22] can be written as follow:

$$\max_{f \in \mathcal{C}^c} \Phi_\lambda(f) = \max_{f \in \mathcal{C}_0^c} \Phi_\lambda(f) = \begin{cases} 5/3 - 9\lambda/4, & \lambda \in [0, 2/9], \\ 2/3 + 1/(9\lambda), & \lambda \in [2/9, 2/3], \end{cases}$$

and

$$\max_{f \in \mathcal{C}_0^c} \Phi_\lambda(f) \leq \max_{f \in \mathcal{C}^c} \Phi_\lambda(f) \leq \frac{5}{6}, \quad \lambda \in (2/3, 1].$$

Given $\alpha \in [0, 1]$ let, for $z \in \mathbf{D}$,

$$g_\alpha(z) := \frac{z}{(1 - \alpha z)^2}, \quad h_\alpha(z) := \frac{z}{1 - \alpha z}.$$

Clearly, $g_\alpha \in \mathcal{S}^*$ and $h_\alpha \in \mathcal{S}^c$ for $\alpha \in [0, 1]$. The corresponding classes $\mathcal{C}(g_\alpha)$ and $\mathcal{C}(h_\alpha)$ are defined, respectively, as: for $\delta \in (-\pi/2, \pi/2)$,

$$(1.2) \quad \operatorname{Re} \{e^{i\delta}(1 - \alpha z)^2 f'(z)\} > 0, \quad z \in \mathbf{D},$$

and

$$(1.3) \quad \operatorname{Re} \{e^{i\delta}(1 - \alpha z) f'(z)\} > 0, \quad z \in \mathbf{D}.$$

For the class $\mathcal{C}(g_\alpha)$ it was shown in [18] that

$$(1.4) \quad \begin{aligned} & \max_{f \in \mathcal{C}(g_\alpha)} \Phi_\lambda(f) \\ & \leq \begin{cases} \left| \frac{2}{3} + \frac{4}{3}\alpha + \alpha^2 - (1 + \alpha)^2 \lambda \right|, & \lambda \in \mathbf{R} \setminus (\tau_1(\alpha), \tau_2(\alpha)), \\ \alpha^2 \left(\frac{(2 - 3\lambda)^2}{3(2 - |2 - 3\lambda|)} + |1 - \lambda| \right) + \frac{2}{3}, & \lambda \in [\tau_1(\alpha), \tau_2(\alpha)], \end{cases} \end{aligned}$$

where

$$(1.5) \quad \tau_1(\alpha) := \frac{2\alpha}{3(1 + \alpha)}, \quad \tau_2(\alpha) := \frac{2(2 + \alpha)}{3(1 + \alpha)}.$$

The sharpness holds for each $\alpha \in (0, 1]$ and each $\lambda \in \mathbf{R} \setminus (2/3, \tau_2(\alpha))$ as well as for $\alpha := 0$ and each $\lambda \in \mathbf{R}$.

As it is known, the Koebe function $k := g_1$ ($\alpha := 1$) is extremal for various computational problems in the class $\mathcal{S}^*$ of starlike functions. Moreover the class $\mathcal{C}_0(k)$ of functions called *convex in the positive direction of the real axis* plays an important role as the subclass of functions convex in one direction defined by Robertson [30] and it was intensively recently studied (see e.g., [3], [24], [4]). For the class $\mathcal{C}(k)$ the Fekete-Szegő problem was separately considered in [19] where it was shown that

$$\max_{f \in \mathcal{C}(k) \cup \{k_0\}} \Phi_\lambda(f) = \max_{f \in \mathcal{C}} \Phi_\lambda(f), \quad \lambda \in \mathbf{R},$$

where

$$k_0(z) := \frac{z}{1 - z^2}, \quad z \in \mathbf{D},$$

is the odd close-to-convex function.

For the class $\mathcal{C}(h_\alpha)$ it was shown in [21] that

$$(1.6) \quad \max_{f \in \mathcal{C}(h_\alpha)} \Phi_\lambda(f) \leq \begin{cases} \alpha^2 \left| \frac{1}{3} - \frac{\lambda}{4} \right| + (1 + \alpha) \left| \frac{2}{3} - \lambda \right|, & \lambda \in \mathbf{R} \setminus [\tau'_1(\alpha), \tau'_2(\alpha)], \\ \alpha^2 \left(\frac{(2 - 3\lambda)^2}{12(2 - |2 - 3\lambda|)} + \left| \frac{1}{3} - \frac{\lambda}{4} \right| \right) + \frac{2}{3}, & \lambda \in [\tau'_1(\alpha), \tau'_2(\alpha)], \end{cases}$$

where

$$\tau'_1(\alpha) := \frac{2\alpha}{3(2 + \alpha)}, \quad \tau'_2(\alpha) := \frac{2(4 + \alpha)}{3(2 + \alpha)}.$$

The sharpness holds for each $\alpha \in (0, 1]$ and each $\lambda \in \mathbf{R} \setminus (2/3, 4/3)$, as well as for $\alpha := 0$ and each $\lambda \in \mathbf{R}$.

As it is known, the function $h := h_1$ ($\alpha := 1$) is extremal for computational problems in the class $\mathcal{S}^c$ of convex functions. For the first time the inequality (1.3) with $\alpha = 1$, treated as the univalence criterium, was distinguished explicitly in [27, p. 185]. For the class $\mathcal{C}(h)$ the Fekete-Szegő problem was separately considered in [20] where it was shown (1.6) for $\alpha = 1$; particularly, it was proved that for $\lambda \in [0, 2/3]$,

$$\max_{f \in \mathcal{C}(h)} \Phi_\lambda(f) = \max_{f \in \mathcal{C}_0^c} \Phi_\lambda(f) = \max_{f \in \mathcal{C}^c} \Phi_\lambda(f).$$

For $\alpha := 0$ the conditions (1.2) and (1.3) reduce to

$$(1.7) \quad \operatorname{Re} \{e^{i\delta} f'(z)\} > 0, \quad z \in \mathbf{D}.$$

Functions f having such a property are called *of bounded turning with argument δ* and form the class denoted usually as $\mathcal{P}'(\delta)$, and further the class $\mathcal{P}' := \mathcal{C}(g_0)$ of functions called *of bounded turning* (cf. [7, Vol. I, p. 101]). On the other hand, the condition (1.7) is known as a famous criterium of univalence due to Noshiro [26] and Warschawski [33]. By setting $\alpha := 0$ into (1.4) or (1.6) we get the following result published, among other results, in [13, Theorem 2.3]: for $\lambda \in [0, 4/3]$,

$$\max_{f \in \mathcal{P}'} \Phi_\lambda(f) = \frac{2}{3}.$$

In this paper we unify results recalled above for the classes $\mathcal{C}(g_\alpha)$ and $\mathcal{C}(h_\alpha)$ with $\alpha \in [0, 1]$. Given $\alpha, \beta \in [0, 1]$, let

$$g_{\alpha, \beta}(z) := \frac{z}{(1 - \alpha z)(1 - \beta z)}, \quad z \in \mathbf{D}.$$

Thus the class $\mathcal{C}(g_{\alpha, \beta})$ is defined as

$$\operatorname{Re} \{e^{i\delta} (1 - \alpha z)(1 - \beta z) f'(z)\} > 0, \quad z \in \mathbf{D}.$$

Clearly, $\mathcal{C}(g_{\alpha, \alpha}) = \mathcal{C}(g_\alpha)$ and $\mathcal{C}(g_{\alpha, 0}) = \mathcal{C}(g_{0, \alpha}) = \mathcal{C}(h_\alpha)$ for $\alpha \in [0, 1]$. The class $\mathcal{C}(g_{\alpha, \beta})$ appeared in [23] and [25] as a generalization of convexity in one direction (see [30]). In the main result we show the upper bound for the Fekete-Szegő functional for the class $\mathcal{C}(g_{\alpha, \beta})$ generalizing (1.4) and (1.6).

2. Main result

Let $\mathcal{P}$ denote the class of analytic functions in $\mathbf{D}$ of the form

$$(2.1) \quad p(z) := 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbf{D},$$

having a positive real part in $\mathbf{D}$. Let

$$L(z) := \frac{1+z}{1-z}, \quad z \in \mathbf{D}.$$

Lemma 2.1. ([25, pp. 41,46]) *If $p \in \mathcal{P}$ is of the form (2.1), then*

$$(2.2) \quad |c_n| \leq 2, \quad n \in \mathbf{N},$$

and

$$(2.3) \quad |c_2 - c_1^2/2| \leq 2 - |c_1|^2/2.$$

Both inequalities are sharp. The equality in (2.2) holds for L and in (2.3) for every function

$$(2.4) \quad p_{t,\theta}(z) := tL(e^{i\theta}z) + (1-t)L(e^{2i\theta}z^2) = 1 + 2te^{i\theta}z + 2e^{2i\theta}z^2 + \dots, \quad z \in \mathbf{D},$$

where $t \in [0, 1]$ and $\theta \in \mathbf{R}$.

The details of the proof of the main theorem are almost exactly the same as of Theorem 2.4 of [18]. Therefore here we present only a sketch of the proof. Similar method of proof with all details of computing appeared in [21]. The main theorem of the paper is

Theorem 2.2. *Let $\alpha, \beta \in [0, 1]$. Then*

$$(2.5) \quad \max_{f \in \mathcal{C}(g_{\alpha,\beta})} \Phi_{\lambda}(f) \leq \begin{cases} \frac{1}{6} \left| \alpha^2 + \beta^2 + (\alpha + \beta)^2 \left(1 - \frac{3}{2}\lambda \right) \right| \\ \frac{1}{6} \left| \alpha^2 + \beta^2 + (\alpha + \beta)^2 \left(1 - \frac{3}{2}\lambda \right) \right| \\ + (1 + \alpha + \beta) \left| \frac{2}{3} - \lambda \right|, & \lambda \in \mathbf{R} \setminus (\tau_1(\alpha, \beta), \tau_2(\alpha, \beta)) \\ + \frac{(\alpha + \beta)^2(2 - 3\lambda)^2}{12(2 - |2 - 3\lambda|)} + \frac{2}{3}, & \lambda \in [\tau_1(\alpha, \beta), \tau_2(\alpha, \beta)], \end{cases}$$

where

$$\tau_1(\alpha, \beta) := \frac{2(\alpha + \beta)}{3(2 + \alpha + \beta)}, \quad \tau_2(\alpha, \beta) := \frac{2(4 + \alpha + \beta)}{3(2 + \alpha + \beta)}.$$

For each $\alpha, \beta \in [0, 1]$, $(\alpha, \beta) \neq (0, 0)$, and each $\lambda \in \mathbf{R} \setminus (2/3, \lambda(\alpha, \beta))$, where

$$\lambda(\alpha, \beta) := \frac{2}{3} + \frac{2}{3} \max \left\{ \frac{\alpha^2 + \beta^2}{(\alpha + \beta)^2}, \frac{2}{2 + \alpha + \beta} \right\},$$

as well as for $\alpha = \beta := 0$ and each $\lambda \in \mathbf{R}$, the inequality is sharp and the equality is attained by a function in $\mathcal{C}_0(g_{\alpha,\beta})$. In particular,

(i) when $\alpha, \beta \in [0, 1]$, $(\alpha, \beta) \neq (0, 0)$, then for each $\lambda \in [\tau_1(\alpha, \beta), 2/3]$ the second equality in (2.5) is attained by a function $f_{\alpha,\beta,t_{\alpha,\beta,\lambda}}$ such that

$$(2.6) \quad f'_{\alpha,\beta,t_{\alpha,\beta,\lambda}}(z) = \frac{p_{t_{\alpha,\beta,\lambda},0}(z)}{(1-\alpha z)(1-\beta z)}, \quad z \in \mathbf{D},$$

with $f_{\alpha,\beta,t_{\alpha,\beta,\lambda}}(0) := 0$ and

$$t_{\alpha,\beta,\lambda} := (\alpha + \beta) (1/(3\lambda) - 1/2);$$

(ii) when $\alpha, \beta \in [0, 1]$, $(\alpha, \beta) \neq (0, 0)$, then for each $\lambda \in \mathbf{R} \setminus (\tau_1(\alpha, \beta), \lambda(\alpha, \beta))$ the first equality in (2.5) is attained by the function $f_{\alpha,\beta,1}$, given by (2.6) with $t_{\alpha,\beta,\lambda} := 1$, i.e., when $\alpha, \beta \in (0, 1)$, $\alpha \neq \beta$, by the function:

$$(2.7) \quad f_{\alpha,\beta,1}(z) := \frac{1}{\alpha - \beta} \left(\frac{1 + \alpha}{\alpha(1 - \alpha)} \log(1 - \alpha z) - \frac{1 + \beta}{\beta(1 - \beta)} \log(1 - \beta z) \right) \\ - \frac{2}{(1 - \alpha)(1 - \beta)} \log(1 - z), \quad z \in \mathbf{D}, \log 1 := 0;$$

when $\beta := 1$, $\alpha \in [0, 1)$, by the function

$$(2.8) \quad f_{\alpha,1,1}(z) := \frac{1}{1 - \alpha} \left(\frac{1 + \alpha}{1 - \alpha} \log \frac{1 - z}{1 - \alpha z} + \frac{2z}{1 - z} \right), \quad z \in \mathbf{D}, \log 1 := 0;$$

when $\alpha := 1$, $\beta \in [0, 1)$, by the function $f_{1,\beta,1} := f_{\beta,1,1}$; when $\beta := 0$ and $\alpha \in (0, 1)$ by the function

$$(2.9) \quad f_{\alpha,0,1}(z) := \frac{1}{1 - \alpha} \left(\frac{1 + \alpha}{\alpha} \log(1 - \alpha z) - 2 \log(1 - z) \right), \quad z \in \mathbf{D}, \log 1 := 0;$$

when $\alpha := 0$ and $\beta \in (0, 1)$ by the function $f_{0,\beta,1} := f_{\beta,0,1}$; when $\beta := \alpha \in (0, 1)$ by the function

$$(2.10) \quad f_{\alpha,\alpha,1}(z) := \frac{1}{(1 - \alpha)^2} \log \frac{1 - \alpha z}{1 - z} - \frac{1 + \alpha}{1 - \alpha} \cdot \frac{z}{1 - \alpha z}, \quad z \in \mathbf{D}, \log 1 := 0;$$

when $\beta = \alpha := 1$ by the Koebe function $f_{1,1,1} := k$.

(iii) when $\beta = \alpha := 0$, then for each $\lambda \in [0, 4/3]$ the second equality in (2.5) is attained by the function

$$(2.11) \quad f_{0,0,0}(z) := -z + \log \frac{1 + z}{1 - z}, \quad z \in \mathbf{D}, \log 1 := 0;$$

for each $\lambda \in \mathbf{R} \setminus (0, 4/3)$ the first equality in (2.5) is attained by the function

$$(2.12) \quad f_{0,0,1}(z) := -z - 2 \log(1 - z), \quad z \in \mathbf{D}, \log 1 := 0.$$

Proof. Fix $\alpha, \beta \in [0, 1]$. Observe that $f \in \mathcal{C}(g_{\alpha,\beta})$ if and only if for $z \in \mathbf{D}$,

$$(2.13) \quad z f'(z) = e^{-i\delta} g_{\alpha,\beta}(z) (p(z) \cos \delta + i \sin \delta)$$

for $\delta \in (-\pi/2, \pi/2)$ and $p \in \mathcal{P}$. For $z \in \mathbf{D}$ we have

$$g_{\alpha, \beta}(z) = z + (\alpha + \beta)z^2 + (\alpha^2 + \alpha\beta + \beta^2)z^3 + \dots$$

Setting the above series with the series (1.1) and (2.1) into (2.13) by comparing coefficients we get

$$(2.14) \quad \begin{aligned} a_2 &= \frac{1}{2} (c_1 e^{-i\delta} \cos \delta + \alpha + \beta), \\ a_3 &= \frac{1}{3} (c_2 e^{-i\delta} \cos \delta + (\alpha + \beta) c_1 e^{-i\delta} \cos \delta + \alpha^2 + \alpha\beta + \beta^2). \end{aligned}$$

Let $\lambda \in \mathbf{R}$. By (2.14) and (2.3) we have

$$(2.15) \quad \begin{aligned} \Phi_\lambda(f) &= \left| \frac{1}{3}(\alpha^2 + \alpha\beta + \beta^2) - \frac{1}{4}(\alpha + \beta)^2 \lambda + \frac{1}{3} \left(c_2 - \frac{c_1^2}{2} \right) e^{-i\delta} \cos \delta \right. \\ &\quad \left. + \frac{c_1^2}{6} \left(1 - \frac{3}{2} \lambda e^{-i\delta} \cos \delta \right) e^{-i\delta} \cos \delta + \frac{1}{2}(\alpha + \beta) \left(\frac{2}{3} - \lambda \right) c_1 e^{-i\delta} \cos \delta \right| \\ &\leq \left| \frac{1}{3}(\alpha^2 + \alpha\beta + \beta^2) - \frac{1}{4}(\alpha + \beta)^2 \lambda \right| + \frac{1}{3} \left(2 - \frac{|c_1|^2}{2} \right) \cos \delta \\ &\quad + \frac{|c_1|^2}{6} \left| 1 - \frac{3}{2} \lambda e^{-i\delta} \cos \delta \right| \cos \delta + \frac{1}{2}(\alpha + \beta) \left| \frac{2}{3} - \lambda \right| |c_1| \cos \delta \\ &= \frac{1}{6} \left| \alpha^2 + \beta^2 + \frac{1}{2}(\alpha + \beta)^2 \gamma \right| + \frac{1}{6} (4 + x^2 (s_\gamma(y) - 1) + (\alpha + \beta) |\gamma| x) y, \end{aligned}$$

where $x := |c_1|$, $y := \cos \delta$, $\gamma := 2 - 3\lambda$ and

$$s_\gamma(y) := \sqrt{1 - (1 - \gamma^2/4) y^2}.$$

In view of (2.2), $x \in [0, 2]$ and clearly, $y \in (0, 1]$. Set $\mu := (\alpha + \beta)/2$ and $R := [0, 2] \times [0, 1]$. Thus $\mu \in [0, 1]$ and with $\gamma \in \mathbf{R}$ define

$$F_{\mu, \gamma}(x, y) := (4 + x^2 (s_\gamma(y) - 1) + 2\mu |\gamma| x) y, \quad (x, y) \in R.$$

Hence and by (2.15) we have

$$(2.16) \quad \max_{f \in \mathcal{C}(g_{\alpha, \beta})} \Phi_\lambda(f) \leq \frac{1}{6} \left| \alpha^2 + \beta^2 + \frac{1}{2}(\alpha + \beta)^2 \gamma \right| + \frac{1}{6} \max_{(x, y) \in R} F_{\mu, \gamma}(x, y).$$

Now for each $\mu \in [0, 1]$ and $\gamma \in \mathbf{R}$ we find the maximum value of $F_{\mu, \gamma}$ on the rectangle R . Since from now the computing is exactly identical as in [21] we demonstrate the short sketch of the proof only.

In the corners of R we have

$$(2.17) \quad \begin{aligned} F_{\mu, \gamma}(0, 0) &= F_{\mu, \gamma}(2, 0) = 0, \\ F_{\mu, \gamma}(0, 1) &= 4, \quad F_{\mu, \gamma}(2, 1) = 2(1 + 2\mu) |\gamma|. \end{aligned}$$

For $x := 0$ and $y \in (0, 1)$ we have a linear function and for $x \in (0, 2)$ and $y := 0$ we have a constant function.

For $x \in (0, 2)$ and $y := 1$ we have a function

$$G_{\mu,\gamma}(x) := F_{\mu,\gamma}(x, 1) = \left(\frac{|\gamma|}{2} - 1 \right) x^2 + 2\mu|\gamma|x + 4$$

which for $|\gamma| = 2$ reduces to the linear function and for $|\gamma| \neq 2$ has the unique critical point at

$$x = \frac{2\mu|\gamma|}{2 - |\gamma|} =: x_{\mu,\gamma} \in (0, 2)$$

if and only if

$$(2.18) \quad \mu \neq 0 \wedge 0 < |\gamma| < \frac{2}{1 + \mu}.$$

Moreover

$$(2.19) \quad F_{\mu,\gamma}(x_{\mu,\gamma}, 1) = G_{\mu,\gamma}(x_{\mu,\gamma}) = \frac{2\mu^2\gamma^2}{2 - |\gamma|} + 4.$$

For $x := 2$ and $y \in (0, 1)$ we have a function

$$H_{\mu,\gamma}(y) := F_{\mu,\gamma}(2, y) = 4ys_{\gamma}(y) + 4\mu|\gamma|y$$

which for $|\gamma| = 2$ reduces to the linear function and for $|\gamma| \neq 2$ has the unique critical point at

$$y = \sqrt{\frac{4 - \mu^2\gamma^2 + \mu|\gamma|\sqrt{\mu^2\gamma^2 + 8}}{2(4 - \gamma^2)}} =: y_{\mu,\gamma} \in (0, 1)$$

if and only if

$$(2.20) \quad |\gamma| < \sqrt{\frac{2}{1 + \mu}}.$$

Moreover

$$(2.21) \quad \begin{aligned} F_{\mu,\gamma}(2, y_{\mu,\gamma}) &= H_{\mu,\gamma}(y_{\mu,\gamma}) \\ &= \sqrt{\frac{4 - \mu^2\gamma^2 + \mu|\gamma|\sqrt{\mu^2\gamma^2 + 8}}{2(4 - \gamma^2)}} \left(\sqrt{\mu^2\gamma^2 + 8} + 3\mu|\gamma| \right). \end{aligned}$$

Repeating exactly argumentation of [18, pp. 8-10] we show that for each $\mu \in [0, 1]$ and each $\gamma \in \mathbf{R}$ the function $F_{\mu,\gamma}$ has no critical point in $(0, 2) \times (0, 1)$.

Summarizing, we conclude that the maximum value of $F_{\mu,\gamma}$ is attained on the boundary of R . Taking into account (2.18) and (2.20), as in [18, p. 10] the following cases hold. For $|\gamma| \geq 2/(1 + \mu)$ the maximum value of $F_{\mu,\gamma}$ is attained in a corner of R , namely,

$$\max_{(x,y) \in R} F_{\mu,\gamma}(x, y) = F_{\mu,\gamma}(2, 1) = 2(1 + 2\mu)|\gamma|.$$

For $\sqrt{2/(1 + \mu)} \leq |\gamma| < 2/(1 + \mu)$ the maximum value of $F_{\mu,\gamma}$ is attained in $(x_{\mu,\gamma}, 1)$, i.e.,

$$\max_{(x,y) \in R} F_{\mu,\gamma}(x, y) = F_{\mu,\gamma}(x_{\mu,\gamma}, 1).$$

For $0 < |\gamma| < \sqrt{2/(1+\mu)}$ we compare all values (2.17), and by (2.19) and (2.21), the values $F_{\mu,\gamma}(x_{\mu,\gamma}, 1)$ and $F_{\mu,\gamma}(2, y_{\mu,\gamma})$ and we show that the maximum value of $F_{\mu,\gamma}$ is attained in $(x_{\mu,\gamma}, 1)$. The key of the computation is to show that

$$F_{\mu,\gamma}(x_{\mu,\gamma}, 1) \geq F_{\mu,\gamma}(2, y_{\mu,\gamma}).$$

Putting (2.19) and (2.21) into above, we get the inequality which is identical as the inequality (2.42) of [18] and further the proof follows exactly as in [18, pp. 10–13] (similar method of proof with all details can be found in [21]). Going back to (2.16) with $\mu = (\alpha + \beta)/2$ we conclude that the following inequality holds:

$$\begin{aligned} & \max_{f \in \mathcal{C}(g_{\alpha,\beta})} \Phi_\lambda(f) \\ & \leq \begin{cases} \frac{1}{6} \left| \alpha^2 + \beta^2 + \frac{1}{2}(\alpha + \beta)^2 \gamma \right| + \frac{1}{3}(1 + \alpha + \beta)|\gamma|, & |\gamma| \geq \frac{4}{2 + \alpha + \beta}, \\ \frac{1}{6} \left| \alpha^2 + \beta^2 + (\alpha + \beta)^2 \gamma \right| + \frac{(\alpha + \beta)^2 \gamma^2}{12(2 - |\gamma|)} + \frac{2}{3}, & |\gamma| \leq \frac{4}{2 + \alpha + \beta}. \end{cases} \end{aligned}$$

Setting $\gamma = 2 - 3\lambda$ the above result yields the inequality (2.5).

Now we discuss the sharpness of the result. Let $\alpha, \beta \in [0, 1]$, $(\alpha, \beta) \neq (0, 0)$. Let $\lambda \in [\tau_1(\alpha, \beta), 2/3]$. Then we consider the second inequality in (2.5) which after simple computing is

$$(2.22) \quad \max_{f \in \mathcal{C}(g_{\alpha,\beta})} \Phi_\lambda(f) \leq \frac{(\alpha + \beta)^2}{9\lambda} + \frac{2 - \alpha\beta}{3}.$$

Let $t_{\alpha,\beta,\lambda} := (\alpha + \beta)(1/(3\lambda) - 1/2)$. Since $\tau_1(\alpha, \beta) \leq \lambda \leq 2/3$, so $0 \leq t_{\alpha,\beta,\lambda} \leq 1$. Thus in view of (2.4), $p_{t_{\alpha,\beta,\lambda},0} \in \mathcal{P}$ with $c_1 = 2t_{\alpha,\beta,\lambda}$ and $c_2 = 2$. Setting $\delta := 0$ and $p := p_{t_{\alpha,\beta,\lambda},0}$ into (2.13) we get the function $f_{\alpha,\beta,t_{\alpha,\beta,\lambda}}$ given by (2.6) for which, by (2.14),

$$\begin{aligned} (2.23) \quad & a_2 = t_{\alpha,\beta,\lambda} + (\alpha + \beta)/2 = (\alpha + \beta)/(3\lambda), \\ & a_3 = (2 + 2(\alpha + \beta)t_{\alpha,\beta,\lambda} + \alpha^2 + \alpha\beta + \beta^2)/3 \\ & = 2(\alpha + \beta)^2/(9\lambda) + (2 - \alpha\beta)/3, \end{aligned}$$

and which makes the equality in (2.22).

Let now $\lambda \in \mathbf{R} \setminus (\tau_1(\alpha, \beta), \lambda(\alpha, \beta))$. Since $\lambda(\alpha, \beta) \geq \tau_2(\alpha, \beta)$, we consider the first inequality in (2.5) which, taking also into account that $\tau_1(\alpha, \beta) \leq 2/3$, after computing, is

$$(2.24) \quad \max_{f \in \mathcal{C}(g_{\alpha,\beta})} \Phi_\lambda(f) \leq \left| \frac{2}{3} + \frac{2}{3}(\alpha + \beta) + \frac{1}{3}(\alpha^2 + \alpha\beta + \beta^2) - \frac{1}{4}(2 + \alpha + \beta)^2 \lambda \right|.$$

Setting $\delta := 0$ and $p := L$ into (2.13) we get the function $f_{\alpha,\beta,1}$ given by (2.6) with $t_{\alpha,\beta,\lambda} := 1$ and with the coefficients a_2 and a_3 given by (2.23), which makes the equality in (2.24). In particular, the function $f_{\alpha,\beta,1}$ is one of the form (2.7)–(2.10).

Let $\alpha = \beta := 0$. For $\lambda \in [\tau_1(0, 0), \tau_2(0, 0)] = [0, 4/3]$ the inequality (2.5) reduces to

$$\max_{f \in \mathcal{C}(g_{0,0})} \Phi_\lambda(f) = \max_{f \in \mathcal{P}'} \Phi_\lambda(f) \leq \frac{2}{3}.$$

Setting $\delta := 0$ and by (2.4), $p := p_{0,0}$ into (2.13) we get the function (2.11) with $a_2 = 0$ and $a_3 = 2/3$, which makes the equality above. For $\lambda \in \mathbf{R} \setminus (0, 4/3)$ the inequality (2.5) reduces to

$$\max_{f \in \mathcal{C}(g_{0,0})} \Phi_\lambda(f) = \max_{f \in \mathcal{P}'} \Phi_\lambda(f) \leq \left| \frac{2}{3} - \lambda \right|.$$

Setting $\delta := 0$ and by (2.4), $p := L$ into (2.13) we get the function (2.12) with $a_2 = 1$ and $a_3 = 2/3$, which makes the equality above. $\square$

Remark 2.3. Let $\beta := \alpha \in (0, 1]$. Then by (1.5) we have

$$\lambda(\alpha, \alpha) = \frac{2}{3} + \frac{2}{3} \max \left\{ \frac{1}{2}, \frac{1}{1+\alpha} \right\} = \tau_2(\alpha),$$

so (2.5) with sharpness reduces to (1.4) (Theorem 2.4 of [18]). Let $\beta := 0$ and $\alpha \in (0, 1]$. Then

$$\lambda(\alpha, 0) = \frac{2}{3} + \frac{2}{3} \max \left\{ 1, \frac{2}{2+\alpha} \right\} = \frac{4}{3},$$

so (2.5) with sharpness reduces to (1.6) (Theorem 2.4 of [21]).

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Presented by Zbigniew Jakubowski at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on May 23, 2016.

O PEWNEJ NIERÓWNOŚCI DLA WSPÓŁCZYNNIKÓW W PODKLASIE FUNKCJI PRAWIE WYPUKŁYCH

S t r e s z c z e n i e

Dla $\alpha, \beta \in [0, 1]$ niech $g_{\alpha, \beta}(z) := z / ((1 - \alpha z)(1 - \beta z))$, $z \in \mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$. Funkcja analityczna unormowana $f : \mathbf{D} \rightarrow \mathbf{C}$ nazywana jest *prawie wypukłą względem funkcji* $g_{\alpha, \beta}$, jeśli dla pewnego $\delta \in (-\pi/2, \pi/2)$ zachodzi nierówność

$$\operatorname{Re} \left\{ e^{i\delta} \frac{z f'(z)}{g_{\alpha, \beta}(z)} \right\} > 0, \quad z \in \mathbf{D}.$$

Dla klasy $\mathcal{C}(g_{\alpha, \beta})$ funkcji prawie wypukłych względem funkcji $g_{\alpha, \beta}$ badany jest problem Fekete-Szegö-Goluzina.

Słowa kluczowe: nierówności współczynnikowe, funkcje prawie wypukłe, problem Fekete-Szegö-Goluzina

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2017

Vol. LXVII

Recherches sur les déformations

no. 1

pp. 91–106

*Stanisław Bednarek and Julian Płoszajski***THE MAGNETIC AND ELECTRIC FIELD PRODUCED
BY A PROTON BUNCH FROM THE LHC AND PROJECT
OF THE CHAMBER FOR INVESTIGATIONS IN THESE FIELDS****Summary**

This paper describes the estimation method of the spatial distribution of magnetic field induction and electric field intensity produced by a relativistic beam of charged particles. The described method is applied to estimation of this field distribution in surroundings of proton bunches which are accelerated inside the Large Hadron Collider (LHC). It is demonstrated that such bunches generate pulsed high magnetic fields with induction from a few dozen to a few hundred T within the space of a few cm. The fields are relevant to the successful application of research of materials properties in time intervals in the order of fs, which have not been available until now. We additionally present a design for a special chamber allowing easy introduction of samples into the accelerator, remote measurement and data acquisition.

Keywords and phrases: magnetic field, high pulse, accelerator, particle, beam, application

1. Introduction

Highly pulsed magnetic fields have many important applications in scientific research and modern technology. Generation of these fields is most often achieved by use of coils supplied with high intensity electric current [1–3]. Unfortunately, this method has several deficiencies which limit the upper values of magnetic induction and time duration of these pulses. The most important limitation is the result of the relatively high electric resistivity and limited mechanical strength of coil material as well as its temperature increases. It is for these reasons that during the generation of magnetic fields exceeding magnitudes of 200–300 T, the coil is effectively destroyed. Problems arise additionally from the inductivity of the coil, its connecting conductors and

capacitor banks, all of which limit the pulse rise time. The strongest pulsed magnetic fields with magnitude of 800-1000 T induction are generated using the magnetic flux compression method. Unfortunately, this method does require the application of large quantities of high-explosive materials, which leads to the inevitable destruction of a large part of the experimental device [4, 5].

An original and promising approach to generate high pulsed magnetic fields is the application of particle beams from an accelerator. The first experiments of such type were described by H. C. Siegman and co-researchers [6-8]. The authors of the cited articles focused on revealing the experimental interaction between an electron beam and a magnetized sample. Relativistic electron beams generated by the 50 GeV energy Linear Stanford Accelerator were used in these experiments. The accelerator beam was passed through the sample causing its damage. Therefore, the research method used was a destructive one. Moreover, the cited works provided only an estimated value of the magnetic field induction to which the sample was subjected while the electric field was not taken into account in spite of its apparent presence. Ampere's law, which may be applied to non-relativistic particles only, was used for the purposes of magnetic induction calculations.

In this paper we show a proprietary method of calculating the spatial distribution of the magnetic field induction, which is generated by a relativistic particle beam coming from an accelerator. The method is universal because it uses the Lorentz transformation. Moreover, we additionally calculate the spatial distribution of the electric field intensity. We analyse both fields outside the particle beam, and therefore it is possible to conduct non-destructive research. The developed method has been applied to calculations of fields, which are generated by the proton beam accelerated in the LHC. In the final part of the article we have described the design for a special chamber which may be installed next to the accelerator beam tube. The chamber allows for the remote controlled exchange of the examined samples as well as the transfer of the measurement results. Construction of such a chamber would facilitate wide applications of accelerator particle beams in materials research. The proposed method would permit the use of high and ultra-short pulsed magnetic fields, in up to now unprecedented time intervals.

2. Calculation method

At the beginning we estimated the spatial field distribution within the coordinate system connected with a charged particle bunch in motion. Only an electric field exists within this coordinate system because the particles are not moving with regard to the system. The particle beam consists of separated bunches which have an axial symmetry. For purposes of calculations we assumed a rectangular Oxy coordinate system moving together with the bunch. Because of the bunch's axial symmetry, we may initially analyse the distribution in two dimensions. We assign the Ox axis of the coordinate system to be parallel to the bunch axis and direction of movement, Fig. 1.

Fig. 1: Frame of reference connected with the particle bunch accepted for calculations; O_1 – the centre of the bunch, O – the point chosen for calculations of the field intensity, $n(x)$ – linear particles density, $l/2$ – half length of the bunch, dq – the charge element, dE_x , dE_y , dE – elements of electric field intensity respectively: longitudinal, transversal and the resultant, r_0 – the distance of the point of calculations from the beam (also bunch) axis, x_0 – translation of the calculation point due to the centre of the bunch calculated in the bunch frame of reference.

Axis Oy will then have a radial direction and one which is perpendicular to the direction of the movement. Each bunch has a length of l while the charge density in the bunch is described by the Gauss distribution as having a maximum in the centre [9–11]. The transversal dimensions of the bunch are measured in tens of μm . The examined sample will be placed at a distance of r_0 from the bunch axis amounting to several mm. It means that transversal dimensions of the bunch are considerably less than r_0 . We may in this situation omit the change of charge density in the radial direction and assume for the calculations a one-dimensional normal distribution. The linear charge density $\rho(x_1)$ is for this distribution a function of the distance $x_1 =$

$x + x_0$ from the centre of the bunch 0_1 and is expressed by the formula:

$$(1) \quad \rho(x_1) = \frac{n_1 e}{\sigma_x \sqrt{2\pi}} \exp\left[-\frac{(x + x_0)^2}{2\sigma_x^2}\right],$$

where: n_1 represents the number of all charges within the bunch, σ_x is the standard deviation for a normal distribution while the symbol e represents an elementary charge.

As mentioned earlier, the coordinate system connected with the bunch includes only an electrical field. The element of the electric field vector dE , generated by the charge element of the bunch $dq = \rho(x_1) \cdot dx$ at the point 0, which is at the distance r from the centre of the element, is expressed by the well-known in electrostatics formula [12].

$$(2) \quad dE = \frac{1}{4\pi\epsilon_0} \frac{dq}{r_0^2 + x^2}.$$

Field components dE_x , dE_y , are respectively directed along the $0x$ and $0y$ axes and are expressed by the formulas:

$$(3) \quad dE_x = dE \cos \alpha = dE \frac{r_0}{\sqrt{r_0^2 + x^2}},$$

$$(4) \quad dE_y = dE \sin \alpha = dE \frac{x}{\sqrt{r_0^2 + x^2}}.$$

After substitution of both formulas (1) and (2) respectively into formulas (3) and (4), the following formulas for calculated components dE_x , dE_y at point 0 are obtained:

$$(5) \quad dE_x = \frac{n_1 e}{4\pi\sqrt{2\pi}\epsilon_0\sigma_x} \frac{x \exp\left[-\frac{(x+x_0)^2}{2\sigma_x^2}\right]}{(r_0^2 + x^2)^{3/2}} dx,$$

$$(6) \quad dE_y = \frac{n_1 r_0 e}{4\pi\sqrt{2\pi}\epsilon_0\sigma_x} \frac{\exp\left[-\frac{(x+x_0)^2}{2\sigma_x^2}\right]}{(r_0^2 + x^2)^{3/2}} dx.$$

The complete components of the electrical field intensity E_x , E_y at point 0 are calculated by integrating expressions (5) and (6) after the entire length l of the given bunch. These integrals are expressed by means of the following formulas:

$$(7) \quad E_x = \frac{n_1 e}{4\pi\sqrt{2\pi}\epsilon_0\sigma_x} \int_{-(\frac{l}{2}+x_0)}^{(\frac{l}{2}-x_0)} \frac{x \exp\left[-\frac{(x+x_0)^2}{2\sigma_x^2}\right]}{(r_0^2 + x^2)^{3/2}} dx,$$

$$(8) \quad E_y = \frac{n_1 e r_0}{4\pi\sqrt{2\pi}\epsilon_0\sigma_x} \int_{-(\frac{l}{2}+x_0)}^{(\frac{l}{2}-x_0)} \frac{\exp\left[-\frac{(x+x_0)^2}{2\sigma_x^2}\right]}{(r_0^2 + x^2)^{3/2}} dx.$$

Further we designate the spatial distribution of the magnetic and electric fields within the coordinate system $0x'y'z'$ with the analysed sample at rest. The axes of this system are directed in parallel to the appropriate axes of the previously

accepted Oxy system, which is associated with the bunch. It is for this purpose that the Lorentz transformation should be applied to the calculated E_x , E_y components. This transformation is expressed by means of the following general formulas [13]:

$$(9) \quad E'_x = E_x \quad E'_y = \gamma(E_y - vB_z) \quad E'_z = \gamma(E_z + vB_y),$$

$$(10) \quad B'_x = B_x \quad B'_y = \gamma(B_y + \frac{\beta}{c}E_z) \quad B'_z = \gamma(B_z - \frac{\beta}{c}E_y).$$

All quantities as part of the formulas bearing the „prim” designation refer to the system associated with the sample. In accordance with this B'_x , B'_y , B'_z indicate the components of the magnetic field induction vector within the sample's system while B_x , B_y , B_z represent the magnetic field induction vector components within the reference system of the bunch. Symbols c , β and γ represent respectively: speed of light in a vacuum, ratio of the particle velocity v to the speed of light in a vacuum ($\beta = v/c$) as well as so-called Lorentz factor as expressed by the formula

$$(11) \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}.$$

In accordance with earlier obtained results, the reference system of the bunch includes only two electric field components $E_x, E_y \neq 0$. Therefore, after the application of the Lorentz transformation we obtain as a result of the formulas (9), (10) the following:

$$(12) \quad E'_x = E_x \quad E'_y = \gamma E_y \quad E'_z = 0,$$

$$(13) \quad B'_x = 0 \quad B'_y = 0 \quad B'_z = -\gamma \frac{\beta}{c} E_y.$$

Formulas (7), (8) as well as (12) and (13) have been used for calculating the electric field intensity components as well as the magnetic induction within the sample reference system. For simplification purposes, the B'_z magnetic field induction component will be referred to henceforth in this article as B .

3. Calculations results

For calculations were used data which are collected in Table 1 that was elaborated on the ground of parameters available in literature, describing the proton beam in the LHC [14]. Moreover, the standard deviation of the Gauss distribution $\sigma_x = 0.2690 \cdot (l/2)$ was accepted. It means that outside of an " l " long section the linear charge density is less than 0.1% of the maximum density in the middle of the bunch and it is the value, which we may omit. The spatial distributions of the components of E'_y , E'_x electric field intensity are shown in Figs. 2, 3.

The spatial distribution of B magnetic field induction was shown in Fig. 4.

For better estimation of usefulness of those fields for research, in Figs. 5–7 were given dependencies of components of E'_x , E'_y electric field intensity B and magnetic

Fig. 2: Spatial distribution of transversal component of the E'_y electric field intensity in the sample frame of reference; r_0 – the distance of the point of calculations from the beam axis, x'_0 – translation of the calculation point due to the centre of the bunch calculated in the sample frame of reference ($x'_0 = x_0/\gamma$).

Fig. 3: Spatial distribution of the longitudinal component of the E'_x electric field intensity in the sample frame of reference; coordinates r_0 , x'_0 have the same meaning, like in Fig. 2.

field induction, as a function of the r_0 distance from the bunch axis. The r_0 distance is measured in the transversal plane of the bunch symmetry, perpendicular to its movement direction. In this plane $x_0 = 0$.

At the next Figs. 8–10 dependencies of the components from Figs. 5–7 were shown as functions of x_0 distance from mentioned transversal plane of symmetry for fixed r_0 distance = 0.005 m (5 mm) from the bunch axis. The charts in Figs. 5–10 are cross-sections received from Figs. 2–4. The choice of the distance $r_0 = 0,5$ mm from

Fig. 4: Spatial distribution of B magnetic field induction in the sample frame of reference; coordinates r_0 , x'_0 have the same meaning, like in Fig. 2.

Fig. 5: Dependence of the B magnetic field induction, as a function of the distance from the bunch axis r_0 in the transversal plane of the bunch symmetry ($x'_0 = 0$) in the sample frame of reference.

the bunch axis is motivated by the practical attitude because it is possible to place an examined sample, approximately 1 mm in size, at this particular distance.

4. Discussion of results

The obtained results show that the maximal values of the B magnetic field induction are at the transversal plane of symmetry of the bunch for which $x_0 = 0$ (Figs. 4,

Fig. 6: Dependence of the E'_y component of electric field intensity, as a function of distance from the bunch axis r_0 in the transversal plane symmetry ($x'_0 = 0$) in the sample frame of reference.

5). The maximal values appear at the moment when the centre of the bunch passes in minimal distance from the sample. The proton bunch diameter is approximately $32 \mu\text{m}$ (Tab. 1).

Tab. 1: Parameters of the proton beam in the LHC

No.	Name of parameter	Symbol	Value
1.	Number of particles (protons) in the bunch	n_1	$1,15 \cdot 10^{11}$
2.	Bunch length	l	7,55 cm
3.	Bunch radius	r_p	16 μm
4.	Particles' energy	E	7 TeV
5.	Particle speed	v	$0,999999991c$
6.	Energy increase (Lorentz factor)	γ	$7,46 \cdot 10^3$
7.	Number of bunches in the pipe	N	2808
8.	Length of tube perimeter	L	25,55 km

We assume that dimensions of examined samples will be approximately 1 mm and such samples are going to be placed at the distance of $r_0 = 1 \text{ mm}$ or larger from the bunch axis. For accepted assumption we see in Fig. 7 that the upper limit of the magnetic field induction helpful in research of the samples at millimetre scale is about 160 T. This value decreases with the distance from the bunch axis. As shown in Fig. 7, at the distance $r_0 = 5 \text{ mm}$ from the bunch axis the value is equal to 60

Fig. 7: Dependence of E'_x electric field component intensity, as a function of distance from the bunch axis r_0 in the transversal plane of the bunch symmetry ($x'_0 = 0$) in the sample frame of reference. B magnetic field induction, as a function of the distance x'_0 from the transversal plane of the bunch symmetry for a fixed distance from the bunch axis $r_0 = 0.005$ m in the sample frame of reference.

T. Those magnetic induction values are comparable to field inductions generated by the coils [1, 2]. For comparison, we inform that at the distance $r_0 = 50$ mm from the bunch axis (the distance equal to the radius of the accelerator tube) magnetic field induction decreases to 6 T.

If the sample is placed further from the bunch axis then the maximal field induction value will be less but more homogenous. For the purpose of better quantitative estimation of magnetic field inhomogeneity we consider a cubic sample with a side length of 1 mm. Let the centre of the sample be r_0 at the distance from the bunch axis and its side be parallel to r_0 direction. Magnetic field inhomogeneity at the sample area will be defined by the formula

$$(14) \quad k = \frac{|B_k - B_0|}{B_0} 100\%,$$

where B_k , B_0 mean the magnetic field induction respectively: at the centre and at the edge of the sample. For the mentioned value of $r_0 = 5$ mm, the magnetic field induction at the sample centre is $B_0 = 60$ T but for a distance of $r_0 = 10$ mm the induction at the centre of the sample decreases to $B_0 = 35$ T. It is easy to calculate, using formula (14) and Fig. 5, that in the first case $k < 9\%$ but in the second case $k < 7\%$.

Fig. 8: Dependence of the B magnetic field induction, as a function of the distance x'_0 from the transversal plane of the bunch symmetry for a fixed distance from the bunch axis $r_0 = 0.005$ m in the sample frame of reference.

Fig. 9: Dependence of the E'_y component of the electric field intensity, as a function of the distance x'_0 from the transversal plane of the bunch symmetry for fixed distance from the bunch axis $r_0 = 0.005$ m in the sample frame of reference.

Fig. 10: Dependence of the component of the E'_x electric field intensity, as a function of the distance x'_0 from the transversal plane of the bunch symmetry for fixed distance from the bunch axis $r_0 = 0.005$ m in the sample frame of reference.

We encounter a comparable magnetic field inhomogeneity in the sample in the case of magnetic fields generated by coils. Therefore, this inhomogeneity is not a distinctive feature of fields generated by the particle bunch and it is not a fundamental obstacle in conducting such a research. The radius of the tube in which the bunch moves is 5 cm and it is for this a reason that maximal volume of the sample could be no more than a few cm^3 [14]. Because of the magnetic field inhomogeneity in such a large sample, its optimal volume should be limited to tens of mm^3 . It is not a serious limitation too, because we know of cases of research in high magnetic fields samples even of several μm^3 [15]. The research of small-sized samples is consistent with a current need of miniaturization in electronics and nanotechnology.

A sample placed nearly to the bunch will also be subjected to the effect of an electric field pulse with E'_x , E'_y components (Figs. 2, 3, 6, 7 and 9, 10). The E'_x component has a maximum and minimum which are equal in value and have opposite signs (Fig. 3). Moreover, $E'_x = 0$ at the plane of the bunch symmetry (Fig. 6). Therefore, we may say that E'_x is an antisymmetric function of x_0 . The spatial distribution of the E'_y component is similar to the spatial distribution of the B magnetic field induction due to the Lorentz transformation (see formula 13) where B and E'_y values are directly proportional. An action of the electric field pulse on a sample appears also in the case of magnetic field pulses generated in the usual way with coils or magnetic flux compression. We observe the electric field because, in accordance with Faraday's induction law, variable magnetic field generates the variable electric field.

Screens made of conducting materials provides a simple protection from such a field. Such protection of the measuring devices and samples is practiced in the case of pulsed magnetic fields which are generated in conventional ways. The difference is that the conductor with a current (coil) is electrically neutral and does not generate an electric field but a moving particle bunch does generate one. Hence, the electric field does not constitute any obstacle in research.

Particle bunches have a velocity very close to the speed of light. Therefore, we call them ultra-relativistic particles. It is for this reason that we may not use Ampere's law for calculations of the magnetic field induction generated by the particles. This law may be applied to the calculations of the magnetic field which is generated by the particles moving with small velocities in comparison with the speed of light in a vacuum ($v \ll c$). Then, according to the (11) formula the Lorentz factor $\gamma \approx 1$ and the last formula (13) after the conversions gives a "classic" formula which is applied in technical calculations for magnetic field induction around a straight wire with a current. From (12) and (13) formulas it follows that failure in taking into account the factor γ yields lower values for B magnetic field induction and E'_y electric field intensity in the sample frame of reference. Moreover, for relativistic particles a Lorentz contraction occurs for the spatial distribution of electric field, which is observed in the sample frame of reference. The distribution takes shape of disc, which is very flattened in the direction of particle bunch movement. The Lorentz contraction of the bunch makes it possible to generate unusually short magnetic field pulses which last approximately 4 fs.

5. Project of a chamber

Interior of the accelerator tube is a high vacuum environment with very significant volume while its exterior is subject to atmospheric air pressure. Direct introduction of an examined sample into the tube could be very inconvenient due to the loss of a vacuum and subsequent need for the tube to be purged of air. It would be a time-consuming and labour-intensive procedure. Moreover, during operation of the accelerator, synchrotron radiation is generated which in turn causes material activation within direct surroundings of the tube. As the result, secondary radiation is additionally emitted also after the beam is switched off. To decrease the intensity of this radiation to a secure level, it is necessary to wait for an appropriately long time. Entering the accelerator tunnel during this time is either impossible or requires special protection.

In order to eliminate the described complications, a special chamber for the remote control of exchanging of samples has been designed. A diagram of such a chamber is illustrated in Fig. 11.

Fig. 11: Scheme of the chamber for remote control exchange of the samples; 1 – particle beam, 2 – accelerator tube, 3 – sample chamber, 4 – window, 5 – inner cover, 6 – propulsion of the inner cover, 7 – connectors supplying cover propulsion, 8 – examined sample, 9 – the examined sample arm, 10 – rotary head with stepper motor, 11 – wires powering the stepper motor, 12 – sample waiting for an exam, 13 – arm of the waiting sample, 14 – switches for collecting measured signals, 15 – wiring connecting the measured signals, 16 – outer cover, 17 – pipe connecting the chamber with the vacuum pump, 18 – valve.

The chamber (3) is located directly adjacent to the accelerator tube (2). From the side accelerator tube, the chamber has a well-sealed inside cover (5), which is remotely controlled by a motor (6). The cover can open or close the window (4) in the accelerator tube (2). The chamber contains a rotary head (10), driven by a remote controlled stepper motor. The head is equipped with several radially distributed arms (9, 13). Samples (8, 12) are located at the end of each arm. One of the samples (8) is being examined at the moment while the rest of the samples is awaiting testing. The surface of the head includes switches which are contacted by fixed brushes (14). The switches are connected to wires (15), leading to the samples. Conductors and brushes are assigned to collect signals produced during the examination such as the magneto-resistivity. Moreover, the chamber is equipped with a hermetic outer cover (16), located opposite the rotary head, and pipe (17) with the valve (18) which is connected to a vacuum pump.

The principle of the chamber's operation is fairly uncomplicated. The outer cover (16) is open during the accelerator's off-time. It is possible at such time to place samples designated for examination on the arms of the rotary head (10). The inner cover (5) is closed then and the accelerator pipe is again brought into a state of vacuum. When the samples are placed, the outer cover is closed. Next, the chamber is purged of air to establish a vacuum as inside the accelerator tube and valve (18) is closed. After the accelerator begins operating, the inner cover (5) is moved and window (4) is opened. The sample (8) located on the turning head (10) is entered into the accelerator tube (2) through the window (4). Opening the cover and the head rotation are remotely controlled. The length of the arm (9) makes it possible to place the chosen sample (8) at the required distance from the bunch axis. (In a more advanced model of the chamber design, this distance can be varied by remote control during the measurements using, for example, telescopic arms.) After finishing the examination of the chosen sample, the head (10) is turned in order to introduce the next sample into the accelerator tube. After completion of all the sample tests, the inner cover (5) is opened and the air pressure is increased to match the ambient pressure. At this time, the chamber is ready for the opening of the outer cover (16) and exchanging of all samples.

6. Conclusions

An application of the accelerator beam for generation of high, pulsed magnetic fields eliminates many problems relevant to coil warming, tensions and damage. Using the non-destructive method makes it possible to obtain repeatable pulsed magnetic fields. A crucial advantage of using the particle beam is the low cost involved. In the case of research using this method, it is not necessary to bear costs connected with the building of the field source because the accelerator already exists for other purposes. The research into the magnetic field generated by the beam from an accelerator could be a new "added value" for this device. It is not necessary to pay for electrical energy because it is an element of regular accelerator activity operating costs. It is worthwhile to consider the possibility of installing the suggested experimental chamber during the construction of newly-designed accelerators or during modernization or maintenance down-time of existing accelerators. Because magnetic fields generated by particle bunches from accelerators have unique parameters unachievable by other methods, they open up a new field of exploration and we may expect significant discoveries as its direct result. They allow us to research very quick processes of remagnetisation. An application of those processes could allow development of magnetic materials useful in building data storage discs with very rapid recording times as well as very quick access to the stored data.

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Presented by Leszek Wojtczak at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 3, 2016.

POLA MAGNETYCZNE I ELEKTRYCZNE WYTWARZANE PRZEZ PACZKĘ PROTONÓW Z AKCELERATORA LHC I PROJEKT KOMORY DO BADAŃ W TYCH POLACH

S t r e s z c z e n i e

W artykule opisano metodę wyznaczania rozkładu przestrzennego indukcji pola magnetycznego i natężenia pola elektrycznego, które są wytwarzane przez relatywistyczne wiązki cząstek naładowanych. Opisaną metodę zastosowano do wyznaczania rozkładów tych pól w otoczeniu paczki protonów przyspieszanych w LHC. Wykazano, że takie paczki wytwarzają silne impulsowe pola magnetyczne o indukcji od kilkuset do kilkudziesięciu T w obszarze o rozmiarach kilku cm. Pola te mogą być z powodzeniem wykorzystane do badania właściwości materiałów w nowej, dotychczas nie osiągalnej skali przedziałów czasu rzędu fs. Opisane również projekt specjalnej komory, umożliwiającej zdalną wymianę próbek i zbieranie wyników pomiarów prowadzonych w tych polach.

Słowa kluczowe: pole magnetyczne, silny impuls, akcelerator, cząstka, wiązka, zastosowanie

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ
2017 Vol. LXVII

Recherches sur les déformations

no. 1

pp. 107–118

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THE BOUNDS OF SOME DETERMINANTS FOR FUNCTIONS OF BOUNDED TURNING OF ORDER ALPHA

Summary

In the present paper, the estimates of some determinants over the class $\mathcal{R}(\alpha)$, $0 \leq \alpha < 1$, of analytic functions f standardly normalized such that

$$\operatorname{Re} f'(z) > \alpha, \quad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\},$$

are shown.

Keywords and phrases: univalent functions, functions of bounded turning, functions of bounded turning of order α , Hankel determinant

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ be its subclass of f normalized by $f(0) := 0$ and $f'(0) := 1$, so of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$

Given $n, q \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of a function $f \in \mathcal{A}$ of the form (1.1) is defined as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix},$$

where $a_1 := 1$. To find the growth of the Hankel determinant $H_{q,n}(f)$ dependent on q and n for the whole class $\mathcal{S} \subset \mathcal{A}$ of univalent functions as well as for its subclasses

is one of the main problem to study. For the class $\mathcal{S}$ some important result was shown by Pommerenke [16]. For fixed q and n the growth problem is reduced to find the bound of the Hankel determinant over selected compact subclasses of $\mathcal{A}$. Recently many authors examined the Hankel determinant $H_{2,2}(f)$ of order 2 as well as the Hankel determinant $H_{3,1}(f)$ of order 3 (see e.g., [9], [14], [11], [2]). Note that $H_{2,1}(f) = a_3 - a_2^2$. Thus the Hankel determinant $H_{2,1}(f)$ of order 2 reduces to the well known coefficient functional which for $\mathcal{S}$ was estimated in 1916 by Bieberbach (see e.g., [7, Vol. I, p. 35]).

Given $\alpha \in [0, 1)$, by $\mathcal{R}(\alpha)$ we denote a subclass of $\mathcal{A}$ of functions f such that

$$(1.2) \quad \operatorname{Re} f'(z) > \alpha, \quad z \in \mathbb{D}.$$

Functions in $\mathcal{R}(\alpha)$ are called *of bounded turning of order α* and in $\mathcal{R} := \mathcal{R}(0)$ of *bounded turning* (see e.g., [7, Vol. I, p. 101]).

In this paper we found sharp estimates of the Hankel determinants $H_{2,2}(f)$, $H_{2,1}(f)$ and of the determinant

$$(1.3) \quad \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} = a_4 - a_2 a_3$$

over the class $\mathcal{R}(\alpha)$. Having these results, the Hankel determinant $H_{3,1}(f)$ can be estimated also, however far from sharpness. For the class $\mathcal{R}$ it was done in [1] with some correctness in [2]. This result for $\mathcal{R}$ was recently improved in [17].

Let $\mathcal{P}$ be the class of Carathéodory functions $p \in \mathcal{H}$ of the form

$$(1.4) \quad p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D},$$

having a positive real part in $\mathbb{D}$. The results below for the class $\mathcal{P}$ will be used in further considerations.

Lemma 1.1. [6, p. 41] *If $p \in \mathcal{P}$ is of the form (1.4), then*

$$(1.5) \quad |c_n| \leq 2, \quad n \in \mathbb{N}.$$

The inequality is sharp and the equality holds for the function $p := L$, where

$$L(z) := \frac{1+z}{1-z} = 1 + 2 \sum_{n=1}^{\infty} z^n, \quad z \in \mathbb{D}.$$

Let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

Lemma 1.2. ([12], [13]) *If $p \in \mathcal{P}$ is of the form (1.4) with $c_1 > 0$, then*

$$(1.6) \quad 2c_2 = c_1^2 + \zeta(4 - c_1^2)$$

and

$$(1.7) \quad 4c_3 = c_1^3 + 2c_1(4 - c_1^2)\zeta - c_1(4 - c_1^2)\zeta^2 + 2(4 - c_1^2)(1 - |\zeta|^2)\eta$$

for some $\zeta \in \overline{\mathbb{D}}$ and $\eta \in \overline{\mathbb{D}}$.

The next lemma is a special case of more general results due to Choi, Kim and Sugawa [5] (see also [15]). Define

$$Y(a, b, c) := \max_{z \in \mathbb{D}} (|a + bz + cz^2| + 1 - |z|^2), \quad a, b, c \in \mathbb{R}.$$

Lemma 1.3. ([5]) *If $ac \geq 0$, then*

$$Y(a, b, c) = \begin{cases} |a| + |b| + |c|, & |b| \geq 2(1 - |c|), \\ 1 + |a| + \frac{b^2}{4(1 - |c|)}, & |b| < 2(1 - |c|). \end{cases}$$

If $ac < 0$, then

$$Y(a, b, c) = \begin{cases} 1 - |a| + \frac{b^2}{4(1 - |c|)}, & -4ac(c^{-2} - 1) \leq b^2 \wedge |b| < 2(1 - |c|), \\ 1 + |a| + \frac{b^2}{4(1 + |c|)}, & b^2 < \min \{4(1 + |c|)^2, -4ac(c^{-2} - 1)\}, \\ R(a, b, c), & \text{otherwise,} \end{cases}$$

where

$$R(a, b, c) = \begin{cases} |a| + |b| - |c|, & |c|(|b| + 4|a|) \leq |ab|, \\ -|a| + |b| + |c|, & |ab| \leq |c|(|b| - 4|a|), \\ (|c| + |a|)\sqrt{1 - \frac{b^2}{4ac}}, & \text{otherwise.} \end{cases}$$

2. Main results

We will start with the determinant (1.3).

Theorem 2.1. *Let $\alpha \in [0, 1)$. If $f \in \mathcal{R}(\alpha)$ is the form (1.1), then*

$$(2.1) \quad |a_2 a_3 - a_4| \leq \frac{1}{2}(1 - \alpha).$$

The inequality is sharp and the equality holds for the function

$$(2.2) \quad f(z) := \int_0^z \frac{1 + (1 - 2\alpha)u^3}{1 - u^3} du, \quad z \in \mathbb{D}.$$

Proof. Fix $\alpha \in [0, 1)$ and let $f \in \mathcal{R}(\alpha)$ be of the form (1.1). Then by (1.2) the function

$$(2.3) \quad p(z) := \frac{1}{1 - \alpha}(f'(z) - \alpha), \quad z \in \mathbb{D},$$

belongs to $\mathcal{P}$. Putting the series (1.1) and (1.4) into (2.3) by equating the coefficients we get

$$(2.4) \quad a_{n+1} = \frac{(1 - \alpha)c_n}{n + 1}, \quad n \in \mathbb{N}.$$

Hence

$$(2.5) \quad |a_2a_3 - a_4| = \frac{1}{12}(1 - \alpha)|2(1 - \alpha)c_1c_2 - 3c_3|.$$

Now by using (1.6) and (1.7) we have

$$(2.6) \quad |a_2a_3 - a_4| = \frac{1}{48}(1 - \alpha)|(1 - 4\alpha)c_1^3 + (4 - c_1^2)[(-4\alpha - 2)c_1\zeta + 3c_1\zeta^2 - 6(1 - |\zeta|^2)\eta]|,$$

where $(\zeta, \eta) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. Since the class $\mathcal{R}(\alpha)$ is invariant under the rotations, by (1.5) we may assume that $c_1 =: t \in [0, 2]$.

Assume first that $t := 2$. Then $p = L$ in (2.3) (see e.g., [7, Vol. I, p. 100]), so

$$(2.7) \quad f(z) = \int_0^z \frac{1 + (1 - 2\alpha)u}{1 - u} du, \quad z \in \mathbb{D},$$

with $a_2 = 1 - \alpha$, $a_3 = 2(1 - \alpha)/3$ and $a_4 = (1 - \alpha)/2$. Hence we see that for $\alpha \in [0, 1)$ the inequality

$$|a_2a_3 - a_4| = \frac{1}{6}(1 - \alpha)|1 - 4\alpha| \leq \frac{1}{2}(1 - \alpha),$$

is true, so the inequality (2.1) holds for the function (2.7).

Let now $t \in [0, 2)$. From (2.6) we have

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{1}{48}(1 - \alpha) \\ &\times [|(1 - 4\alpha)t^3 + (4 - t^2)[(-4\alpha - 2)t\zeta + 3t\zeta^2| + 6(4 - t^2)(1 - |\zeta|^2)|\eta|] \\ &\leq \frac{1}{8}(1 - \alpha)(4 - t^2) \left[\left| \frac{(1 - 4\alpha)t^3}{6(4 - t^2)} - \frac{1}{3}(1 + 2\alpha)t\zeta + \frac{1}{2}t\zeta^2 \right| + 1 - |\zeta|^2 \right] \\ &= \frac{1}{8}(1 - \alpha)(4 - t^2) [|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2], \end{aligned}$$

where

$$A := \frac{(1 - 4\alpha)t^3}{6(4 - t^2)}, \quad B := -\frac{1}{3}(1 + 2\alpha)t, \quad C := \frac{1}{2}t.$$

Thus to prove the theorem we will show that

$$(2.8) \quad (4 - t^2) [|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2] \leq 4.$$

I. Consider first the case $AC \geq 0$ which holds when $\alpha \in [0, 1/4]$.

1. Since $|B| \geq 2(1 - |C|)$ holds when $t \in [3/(2 + \alpha), 2)$, by Lemma 1.3 we have

$$(2.9) \quad \begin{aligned} &(4 - t^2) [|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2] \\ &\leq (4 - t^2)(|A| + |B| + |C|) = (4 - t^2) \cdot \frac{(1 - 4\alpha)t^3 + (4\alpha + 5)(4 - t^2)t}{6(4 - t^2)} \\ &= -\frac{2}{3}(1 + 2\alpha)t^3 + \frac{2}{3}(4\alpha + 5)t =: \gamma(t), \quad t \in [3/(2 + \alpha), 2). \end{aligned}$$

Note that

$$\begin{aligned} \gamma'(t) &= -2(1 + 2\alpha)t^2 + \frac{2}{3}(4\alpha + 5) \\ &= -2(1 + 2\alpha)(t + t_0)(t - t_0), \quad t \in [3/(2 + \alpha), 2), \end{aligned}$$

where

$$t_0 := \sqrt{\frac{4\alpha + 5}{3(1 + 2\alpha)}}.$$

But

$$t_0 < \frac{3}{2 + \alpha}, \quad \alpha \in [0, 1).$$

Indeed, the above inequality equivalently written as

$$\frac{4\alpha + 5}{3(1 + 2\alpha)} < \frac{9}{(2 + \alpha)^2}$$

is equivalent to the inequality

$$4\alpha^3 + 21\alpha^2 - 18\alpha - 7 = (\alpha - 1)(4\alpha^2 + 25\alpha + 7) < 0, \quad \alpha \in [0, 1),$$

which clearly holds. Thus the function γ is decreasing, and moreover

$$(2.10) \quad \begin{aligned} \gamma(t) &\leq \gamma\left(\frac{3}{2 + \alpha}\right) \\ &= \frac{8\alpha^3 + 42\alpha^2 + 36\alpha + 22}{(2 + \alpha)^3} \leq 4, \quad t \in [3/(2 + \alpha), 2). \end{aligned}$$

Indeed, the second inequality is equivalent to the inequality

$$2\alpha^3 + 9\alpha^2 - 6\alpha - 5 = (\alpha - 1)(\alpha + 5)(2\alpha + 1) \leq 0, \quad \alpha \in [0, 1),$$

which clearly holds.

2. Since $|B| < 2(1 - |C|)$ holds when $t \in [0, 3/(2 + \alpha))$, by Lemma 1.3 we have

$$(2.11) \quad \begin{aligned} &(4 - t^2) [|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2] \\ &\leq (4 - t^2) \left(1 + |A| + \frac{B^2}{4(1 - |C|)}\right) \\ &= (4 - t^2) \left(1 + \frac{(1 - 4\alpha)t^3}{6(4 - t^2)} + \frac{(1 + 2\alpha)^2 t^2}{18(2 - t)}\right) \\ &= (4 - t^2) \cdot \frac{72 + [2(1 + 2\alpha)^2 - 18]t^2 + [3(1 - 4\alpha) + (1 + 2\alpha)^2]t^3}{18(4 - t^2)} \\ &= \frac{1}{18} (72 + t^2[4(1 - \alpha)^2 t + 8(\alpha - 1)(\alpha + 2)]) \\ &\leq 4 + \frac{1}{2(2 + \alpha)^2} \left[4(1 - \alpha)^2 \cdot \frac{3}{2 + \alpha} + 8(\alpha - 1)(\alpha + 2)\right] \\ &= 4 + \frac{2}{(2 + \alpha)^3} (\alpha - 1)(\alpha + 5)(2\alpha + 1) \leq 4, \quad \alpha \in [0, 1), \quad t \in [0, 3/(2 + \alpha)). \end{aligned}$$

Summarizing, by (2.9) and (2.11) it follows that (2.8) is true for $\alpha \in [0, 1/4]$.

II. Consider the case $AC < 0$ which holds when $\alpha \in (1/4, 1)$.

1. Since

$$B^2 C^2 + 4AC(1 - C^2) = \frac{1}{9}(1 - \alpha)^2 t^4 \geq 0$$

and $|B| < 2(1 - |C|)$ holds when $t < 3/(2 + \alpha)$, by Lemma 1.3, repeating computing as in (2.11) we have

$$\begin{aligned}
 (2.12) \quad & (4 - t^2) [|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2] \\
 & \leq (4 - t^2) \left(1 - |A| + \frac{B^2}{4(1 - |C|)} \right) \\
 & = (4 - t^2) \left(1 + \frac{(1 - 4\alpha)t^3}{6(4 - t^2)} + \frac{(1 + 2\alpha)^2 t^2}{18(2 - t)} \right) \leq 4, \quad t \in [0, 3/(2 + \alpha)).
 \end{aligned}$$

2. Since $4(1 + |C|)^2 = (2 + t)^2$ and $-4AC(C^{-2} - 1) = (4\alpha - 1)t^2/3$, so the inequality

$$\begin{aligned}
 & B^2 = \frac{1}{9}(1 + 2\alpha)^2 t^2 \\
 & < \min \{4(1 + |C|)^2, -4AC(C^{-2} - 1)\} = \frac{1}{3}(4\alpha - 1)t^2
 \end{aligned}$$

is equivalent to the inequality $(\alpha - 1)^2 < 0$ which clearly does not hold.

3.(a) Note that the inequality $|C|(|B| + 4|A|) \leq |AB|$, i.e., the inequality

$$\frac{t}{2} \left(\frac{1}{3}(1 + 2\alpha)t - \frac{2(1 - 4\alpha)t^3}{3(4 - t^2)} \right) \leq \frac{(4\alpha - 1)(1 + 2\alpha)t^4}{18(4 - t^2)}$$

which after computing is equivalent to the inequality

$$t \geq \frac{1}{1 - \alpha} \sqrt{\frac{3}{2}(1 + 2\alpha)}$$

does not hold since the inequality

$$\frac{1}{1 - \alpha} \sqrt{\frac{3}{2}(1 + 2\alpha)} > 2, \quad \alpha \in (1/4, 1),$$

equivalently written as

$$8\alpha^2 - 22\alpha + 5 < 0, \quad \alpha \in (1/4, 1),$$

is clearly true.

(b) Observe that the inequality $|AB| \leq |C|(|B| - 4|A|)$, i.e., the inequality

$$\frac{(4\alpha - 1)(1 + 2\alpha)t^4}{18(4 - t^2)} \leq \frac{t}{2} \left(\frac{1}{3}(1 + 2\alpha)t - \frac{2(4\alpha - 1)t^3}{3(4 - t^2)} \right)$$

for $t \in [3/(2 + \alpha), 2)$, is equivalent to the inequality

$$(2.13) \quad \frac{3}{2 + \alpha} \leq t \leq \sqrt{\frac{3(1 + 2\alpha)}{2\alpha^2 + 8\alpha - 1}}.$$

Note first that $2\alpha^2 + 8\alpha - 1 > 0$ for $\alpha \in (1/4, 1)$. Moreover

$$\sqrt{\frac{3(1 + 2\alpha)}{2\alpha^2 + 8\alpha - 1}} > \frac{3}{2 + \alpha}.$$

Indeed, by squaring both sides of above inequality and by simple computing we equivalently get the inequality

$$2\alpha^3 + 3\alpha^2 - 12\alpha + 7 = (1 - \alpha)^2(2\alpha + 7) > 0, \quad \alpha \in (1/4, 1),$$

which clearly holds. We also see that that the inequality

$$\sqrt{\frac{3(1+2\alpha)}{2\alpha^2 + 8\alpha - 1}} < 2$$

holds, since by squaring and further computing it is equivalent to the true inequality

$$8\alpha^2 + 26\alpha - 7 = 8\left(\alpha + \frac{7}{2}\right)\left(\alpha - \frac{1}{4}\right) > 0, \quad \alpha \in (1/4, 1).$$

Arguing now exactly as of Part I.1 for t satisfying (2.13) we get

$$\begin{aligned} (2.14) \quad & (4 - t^2) [|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2] \\ & \leq (4 - t^2)(-|A| + |B| + |C|) \\ & = -\frac{2}{3}(1 + 2\alpha)t^3 + \frac{2}{3}(4\alpha + 5)t = \gamma(t) \leq 4. \end{aligned}$$

(c) It remains to consider

$$(2.15) \quad \sqrt{\frac{3(1+2\alpha)}{2\alpha^2 + 8\alpha - 1}} < t < 2.$$

Then by by Lemma 1.3 we have

$$\begin{aligned} & (4 - t^2) [|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2] \\ & \leq (4 - t^2)(|C| + |A|)\sqrt{1 - \frac{B^2}{4AC}} \\ & = \left(4 - \frac{4}{3}(1 - \alpha)t^2\right) \sqrt{\frac{(1 - \alpha)^2 t^2 - (1 + 2\alpha)^2}{3(1 - 4\alpha)}} =: \varphi(t), \end{aligned}$$

where t satisfies (2.15). Hence $\varphi'(t) = 0$ iff

$$-\frac{8}{3}(1 - \alpha)t \cdot \frac{(1 - \alpha)^2 t^2 - (1 + 2\alpha)^2}{3(1 - 4\alpha)} + \frac{4}{3} \left(1 - \frac{1}{3}(1 - \alpha)t^2\right) \frac{(1 - \alpha)^2 t}{1 - 4\alpha} = 0$$

i.e., after simplifying iff

$$3(1 - \alpha)^2 t^2 = 8\alpha^2 + 5\alpha + 5,$$

and hence iff

$$t = \pm \frac{1}{\sqrt{3(1 - \alpha)}} \sqrt{8\alpha^2 + 5\alpha + 5} =: \pm t_1,$$

where $\alpha \in (1/4, 1)$ and t satisfies (2.15). But $t_1 > 2$. Indeed, this inequality after simplifying is equivalent to the inequality

$$4\alpha^2 - 29\alpha + 7 = 4\left(\alpha - \frac{1}{4}\right)(\alpha - 7) < 0, \quad \alpha \in (1/4, 1),$$

which is obviously true. Consequently $\varphi'(t) > 0$ iff

$$-3(1-\alpha)^2 t^2 + 8\alpha^2 + 5\alpha + 5 < 0$$

which holds for t satisfying (2.15). Thus the function φ is increasing and hence

$$\begin{aligned} (2.16) \quad \varphi(t) \leq \varphi(2) &= \left(4 - \frac{16}{3}(1-\alpha)\right) \sqrt{\frac{4(1-\alpha)^2 - (1+2\alpha)^2}{3(1-4\alpha)}} \\ &= \frac{4}{3}(4\alpha - 1) \leq 4, \quad \alpha \in (1/4, 1). \end{aligned}$$

Summarizing, by (2.14), (2.15) and (2.16) it follows that (2.8) is true for $\alpha \in (1/4, 1)$. In this way it was shown that the inequality (2.8) is true which ends the proof of the inequality (2.1).

To show the sharpness, fix $\alpha \in [0, 1)$ and take the function f given by (2.2). Clearly, $f \in \mathcal{R}(\alpha)$ with $a_2 = a_3 = 0$ and $a_4 = (1-\alpha)/2$ which make the equality in (2.1). $\square$

For $\alpha := 0$, i.e., for the class $\mathcal{R}$ the above theorem reduces to Theorem 2.1 of [2].

Remark 2.2. The proof of Theorem 2.1 was based on Lemma 1.3 which is useful for such computing. This lemma was also applied in [3] and [4] to find sharp bounds analogous as in Theorem 2.1 for the class of starlike functions of order α and of strongly starlike functions, respectively. Let now remark that the result of Theorem 2.1 can be achieved in a simple way by using Theorem 2.4 of [8] which particularly produces the following result: if $p \in \mathcal{P}$ is of the form (1.4) and $\mu \in [0, 1]$, then for $n, m \in \mathbb{N}$, $m < n$, the following sharp estimate holds

$$(2.17) \quad |c_n - \mu c_{n-m} c_m| \leq 2.$$

Hence, from (2.5) and since $\mu := 2(1-\alpha)/3 \leq 1$ for $\alpha \in [0, 1)$, we have

$$|a_2 a_3 - a_4| = \frac{1}{4}(1-\alpha) \left| c_3 - \frac{2}{3}(1-\alpha) c_1 c_2 \right| \leq \frac{1}{2}(1-\alpha).$$

Theorem 2.4 can be found in [14] as Corollary 3.2. We reprove it by using the result below for the class $\mathcal{R}$ shown in [9].

Lemma 2.3. *If $f \in \mathcal{R}$ is of the form (1.1), then*

$$(2.18) \quad |a_2 a_4 - a_3^2| \leq \frac{4}{9}.$$

The inequality is sharp and the equality holds for the function

$$(2.19) \quad f(z) := \int_0^z \frac{1+u^2}{1-u^2} du = -z + \log \frac{1+z}{1-z}, \quad z \in \mathbb{D}, \quad \log 1 := 0.$$

Theorem 2.4. *Let $\alpha \in [0, 1)$. If $f \in \mathcal{R}(\alpha)$ is the form (1.1), then*

$$(2.20) \quad |a_2 a_4 - a_3^2| \leq \frac{4}{9}(1 - \alpha)^2.$$

The inequality is sharp and the equality holds for the function

$$(2.21) \quad \begin{aligned} f(z) &:= \int_0^z \frac{1 + (1 - 2\alpha)u^2}{1 - u^2} du \\ &= (-1 + 2\alpha)z + (1 - \alpha) \log \frac{1 + z}{1 - z}, \quad z \in \mathbb{D}, \quad \log 1 := 0. \end{aligned}$$

Proof. Fix $\alpha \in [0, 1)$ and let $f \in \mathcal{R}(\alpha)$. Define

$$(2.22) \quad g(z) := \frac{1}{1 - \alpha}(f(z) - \alpha z), \quad z \in \mathbb{D}.$$

Assuming that

$$(2.23) \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

and using (1.1), from (2.22) by comparing the coefficients of the series, we get

$$(2.24) \quad a_n = (1 - \alpha)b_n, \quad n \in \mathbb{N} \setminus \{1\}.$$

Since $g \in \mathcal{R}$, by applying the inequality (2.18) we get

$$|a_2 a_4 - a_3^2| = (1 - \alpha)^2 |b_2 b_4 - b_3^2| \leq \frac{4}{9}(1 - \alpha)^2.$$

Since the equality in (2.18) holds for the function given by (2.19), it follows from (2.22) that the equality in (2.20) holds for the function given by (2.21). $\square$

To prove Theorem 2.6 we use the following result related to the Fekete-Szegő functional for the class $\mathcal{R}$ (see e.g., [10, Corollary 2.7]).

Theorem 2.5. *Let $\lambda \in [0, 4/3]$. If $f \in \mathcal{R}$ is of the form (1.1), then*

$$(2.25) \quad |a_3 - \lambda a_2^2| \leq \frac{2}{3}.$$

The inequality is sharp and for each $\lambda \in [0, 4/3]$ the equality holds for the function (2.19).

Theorem 2.6. *Let $\alpha \in [0, 1)$. If $f \in \mathcal{R}(\alpha)$ is the form (1.1), then*

$$(2.26) \quad |a_3 - a_2^2| \leq \frac{2}{3}(1 - \alpha).$$

The inequality is sharp and the equality holds for the function (2.21).

Proof. Let $\alpha \in [0, 1)$ and $f \in \mathcal{R}(\alpha)$. Let g be a function defined by (2.22) of the form (2.23). Since $g \in \mathcal{R}$ by (2.24) and by applying the inequality (2.25) with $\lambda := 1 - \alpha \in [0, 1]$, we have

$$|a_3 - a_2^2| = (1 - \alpha)|b_3 - (1 - \alpha)b_2| \leq \frac{2}{3}(1 - \alpha).$$

Since the equality in (2.25) holds for the function given by (2.19), it follows from (2.22) that the equality in (2.26) holds for the function (2.21).

The inequality (2.26) follows also from the inequality (2.17). Indeed, by using (2.17), (2.4) and since $\mu := 3(1 - \alpha)/4 \leq 1$ for $\alpha \in [0, 1)$, we have

$$|a_3 - a_2^2| = \frac{1}{3}(1 - \alpha) \left| c_2 - \frac{3}{4}(1 - \alpha)c_1^2 \right| \leq \frac{2}{3}(1 - \alpha). \quad \square$$

Let $\alpha \in [0, 1)$ and let f of the form (1.1) belong to the class $\mathcal{R}(\alpha)$. Then the function p given by (2.3) is in $\mathcal{P}$. Thus by (2.4) and (1.5) we have the following well known result.

Theorem 2.7. *Let $\alpha \in [0, 1)$. If $f \in \mathcal{R}(\alpha)$ is the form (1.1), then*

$$(2.27) \quad |a_n| \leq \frac{2(1 - \alpha)}{n}, \quad n \in \mathbb{N} \setminus \{1\}.$$

The inequality (2.27) is sharp and the equality holds for the function

$$f(z) := (-1 + 2\alpha)z - 2(1 - \alpha)\log(1 - z), \quad z \in \mathbb{D}, \quad \log 1 := 0.$$

Since for $f \in \mathcal{A}$,

$$|H_{3,1}(f)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|,$$

by using (2.1), (2.20), (2.26) and (2.27) we have

Theorem 2.8. *Let $\alpha \in [0, 1)$. If $f \in \mathcal{R}(\alpha)$ is the form (1.1), then*

$$|H_{3,1}(f)| \leq \frac{1}{540}(1 - \alpha)^2(439 - 160\alpha).$$

Particularly, for $\alpha := 0$ we get Theorem 2.2 of [2].

Corollary 2.9. *If $f \in \mathcal{R}$ is the form (1.1), then*

$$|H_{3,1}(f)| \leq \frac{439}{540} \approx 0.813.$$

Recent result [17] improves the above one.

Corollary 2.10. *If $f \in \mathcal{R}$ is the form (1.1), then*

$$|H_{3,1}(f)| \leq \frac{41}{60} \approx 0.683.$$

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Presented by Zbigniew Jakubowski at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on December 1, 2016.

OSZACOWANIA PEWNYCH WYZNACZNIKÓW DLA FUNKCJI O OGRANICZONYM OBROCIE RZĘDU ALFA

S t r e s z c z e n i e

W pracy podane są oszacowania pewnych wyznaczników w klasie $\mathcal{R}(\alpha)$, $0 \leq \alpha < 1$, funkcji analitycznych f standardowo unormowanych takich, że

$$\operatorname{Re} f'(z) > \alpha, \quad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

Słowa kluczowe: funkcje jednoliste, funkcje o ograniczonym obrocie, funkcje o ograniczonym obrocie rzędu α , wyznacznik Hankela

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2017

Vol. LXVII

Recherches sur les déformations

no. 1

pp. 119–131

Adam Różycki

EFFECTIVE CALCULATION OF THE DEGREE OF *-COVERING

Summary

Let $f: \mathbb{C}^n \rightarrow \mathbb{C}^m$, $m \geq n$ be a proper polynomial mapping such that $f^{-1}(0) = \{0\}$. Then the mapping $f: \mathbb{C}^n \rightarrow f(\mathbb{C}^n)$ is a $*$ -covering (in the sense of [7]). In this paper we give an effective method of calculating the degree of this covering.

Keywords and phrases: multiplicity of mapping, intersection multiplicity, $*$ -covering, local degree, effective formula

1. Introduction

R. Draper in [4] defined the multiplicity for proper intersection of analytic sets. Next, in the case of isolated intersection, R. Achilles, P. Tworzewski, T. Winiarski in [1] generalized this definition to improper intersection case. The above multiplicity leads to the definition of multiplicity $i_0(f)$ of a holomorphic mapping at a zero of this mapping (see [14]).

Let $\Omega \subset \mathbb{C}^n$ be a neighbourhood of the point $0 \in \mathbb{C}^n$ and let $f: \Omega \rightarrow \mathbb{C}^m$, where $m \geq n$, be a holomorphic mapping such that 0 is an isolated point of the fiber $f^{-1}(0)$.

S. Spodzieja in [12] (see also [3], [13]) proved the following result which shows the connections between the multiplicity $i_0(f)$ and the degree of a $*$ -covering (in the sense of [7]), inducted by f in a neighbourhood of 0 .

Theorem 1. [12, Theorem 1.2] *There exists a neighbourhood $U \subset \Omega$ of the point 0 such that $f^{-1}(0) \cap U = \{0\}$, $f|_U: U \rightarrow f(U)$ is $*$ -covering and*

$$(1) \quad i_0(f) = d(f|_U) \cdot \deg_0 f(U),$$

where $d(f|_U)$ is the degree of the $*$ -covering $f|_U$ and $\deg_0 f(U)$ is the degree of an analytic set $f(U) \subset \mathbb{C}^m$ at $0 \in \mathbb{C}^n$.

In the paper we will give an effective method to calculate the degree of the $*$ -covering $f: \mathbb{C}^n \rightarrow f(\mathbb{C}^n)$, provided f is a polynomial mapping. Theorem 1 will be the crucial tool in this calculations.

Let $f: \mathbb{C}^n \rightarrow \mathbb{C}^m$, $m \geq n$ be a proper polynomial mapping such that $f^{-1}(0) = \{0\}$. Denote by $\mathbb{L}(n, m)$ the set of all linear mappings $\mathbb{C}^n \rightarrow \mathbb{C}^m$.

We will effectively specify an open set $\hat{U} \subset \mathbb{L}(m, n)$ such that for any $p \in \hat{U}$ the mapping $\pi \circ (F^*, p): \mathbb{C}^m \rightarrow \mathbb{C}^m$ is proper and $i_0(F^*, p) = m_0(\pi \circ (F^*, p))$ for the generic $\pi \in \mathbb{L}(s + n, m)$, where $F^*: \mathbb{C}^m \rightarrow \mathbb{C}^s$ is polynomial mapping such that $C_0(f(\mathbb{C}^n)) = (F^*)^{-1}(0)$, and $\hat{C}_0(f(\mathbb{C}^n))$ is the tangent cone to $f(\mathbb{C}^n)$ at 0 in the sense of [15]. Next, fixing $p \in \hat{U}$, we describe a set $\mathcal{U} \subset \mathbb{L}(n, 1)$ of linear functions such that for any $l \in \mathcal{U}$ the function l is injective on the generic fiber of the mapping $p \circ f$. The main result of this paper is the following theorem.

Theorem 2. Fix any $p \in \hat{U}$. There exist effectively computable: linear function $l \in \mathcal{U}$, non-zero irreducible polynomial of the form $P_{p,l}(y, t) = \sum_{j=0}^k P_{p,l,j}(y)t^j$ which vanishes on the image of the mapping $(H_p, l): \mathbb{C}^n \rightarrow \mathbb{C}^{n+1}$, where $H_p(x) = (p \circ f)(x) + (x_1^{d^n+1}, \dots, x_n^{d^n+1})$, $d = \deg f$ and non-zero irreducible polynomial $P_p^*(\pi, N, w, t) = \sum_{i=0}^l P_{p,i}^*(\pi, N, w)t^i$ which vanishes on the image of the mapping $\Phi_p^*: \mathbb{L}(s + n, m) \times \mathbb{L}(m, 1) \times \mathbb{C}^m \rightarrow \mathbb{L}(s + n, m) \times \mathbb{L}(m, 1) \times \mathbb{C}^{m+1}$ of the form $\Phi_p^*(\pi, N, y) = (\pi, N, H_{p,\pi}^*(y), N(y))$, $H_{p,\pi}^*(y) = (\pi \circ (F^*, p))(y) + (y_1^{d^*m+1}, \dots, y_m^{d^*m+1})$, $d^* = \deg F^*$ such that

$$(2) \quad d(f) = \frac{\min\{j \in \{1, \dots, k\} : \text{ord}_y P_{p,l,j} = 0\}}{\min\{i \in \{1, \dots, l\} : \text{ord}_w P_{p,i}^* = 0\}}.$$

In the paper [10], there was presented a similar result concerning the calculation of the polynomial $P_{p,l}(y, t)$, not for fixed mappings p and l but for the linear mapping with variable coefficients. From the computational point of view, increasing the number of variables made the algorithm slower. The characterizations of the sets U and $\mathcal{U}^*$ are intended to make the algorithm faster.

2. The Jelonek set of polynomial mapping

Let X and Y be locally compact topological spaces. The mapping $f: X \rightarrow Y$ is said to be *proper* if for any compact set $K \subset Y$ the set $f^{-1}(K)$ is a compact subset of X . We say that the mapping $f: X \rightarrow Y$ is *proper at point* $y \in Y$ if there exists an open neighbourhood D of y such that

$$f|_{f^{-1}(D)}: f^{-1}(D) \rightarrow D$$

is a proper mapping.

Proposition 3. [6, Remark 5.2] *Let X and Y be locally compact spaces. Then the mapping f is proper if and only if it is proper at every point $y \in Y$.*

The set of all points at which the mapping f is not proper is called the *set of non-properness of mapping f* (or the *Jelonek set of f*) and is denoted by S_f .

Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial mapping. The mapping f is said to be *dominant* if $\overline{f(\mathbb{C}^n)} = \mathbb{C}^n$.

The mapping f is said to be *finite* if it is proper and all its fibers are finite.

The following Proposition is well known.

Proposition 4. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a proper polynomial mapping. Then f is finite and surjective (hence dominant).*

Let us recall, after [6], the effective construction of the Jelonek set.

Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a dominant mapping such that $f(0) = 0$. Define mapping $\Phi_j : \mathbb{C}^n \rightarrow \mathbb{C}^{n+1}$ by

$$\Phi_j(x) = (f(x), x_j),$$

for any $j = 1, \dots, n$, where $x = (x_1, \dots, x_n)$. Since mapping f is dominant, hence $\overline{\Phi_j(\mathbb{C}^n)}$ is an algebraic set in $\mathbb{C}^{n+1}$ of dimension n , i.e. it is a hypersurface. There exists a polynomial $P_j \in \mathbb{C}[y, t]$, $y = (y_1, \dots, y_n) \in \mathbb{C}^n$, $t \in \mathbb{C}$, irreducible and of minimal degree of the form

$$(3) \quad P_j(y, t) = P_{j,0}(y)t^{d_j} + \dots + P_{j,d_j}(y), \quad P_{j,0} \neq 0, \quad j = 1, \dots, n$$

such that $\overline{\Phi_j(\mathbb{C}^n)} = P_j^{-1}(0)$. We have

Lemma 5. [6, Proposition 7]

$$S_f = \{y \in \mathbb{C}^n : \prod_{j=1}^n P_{j,0}(y) = 0\}.$$

3. Effective Primitive Element Theorem

Now, we will give the effective method of finding the set of linear functions which separate points of fibers of polynomial mappings.

Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $f(0) = 0$, be a proper polynomial mapping (i.e. $S_f = \emptyset$).

Let $l \in \mathbb{L}(n, 1)$ be a linear function of the form $l(x) = a_1x_1 + \dots + a_nx_n$, where $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, and $a = (a_1, \dots, a_n) \in \mathbb{C}^n$. Define mapping

$$F : \mathbb{L}(n, 1) \times \mathbb{C}^n \rightarrow \mathbb{L}(n, 1) \times \mathbb{C}^n \times \mathbb{C}$$

by

$$F(l, x) = (l, f(x), l(x)).$$

Obviously the mapping F is proper. Then by the Chevalley Theorem $F(\mathbb{L}(n, 1) \times \mathbb{C}^n)$ is an algebraic irreducible set of dimension $2n$.

Therefore there exists an irreducible polynomial $P \in \mathbb{C}[l, y, t]$, $y \in \mathbb{C}^n$, $t \in \mathbb{C}$ of the form

$$P(l, y, t) = \sum_{j=0}^d P_j(l, y) t^j$$

such that $P_d \neq 0$ and $F(\mathbb{L}(n, 1) \times \mathbb{C}^n) = P^{-1}(0)$.

The polynomial P , which vanishes exactly on the image of the polynomial map F , could be effectively computed by means of Gröbner bases.

Denote by Δ the discriminant of P i.e.

$$\Delta(l, y) = \text{Res}_t \left(P, \frac{\partial P}{\partial t} \right) (l, y), \quad \text{for any } (l, y) \in \mathbb{L}(n, 1) \times \mathbb{C}^n.$$

Put

$$\mathcal{W} = \{(l, y) \in \mathbb{L}(n, 1) \times \mathbb{C}^n : \Delta(l, y) = 0\}.$$

and

$$\mathcal{V} = \{l \in \mathbb{L}(n, 1) : \Delta(l, y) = 0 \text{ for any } y \in \mathbb{C}^n\}.$$

Put now

$$\mathcal{U} = \mathbb{L}(n, 1) \setminus \mathcal{V}.$$

For any fixed $l \in \mathcal{U}$ denote $\Delta_l(y) = \Delta(l, y)$, and $V_l = \{y \in \mathbb{C}^n : \Delta_l(y) = 0\}$.

We have

Proposition 6. *For any $l \in \mathcal{U}$ there exists $y \in \mathbb{C}^n \setminus V_l$ such that the restriction*

$$l|_{f^{-1}(y)} : f^{-1}(y) \rightarrow \mathbb{C}$$

is injective.

Proof. Let us fix arbitrary $l \in \mathcal{U}$. Then there exists $y \in \mathbb{C}^n$ such that $\Delta_{l,y} := \Delta_l(y) \neq 0$. Thus, for this y , the polynomial

$$P_{l,y}(t) = P(l, y, t) \in \mathbb{C}[t]$$

has no double roots. Let $t_1, \dots, t_d$ be all roots of $P_{l,y}$. Then

$$P_{l,y}(t) = P_{l,y,0} \cdot \prod_{j=1}^d (t - t_j),$$

where $P_{l,y,0} \in \mathbb{C} \setminus \{0\}$ and

$$(4) \quad 0 \neq \Delta_{l,y} = \text{Res}(P_{l,y}, P'_{l,y}) = P_{l,y,0}^{2d-2} \cdot \prod_{i < j} (t_i - t_j)^2.$$

Since f is proper and polynomial mapping, then by Proposition 4 there exist a finite set $\{x^1, \dots, x^k\} \subset \mathbb{C}^n \setminus f^{-1}(V_l)$, $x^i \neq x^j$, such that $f^{-1}(y) = \{x^1, \dots, x^k\}$.

Hence, for any $x^i \in f^{-1}(y)$ we have

$$0 = P_l(f(x^i), l(x^i)) = P_l(y, l(x^i)) = P_{l,y}(l(x^i)).$$

Therefore $t_i = l(x^i)$, for $i = 1, \dots, k$ (after an eventual permutation). Since all t_j are different, therefore $k = d$, and by (4), we have

$$P_{l,y,0}^{2d-2} \cdot \prod_{i < j} (l(x^i) - l(x^j))^2 \neq 0.$$

This ends the proof. $\square$

As an immediate consequence of Proposition 6, we get

Corollary 7. *For any $l \in \mathcal{U}$, putting $t = l(x_1, \dots, x_n)$, we have that t is the primitive element of the extension $\mathbb{C}(f_1, \dots, f_n) \subset \mathbb{C}(x_1, \dots, x_n)$ i.e.*

$$\mathbb{C}(x_1, \dots, x_n) = \mathbb{C}(f_1, \dots, f_n)(t).$$

4. Index of overdetermined mapping

Let us recall (see [7]) the definition of the multiplicity of a holomorphic mapping.

Let M and N be complex manifolds of the same dimension $n > 0$, and let $f : M \rightarrow N$ be a holomorphic mapping. Let $a \in M$ be an isolated point of its fiber $f^{-1}(f(a))$. We define the *multiplicity of the mapping f at the point a* by

$$m_a(f) = \sup\{\#(f|_U)^{-1}(y) : y \in D\},$$

where U and D are sufficiently small neighbourhoods of the points a and $f(a)$, respectively.

Let us recall [2] (see also [4]) the definition of the multiplicity of the proper, isolated intersection of the analytic sets.

Let $X_1, \dots, X_k$ be analytic sets in a domain $D \in \mathbb{C}^n$ of pure dimensions $p_1, \dots, p_k$, respectively. Assume that

$$0 = \dim \bigcap_{j=1}^k X_j = \sum_{j=1}^k p_j - (k-1)n,$$

i.e. the intersection $\bigcap_{j=1}^k X_j$ is proper and isolated. Denote by

$$\Delta = \{(x^1, \dots, x^{kn}) \in \mathbb{C}^{kn} : x^1 = \dots = x^{kn}\}$$

the diagonal set in $\mathbb{C}^{kn}$.

If $a \in \mathbb{C}^n$ is an isolated point of $\bigcap_{j=1}^k X_j$, then $(a)^k := (a, \dots, a) \in \mathbb{C}^{kn}$ is the isolated point of $(X_1 \times \dots \times X_k) \cap \Delta$. Hence the projection

$$\pi_\Delta|_{X_1 \times \dots \times X_k} : X_1 \times \dots \times X_k \rightarrow \Delta^\perp \subset \mathbb{C}^{kn}$$

along Δ is an analytic cover in some neighbourhood of $(a)^k$.

The multiplicity

$$\mu_{(a)^k}(\pi_\Delta|_{X_1 \times \dots \times X_k})$$

of the projection $\pi_\Delta|_{X_1 \times \dots \times X_k}$ at the point $(a)^k \in \mathbb{C}^{kn}$ is called the *intersection multiplicity* or *intersection index* of the sets $X_1, \dots, X_k$ at a point $a \in \bigcap_{j=1}^k X_j$ and we denote it by $i(X_1 \cdot \dots \cdot X_k; a)$. For $a \in D \setminus \bigcap_{j=1}^k X_j$ we put $i(X_1 \cdot \dots \cdot X_k; a) = 0$.

Recall after [1] some fact concerning the improper intersection of the analytic sets.

Let X be a pure k -dimensional analytic subset of a complex manifold M of dimension m . Let N be a submanifold of M of dimension n such that N intersects X at an isolated point $a \in M$. We denote by $\mathcal{F}_a(X, N)$ the set of all locally analytic subsets V of M satisfying the following conditions:

- (i) V has pure dimension $m - k$,
- (ii) $N_a \subset V_a$, where N_a, V_a denote the germs at a of N and V , respectively,
- (iii) a is an isolated point of $V \cap X$.

We define the *multiplicity of improper isolated intersection the analytic set X with submanifold N at the point a* by

$$\tilde{i}(X \cdot N; a) = \min\{i(X \cdot V; a) : V \in \mathcal{F}_a(X, N)\}.$$

By *index of a holomorphic mapping $f: D \rightarrow \mathbb{C}^m$ at 0*, where $D \subset \mathbb{C}^n$ is a neighbourhood of 0 such that 0 is an isolated zero of f , we mean the multiplicity

$$\tilde{i}(\Gamma_f \cdot (\mathbb{C}^n \times \{0\}), (0, 0))$$

of improper intersection of the graph Γ_f (i.e. $\Gamma_f = \{(x, f(x)) : x \in D\}$) of f and the set $\mathbb{C}^n \times \{0\} \subset \mathbb{C}^n \times \mathbb{C}^m$ at the point $(0, 0) \in \mathbb{C}^n \times \mathbb{C}^m$ and denote it by $i_0(f)$.

Let $D \subset \mathbb{C}^n$ be an open neighbourhood of the point $0 \in \mathbb{C}^n$ and let $f: D \rightarrow \mathbb{C}^m$ be a holomorphic mapping such that $f(0) = 0$ and 0 is an isolated point of the set $f^{-1}(0)$.

Theorem 8. [12, Theorem 1.1] *For any $p \in \mathbb{L}(m, n)$ such that the point 0 is an isolated zero of $p \circ f$ we have*

$$i_0(f) \leq m_0(p \circ f).$$

Moreover, for the generic $p \in \mathbb{L}(m, n)$, the point 0 is an isolated zero of $p \circ f$ and

$$(5) \quad i_0(f) = m_0(p \circ f).$$

Let $f: \mathbb{C}^n \rightarrow \mathbb{C}^m$, $m \geq n$ be a polynomial mapping such that 0 is an isolated point of the set $f^{-1}(0)$ and $f(0) = 0$. Then there exists a neighbourhood U of the point 0 such that $f(U) \ni 0$ is an algebraic set in $\mathbb{C}^m$ of the pure dimension n . Let $f|_U: U \rightarrow f(U)$. Put

$$(6) \quad \tilde{\mathcal{U}} = \{p \in \mathbb{L}(m, n) : i_0(f) = m_0(p \circ f)\}$$

and

$$(7) \quad \mathcal{U}' = \{p \in \mathbb{L}(m, n) : C_0(f(U)) \cap \ker p = \{0\}\},$$

where $C_0(f(U))$ is a tangent cone of $f(U)$ at the point 0 (in the sense of [15]).

We have the following

Proposition 9. [11, Theorem 2.2]

$$(8) \quad \mathcal{U}' \subset \tilde{\mathcal{U}}.$$

The set $\mathcal{U}'$ can be effectively calculated (see [11] or Proposition 11).

5. Local degree of algebraic set

Let X be a pure s -dimensional analytic set in $\mathbb{C}^n$ and $a \in X$. Denote by $G(n-s, n)$ the set of all $(n-s)$ -dimensional linear subspaces of $\mathbb{C}^n$. Let $L \in G(n-s, n)$ be such that a is an isolated point of the set $X \cap (a + L)$. It is well known, that there exists a neighbourhood $U \subset \mathbb{C}^n$ of the point a such that $X \cap U \cap (a + L) = \{a\}$, and such that the projection

$$\pi_L : X \cap U \rightarrow U' \subset L^\perp$$

along L is a k -sheeted analytic cover, for some $k \in \mathbb{N}$, where $L^\perp$ is subspace of $\mathbb{C}^n$ orthogonal to $L \subset \mathbb{C}^n$. This number k is called the *multiplicity of the projection* $\pi_L|X$ at the point a , and is denoted by $\mu_a(\pi_L|X)$.

We put $\mu_a(\pi_L|X) = +\infty$ if there exists $b \in U'$ such that $a \in (\pi_L)^{-1}(b)$ and $\dim(\pi_L)^{-1}(b) > 0$.

We put $\mu_a(\pi_L|X) = 0$ if $a \notin X$.

Denote by $\mathcal{G}(n-s, n)$ the set of all linear subspaces $L \in G(n-s, n)$, $L \ni 0$ such that a is an isolated point of the set $X \cap (a + L)$. Then for every $L \in \mathcal{G}(n-s, n)$ the multiplicity of the projection, $\mu_a(\pi_L|X)$, is finite. The number

$$\min\{\mu_a(\pi_L|X) : L \in \mathcal{G}(n-s, n)\}$$

is said to be the *local degree of X at the point a* and is denoted by $\deg_a(X)$.

Proposition 10. [2, Proposition 11.2] *Let X be a pure s -dimensional analytic set in a neighbourhood of 0 in $\mathbb{C}^n$, $0 \in X$, and let $L \in G(n-s, n)$. Then*

$$\mu_0(\pi_L|X) = \deg_0(X) \iff L \cap C_0(X) = \{0\},$$

where $C_0(X)$ is a tangent cone of X at the point 0 (in the sense of [15]).

6. Effective calculations of local degree

Now, we will present the effective procedure of calculation of the local degree of an algebraic set.

Let $F = (f_1, \dots, f_n) : \mathbb{C}^m \rightarrow \mathbb{C}^n$, be a polynomial mapping such that $F(0) = 0$. Assume that $\dim F^{-1}(0) = k$, $0 < k < m$.

Let $I \subset \mathbb{C}[x_1, \dots, x_m]$ be the ideal generated by F i.e. $I = \langle f_1, \dots, f_n \rangle$ and let $B = \{\tilde{f}_1, \dots, \tilde{f}_s\}$ be the standard base of I . Then, by [5, Lemma 5.5.11]

$$\text{in } I = \langle \text{in } \tilde{f}_1, \dots, \text{in } \tilde{f}_s \rangle,$$

where $\text{in } \tilde{f}_j$ denotes the initial form of $\tilde{f}_j$, for $j = 1, \dots, s$. Put

$$F^* = (\text{in } \tilde{f}_1, \dots, \text{in } \tilde{f}_s) : \mathbb{C}^m \rightarrow \mathbb{C}^s.$$

Then, by [15, Theorem 10.6] we have

$$C_0(F^{-1}(0)) = (F^*)^{-1}(0),$$

and by [15, Lemma 8.11] we get that

$$\dim(F^*)^{-1}(0) = \dim F^{-1}(0) = k.$$

Define mapping

$$\tilde{F}^* : \mathbb{C}^m \times \mathbb{L}(m, k) \rightarrow \mathbb{C}^{s+k} \times \mathbb{L}(m, k)$$

by

$$\tilde{F}^*(x, p) = (F^*(x), p(x), p).$$

Observe that for the generic $p \in \mathbb{L}(m, k)$, denoting $\tilde{F}_p^*(x) = \tilde{F}^*(x, p)$, we have

$$\dim(\tilde{F}_p^*)^{-1}(0) = 0.$$

Hence $s + k \geq m$. Next, define

$$F' : \mathbb{C}^m \times \mathbb{L}(m, k) \times \mathbb{L}(s + k, m) \rightarrow \mathbb{C}^m \times \mathbb{L}(m, k) \times \mathbb{L}(s + k, m)$$

by

$$F'(x, p, \pi) = (\pi \circ \tilde{F}^*(x, p), \pi)$$

and mapping

$$\tilde{\Phi}_j : \mathbb{C}^m \times \mathbb{L}(m, k) \times \mathbb{L}(s + k, m) \rightarrow \mathbb{C}^m \times \mathbb{L}(m, k) \times \mathbb{L}(s + k, m) \times \mathbb{C}$$

by

$$\tilde{\Phi}_j(x, p, \pi) = (F'(x, p, \pi), x_j),$$

for $j = 1, \dots, m$. Since the mapping F' is dominant for the generic (p, π) , there exists a polynomial $P_j^* \in \mathbb{C}[y, p, \pi, t]$ of the form

$$P_j^*(y, p, \pi, t) = P_{j,0}^*(p, \pi)t^{d_j} + P_{j,1}^*(y, p, \pi)t^{d_j-1} + \dots + P_{j,d_j}^*(y, p, \pi),$$

such that

$$\overline{\tilde{\Phi}_j(\mathbb{C}^m \times \mathbb{L}(m, k) \times \mathbb{L}(s + k, m))} = (P_j^*)^{-1}(0).$$

Similarly as in Section 3, each of the polynomial P_j^* , $j = 1, \dots, m$ can be effectively calculated.

Let

$$\mathcal{V}^* = \left\{ (p, \pi) \in \mathbb{L}(m, k) \times \mathbb{L}(s + k, m) : \prod_{j=1}^m P_{j,0}^*(p, \pi) = 0 \right\}.$$

Put $\mathcal{U}^* = (\mathbb{L}(m, k) \times \mathbb{L}(s + k, m)) \setminus \mathcal{V}^*$. Denote

$$\hat{\mathcal{U}} = \{p \in \mathbb{L}(m, k) : \prod_{j=1}^m P_{j,0}^*(p, \pi) \neq 0 \text{ for some } \pi \in \mathbb{L}(s + k, m)\}.$$

We have the following

Proposition 11. *For any $p \in \hat{\mathcal{U}}$ putting $L = \ker p$ we have*

$$\mu_0(\pi_L|F^{-1}(0)) = \deg_0(F^{-1}(0)).$$

Proof. Let us fix arbitrary $p \in \hat{\mathcal{U}}$. Then there exists $\pi \in \mathbb{L}(s + k, m)$ such that $(p, \pi) \in \mathcal{U}^*$. Hence

$$P_{j,0}^*(p, \pi) \neq 0,$$

for each $j = 1, \dots, m$. Therefore the mapping $\pi \circ (F^*, p)$ is proper. This means that the fiber $(\pi \circ (F^*, p))^{-1}(0)$ is finite. Since

$$(\pi \circ (F^*, p))^{-1}(0) \supset (F^*, p)^{-1}(0),$$

hence the fiber $(F^*, p)^{-1}(0)$ is also finite. But the mapping F^* is homogeneous. Therefore $(F^*, p)^{-1}(0) = \{0\}$. On the other hand we have

$$(F^*, p)^{-1}(0) = (F^*)^{-1}(0) \cap p^{-1}(0) = C_0(F^{-1}(0)) \cap p^{-1}(0).$$

Putting $L = \ker p$ we obtain

$$(9) \quad C_0(F^{-1}(0)) \cap L = \{0\}.$$

This together with the Proposition 10 gives the assertion. $\square$

Let us fix $p \in \hat{\mathcal{U}}$. We will calculate $i_0(F^*, p)$. Define mapping $H_{p,\pi}^*: \mathbb{C}^m \rightarrow \mathbb{C}^m$ by

$$H_{p,\pi}^*(y) = (\pi \circ (F^*, p))(y) + (y_1^{d^*m+1}, \dots, y_m^{d^*m+1}),$$

where $d^* = \max\{\deg F_1^*, \dots, \deg F_s^*\}$, $\pi \in \mathbb{L}(s + n, m)$ and $N \in \mathbb{L}(m, 1)$. Next define mapping $\Phi_p^*: \mathbb{L}(s + n, m) \times \mathbb{L}(m, 1) \times \mathbb{C}^m \rightarrow \mathbb{L}(s + n, m) \times \mathbb{L}(m, 1) \times \mathbb{C}^{m+1}$ by

$$\Phi_p^*(\pi, N, y) = (\pi, N, H_{p,\pi}^*(y), N(y)).$$

Since the mapping Φ_p^* is proper we can find an irreducible polynomial

$$P_p^* \in \mathbb{C}[\pi, N, w, t]$$

of the form $P_p^*(\pi, N, w, t) = \sum_{i=0}^l P_{p,i}^*(\pi, N, w) t^i$ such that $P_{p,l}^* \neq 0$ and

$$\Phi_p^*(\mathbb{L}(s + n, m) \times \mathbb{L}(m, 1) \times \mathbb{C}^m) = P_p^{*-1}(0).$$

Then by [9, Theorem 7] there exists $r^* \in \mathbb{N}$, $0 \leq r^* < l$ such that

$$\text{ord}_w P_{p,i}^* > 0 \text{ for } i = 0, \dots, r^* \text{ and } \text{ord}_w P_{p,r^*+1}^* = 0.$$

Hence and by [10, Theorem 4] we have

Corrolary 12.

$$\deg_0(F^{-1}(0)) = i_0(F^*, p) = \min\{i \in \{1, \dots, l\} : \text{ord}_w P_{p,i}^* = 0\}.$$

7. Proof of the main theorem

Let M and N be arbitrary complex manifolds and $X \subset M$ and $Y \subset N$ be non-empty analytic subsets. Let $f : X \rightarrow Y$ be a proper holomorphic mapping such that its fibres are finite.

We say that the mapping f is **-covering* if there exist nowhere dense analytic sets $Z \subset X$ and $\Sigma \subset Y$ such that $X \setminus Z$ and $Y \setminus \Sigma$ are manifolds, $Y \setminus \Sigma$ is connected, $f(X \setminus Z) \subset Y \setminus \Sigma$ and restriction

$$(10) \quad f_{X \setminus f^{-1}(Z)} : X \setminus f^{-1}(Z) \rightarrow Y \setminus \Sigma$$

is a finite covering. The degree of the covering (10) is called the *degree* of the **-covering* and denoted by $d(f)$.

Let $f = (f_1, \dots, f_m) : \mathbb{C}^n \rightarrow \mathbb{C}^m$, $m \geq n$, be a proper polynomial mapping such that $f^{-1}(0) = \{0\}$. By Theorem 1 we have that

$$(11) \quad i_0(f) = d(f) \cdot \deg_0 f(\mathbb{C}^n).$$

In this Section we give effective method to calculate the number $d(f)$ in (11).

Let I_f be the ideal of the graph of the mapping f , i.e.

$$I_f = \langle y_1 - f_1, \dots, y_m - f_m \rangle,$$

where $y_1, \dots, y_m$ are new variables. Equip $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$ with an eliminating local ordering $<$ for $x_1, \dots, x_n$. Let G be the Gröbner basis of the ideal I_f with respect to this ordering. Such base can always be effectively computed. Let $J_f = I_f \cap \mathbb{C}[y_1, \dots, y_m]$. Then the ideal J_f is generated by a set $G' = G \cap \mathbb{C}[y_1, \dots, y_m]$. Let $G' = \{g_1, \dots, g_u\}$, $u \geq m - n$. Then $J_f = \langle g_1, \dots, g_u \rangle$. Let $G = (g_1, \dots, g_u) : \mathbb{C}^m \rightarrow \mathbb{C}^u$. Then

$$f(\mathbb{C}^n) = G^{-1}(0).$$

Let $\{h_1, \dots, h_s\}$ be the standard basis of ideal J_f with respect to the degree ordering given by $<$. Then by [5, Lemma 5.5.11]

$$\text{in}(J_f) = \langle \text{in}(h_1), \dots, \text{in}(h_s) \rangle, \quad s \geq m - n,$$

where $\text{in}(h_j)$ denotes the initial form of h_j , $j = 1, \dots, s$. Set

$$F^* = (F_1^*, \dots, F_s^*) : \mathbb{C}^m \rightarrow \mathbb{C}^s,$$

where $F_j^* = \text{in}(h_j)$, for $j = 1, \dots, s$. By [15, Theorem 10.6] we have

$$(F^*)^{-1}(0) = C_0(f(\mathbb{C}^n)).$$

Repeating argument used in Section 5 we can find open sets $\hat{U} \subset \mathbb{L}(m, n)$ such that for any $p \in \hat{U} \subset \mathbb{L}(m, n)$ we have

$$(F^*)^{-1}(0) \cap \ker p = \{0\}.$$

Let us fix $p \in \hat{U}$. Define mapping $H_{p,\pi}^*: \mathbb{C}^m \rightarrow \mathbb{C}^m$ by

$$H_{p,\pi}^*(y) = (\pi \circ (F^*, p))(y) + (y_1^{d^*m+1}, \dots, y_m^{d^*m+1}),$$

where $d^* = \max\{\deg F_1^*, \dots, \deg F_s^*\}$, $\pi \in \mathbb{L}(s+n, m)$ and $N \in \mathbb{L}(m, 1)$. Next define mapping $\Phi_p^*: \mathbb{L}(s+n, m) \times \mathbb{L}(m, 1) \times \mathbb{C}^m \rightarrow \mathbb{L}(s+n, m) \times \mathbb{L}(m, 1) \times \mathbb{C}^{m+1}$ by

$$\Phi_p^*(\pi, N, y) = (\pi, N, H_{p,\pi}^*(y), N(y)).$$

Since the mapping Φ_p^* is proper we can find an irreducible polynomial

$$P_p^* \in \mathbb{C}[\pi, N, w, t]$$

of the form $P_p^*(\pi, N, w, t) = \sum_{i=0}^l P_{p,i}^*(\pi, N, w) t^i$ such that $P_{p,l}^* \neq 0$ and

$$\Phi_p^*(\mathbb{L}(s+n, m) \times \mathbb{L}(m, 1) \times \mathbb{C}^m) = P_p^{*-1}(0).$$

Again by [9, Theorem 7] there exists $r^* \in \mathbb{N}$, $0 \leq r^* < l$ such that

$$\text{ord}_w P_{p,i}^* > 0 \text{ for } i = 0, \dots, r^* \text{ and } \text{ord}_w P_{p,r^*+1}^* = 0.$$

and by [10, Theorem 4]

$$\deg_0 f(\mathbb{C}^n) = \deg_0(F^{-1}(0)) = \min\{i \in \{1, \dots, l\} : \text{ord}_w P_{p,i}^* = 0\}.$$

This gives the effective calculation of the number $\deg_0 f(\mathbb{C}^n)$.

On the other hand, by the choice of $p \in \hat{U}$, and by Proposition 9, $i_0(f) = m_0(p \circ f)$. Then the mapping $H_p: \mathbb{C}^n \rightarrow \mathbb{C}^n$ by

$$H_p(x) = (p \circ f)(x) + (x_1^{d^n+1}, \dots, x_n^{d^n+1}),$$

where $d = \max\{\deg f_1, \dots, \deg f_m\}$, is proper (i.e. $S_{H_p} = \emptyset$) and $i_0(f) = m_0(H_p)$. Therefore, repeating argument used in Section 3 for mapping H_p , we can find the open set $\mathcal{U} \subset \mathbb{L}(n, 1)$ of linear functions which separate the fibers of the mapping H_p . Let us fix such $l \in \mathcal{U}$ and define

$$\Phi_{p,l}: \mathbb{C}^n \rightarrow \mathbb{C}^n \times \mathbb{C}$$

by

$$\Phi_{p,l}(x) = (H_p(x), l(x)).$$

The mapping $\Phi_{p,l}$ is proper and consequently $\Phi_{p,l}(\mathbb{C}^n)$ is an algebraic set of pure dimension n . So, there exists an irreducible polynomial $P_{p,l} \in \mathbb{C}[y, t]$, where $y = (y_1, \dots, y_n)$ and $y_1, \dots, y_n, t$ are independent variables, of the form

$$(12) \quad P_{p,l}(y, t) = \sum_{j=0}^k P_{p,l,j}(y) t^j$$

such that $P_{p,l,k} \neq 0$ and $\Phi_{p,l}(\mathbb{C}^n) = P_{p,l}^{-1}(0)$. Hence, by [9, Theorem 7] we have that there exist $r \in \mathbb{N}$ with $0 \leq r < k$ such that

$$(13) \quad \text{ord}_y P_{p,l,j} > 0 \text{ for } j = 0, \dots, r \text{ and } \text{ord}_y P_{p,l,r+1} = 0.$$

Hence and by [10, Theorem 4] we have

$$i_0(f) = m_0(H_p) = \min\{j \in \{1, \dots, k\} : \text{ord}_y P_{p,l,j} = 0\}.$$

The above proposition gives an effective calculation method of the number $i_0(f)$.

In summary, we can find effectively $i_0(f)$ and $\deg_0 f(\mathbb{C}^n)$ in (11). Hence the third number in (11) can be effectively determined. The above considerations and Theorem 1 ends the proof of Theorem 2.

Acknowledgements

I am very grateful to Professor Tadeusz Krasieński and Professor Stanisław Spodzieja for their valuable comments and advices.

This research was partially supported by the Polish National Science Centre, grant 2012/07/B/ST1/03293.

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Presented by Adam Paszkiewicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 3, 2016.

EFEKTYWNE WYLICZANIE STOPNIA $*$ -NAKRYCIA

S t r e s z c z e n i e

Niech $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$, $m \geq n$ będzie odwzorowaniem wielomianowym takim, że $f(0) = 0$ oraz 0 jest punktem izolowanym zbioru $f^{-1}(0)$. Wówczas odwzorowanie $f|_D : D \rightarrow f(D)$ jest $*$ -nakryciem (w rozumieniu definicji [7]) w pewnym otoczeniu D punktu $0 \in \mathbb{C}^n$. W pracy podajemy efektywną metodę wyliczania stopnia tego nakrycia.

Słowa kluczowe: krotność odwzorowania, krotność przecięcia, $*$ -nakrycie, efektywny wzór

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2017

Vol. LXVII

Recherches sur les déformations

no. 1

pp. 133–148

*Dorota Pylak***APPLICATION OF JACOBI POLYNOMIALS TO APPROXIMATE SOLUTION OF A COMPLETE SINGULAR INTEGRAL EQUATION WITH CAUCHY KERNEL ON THE REAL HALF-LINE****Summary**

Jacobi polynomials are used to derive approximate solutions of the complete singular integral equation with Cauchy-type kernel defined on the real half-line in the case of constant complex coefficients. Moreover, estimations of errors of the approximated solutions are presented and proved.

Keywords and phrases: Jacobi polynomials, singular integral equations, Cauchy kernel, approximate solutions

1. Introduction

In the theory of singular integral equations (SIEs) with Cauchy-type kernels [9, 12, 13, 26, 30] there is considered the following equation:

$$(1) \quad a(x)\varphi(x) + \frac{1}{\pi i} \int_{-1}^1 \frac{b(t)\varphi(t)}{t-x} dt + \frac{1}{\pi i} \int_{-1}^1 k(x,t)\varphi(t) dt = f(x), \quad -1 < x < 1,$$

$$a^2(x) - b^2(x) \neq 0, \quad \forall x \in [-1, 1].$$

The equation (1) plays a pivotal role in the airfoil theory, aeroelasticity, and other problems of solid and fluid mechanics [3, 4, 16, 22, 27, 28, 37]. A comprehensive survey of applications of SIEs to mechanics can be found in [5]. The first studies on the approximate solutions of this equation appeared in the beginning of the third decade of the previous century and now the number of papers concerning numerical method of solving singular integral equations with Cauchy-type kernels is very large. For the details one can consult the papers [6, 7, 10, 11, 14, 17–21, 23, 38, 41, 43], monographs [2, 8, 15] and citations therein.

In the present paper, we consider the complete singular integral equation

$$(2) \quad a\varphi(x) + \frac{1}{\pi i} \int_0^{+\infty} \frac{b\varphi(\sigma)}{\sigma - x} d\sigma + \frac{1}{\pi i} \int_0^{+\infty} k(x, \sigma)\varphi(\sigma) d\sigma = f(x), \quad x > 0,$$

where $k(x, \sigma)$, $f(x)$ are given complex-valued Hölder continuous functions, the coefficients a and b are given complex numbers satisfying the conditions $a^2 - b^2 \neq 0$, $b \neq 0$, and $\varphi(x)$ is the unknown function. We assume that the behavior of the kernel $k(x, \sigma)$ as $|\sigma| \rightarrow \infty$ is described by the relation $k(x, \sigma) = k_0(x, \sigma)(\sigma+1)^{-\alpha}$, $\operatorname{Re} \alpha > 1$, where $k_0(x, \sigma)$ satisfies Hölder condition with respect to both variables.

The solution $\varphi(x)$, $x > 0$, will be sought in the class of Hölder functions, vanishing at infinity and having an integrable singularity in a neighbourhood of $x = 0$ ($h(\infty)$ class) or bounded at $x = 0$ ($h(0, \infty)$ class) (cf. [34, 35]).

The theory of singular integral equations defined on the infinite curve can be slightly different from the theory of corresponding equations on the finite curve, see, eg. the works [1, 9, 24, 29, 33, 35, 36]. Equations of this type occur i.a. in quantum mechanics [31].

The exact solution of dominant equation

$$(3) \quad a\varphi(x) + \frac{1}{\pi i} \int_0^{+\infty} \frac{b\varphi(\sigma)}{\sigma - x} d\sigma = f(x), \quad x > 0,$$

which is a special case of equation (2), was presented i.a. in [9, 34]. Moreover, in [34] Jacobi polynomials were applied to derive approximate solutions of (3).

The main objective of the present paper is to build the approximate solutions of (2) in the $h(\infty)$ and $h(0, \infty)$ function classes.

The paper is organized as follows. First we reduce the equation (2) to a Fredholm equation using Carleman-Vekua regularization and we find the conditions for the unique solvability of (2). Next, we present efficient numerical schemes for equation (2) based on Jacobi polynomials and estimate the order of accuracy of the approximate solution.

2. Exact solution

In this section, we reduce the singular integral equation (2) to a Fredholm equation by Carleman-Vekua regularization [12, 30, 40]. Setting

$$(4) \quad x = \frac{1+t}{1-t}, \quad \sigma = \frac{1+\tau}{1-\tau},$$

we can present the equation (2) in the form

$$(5) \quad a\varphi^*(t) + \frac{b}{\pi i} \int_{-1}^1 \frac{\varphi^*(\tau)}{\tau - t} d\tau - \frac{b}{\pi i} \int_{-1}^1 \frac{\varphi^*(\tau)}{\tau - 1} d\tau + \frac{1}{\pi i} \int_{-1}^1 k^*(t, \tau)\varphi^*(\tau) d\tau = f^*(t),$$

$$-1 < t < 1, \quad \lim_{t \rightarrow 1^-} \varphi^*(t) = \lim_{x \rightarrow \infty} \varphi(x) = 0,$$

where

$$\varphi^*(t) = \varphi\left(\frac{1+t}{1-t}\right), \quad f^*(t) = f\left(\frac{1+t}{1-t}\right), \quad k^*(t, \tau) = 2k\left(\frac{1+t}{1-t}, \frac{1+\tau}{1-\tau}\right)(1-\tau)^{-2}.$$

Passing in (5) to the new unknown function $u(t)$ by the rule $\varphi^*(t) = \frac{Z(t)}{a^2-b^2} u(t)$, and introducing notation $A = \frac{a}{a^2-b^2}$, $B = \frac{b}{a^2-b^2}$, we reduce Eq. (5) to the form

$$(6) \quad \begin{aligned} AZ(t)u(t) + \frac{1}{\pi i} \int_{-1}^1 \frac{BZ(\tau)u(\tau)}{\tau-t} d\tau - \frac{1}{\pi i} \int_{-1}^1 \frac{BZ(\tau)u(\tau)}{\tau-1} d\tau \\ + \frac{1}{\pi i} \int_{-1}^1 k^*(t, \tau) \frac{Z(\tau)}{a^2-b^2} u(\tau) d\tau = f^*(t), \quad t \in (-1, 1). \end{aligned}$$

Note that $k^*(t, \tau)$ as $\tau \rightarrow 1$ has the form

$$(7) \quad k^*(t, \tau) = 2^{1-\alpha} k\left(\frac{1+t}{1-t}, \frac{1+\tau}{1-\tau}\right)(1-\tau)^{\alpha-2}.$$

Function $Z(t)$ depends on the function class in which we are looking for its solution. If the solution $\varphi^*(t)$ belongs to $h(1)$ function class [30] (i.e. the class of Hölder continuous functions on $(-1, 1)$, bounded in a neighbourhood of the point $z = 1$, and admitting an integrable singularity at the point $z = -1$), then for $0 < \theta < \pi$ we have (cf. [34])

$$(8) \quad Z(t) = \sqrt{a^2-b^2} (1-t)^\alpha (1+t)^\beta, \quad \alpha = \omega_1 + i\omega_2, \quad \beta = -\omega_1 - i\omega_2,$$

and for $-\pi < \theta < 0$ we obtain

$$(9) \quad Z(t) = -\sqrt{a^2-b^2} (1-t)^\alpha (1+t)^\beta, \quad \alpha = 1 + \omega_1 + i\omega_2, \quad \beta = -1 - \omega_1 - i\omega_2,$$

where

$$G = \frac{a-b}{a+b}, \quad \theta = \arg G, \quad \omega_1 = \frac{\theta}{2\pi} \quad \text{and} \quad \omega_2 = -\frac{1}{2\pi} \ln |G|.$$

The index of the characteristic operator is equal to $\kappa = -(\alpha + \beta) = 0$.

If the solution $\varphi^*(t)$ is sought in the class $h(-1, 1)$ [30] (i.e. the class of Hölder continuous functions on $(-1, 1)$, bounded at neighbourhoods of points $z = \pm 1$), then for $0 < \theta < \pi$ we have (cf. [34])

$$(10) \quad Z(t) = \sqrt{a^2-b^2} (1-t)^\alpha (1+t)^\beta, \quad \alpha = \omega_1 + i\omega_2, \quad \beta = 1 - \omega_1 - i\omega_2,$$

and for $-\pi < \theta < 0$ we obtain

$$(11) \quad Z(t) = -\sqrt{a^2-b^2} (1-t)^\alpha (1+t)^\beta, \quad \alpha = 1 + \omega_1 + i\omega_2, \quad \beta = -\omega_1 - i\omega_2.$$

In this case, the index κ is equal to $\kappa = -(\alpha + \beta) = -1$.

Moving the regular term of Eq. (6) to the right-hand side and then solving it as a dominant equation in $h(1)$ class (cf. [34]), after a few elementary transformations

Eq. (6) can be reduced to the Fredholm equation of the form

$$(12) \quad u(t) + \int_{-1}^1 N(t, \tau_1) u(\tau_1) d\tau_1 = F(t),$$

where

$$N(t, \tau_1) = \frac{1}{\pi i} \frac{Z(\tau_1)}{Z(t)(a^2 - b^2)} \left[ak^*(t, \tau_1) - \frac{Z(t)}{\pi i} \int_{-1}^1 \frac{b}{Z(\tau)} \frac{k^*(\tau, \tau_1)}{\tau - t} d\tau \right],$$

$$F(t) = \frac{a}{Z(t)} f^*(t) - \frac{1}{\pi i} \int_{-1}^1 \frac{b}{Z(\tau)} \frac{f^*(\tau)}{\tau - t} d\tau + \gamma_0,$$

where γ_0 is an arbitrary complex constant. The theory of Fredholm equations is valid for Eq. (12) ([12]). Therefore, if the corresponding equation with $F(t) \equiv 0$ is unsolvable (has only the trivial solution), then the solution of the non homogeneous equation (12) is equal to

$$(13) \quad u(t) = F(t) - \int_{-1}^1 \Gamma(t, \tau) F(\tau) d\tau, \quad t \in (-1, 1),$$

where $\Gamma(t, \tau)$ is the resolvent of the kernel $N(t, \tau)$.

The constant γ_0 will be uniquely determined if we supplement the equation (6) by the following orthogonality condition

$$(14) \quad \frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u(\tau)}{\tau - 1} d\tau = A_0^*,$$

where A_0^* is a given complex number. Indeed, taking into account the Poincaré-Bertrand formula [9, 12] and the relation [30, p.140]

$$\Gamma(t, \tau) = N(t, \tau) + \int_{-1}^1 N(t, \tau_3) \Gamma(\tau_3, \tau) d\tau_3,$$

by a straightforward substitution we verify that $\gamma_0 = A_0^*$.

Remark 1. The same result can be retrieved by defining

$$\varphi^*(t) = \varphi\left(\frac{1+t}{1-t}\right) \frac{1}{1-t}, \quad f^*(t) = f\left(\frac{1+t}{1-t}\right) \frac{1}{1-t}$$

and considering the Muskhelishvili class h_0 (index $\kappa = 1$).

In $h(-1, 1)$ class ($\kappa < 0$), Eq. (6) is solvable if and only if

$$\frac{1}{\pi i} \int_{-1}^1 \frac{b}{Z(\tau)} \left[f^*(\tau) - \frac{1}{\pi i} \int_{-1}^1 k^*(\tau, \tau_1) \frac{Z(\tau_1)}{a^2 - b^2} u(\tau_1) d\tau_1 + A_0^* \right] d\tau = 0,$$

where A_0^* is given by (14). Since the above condition is equivalent to the relation

$$(15) \quad \begin{aligned} \frac{1}{\pi i} \int_{-1}^1 \frac{b}{Z(\tau)} f^*(\tau) d\tau &= \frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u(\tau)}{\tau-1} d\tau \\ &+ \frac{1}{(\pi i)^2} \int_{-1}^1 \int_{-1}^1 \frac{b}{Z(\tau)} \frac{Z(\tau_1)}{a^2-b^2} k^*(\tau, \tau_1) u(\tau_1) d\tau_1 d\tau, \end{aligned}$$

Eq.(6) can be rewritten in the form

$$(16) \quad \begin{aligned} AZ(t)u(t) &+ \frac{1}{\pi i} \int_{-1}^1 \frac{BZ(\tau)u(\tau)}{\tau-t} d\tau + \frac{1}{\pi i} \int_{-1}^1 k^*(t, \tau) \frac{Z(\tau)}{a^2-b^2} u(\tau) d\tau \\ &= f^*(t) + \frac{b}{\pi i} \int_{-1}^1 \frac{f^*(\tau)}{Z(\tau)} d\tau - \frac{b}{(\pi i)^2} \int_{-1}^1 \int_{-1}^1 \frac{Z(\tau_1)}{Z(\tau)} \frac{u(\tau_1)}{a^2-b^2} k^*(\tau, \tau_1) d\tau_1 d\tau. \end{aligned}$$

Then, in the same way as in the previous case, one can pass from Eq.(16), to Fredholm Eq.(12), in whose right-hand side one should set $\gamma_0 = 0$.

In the original variables x, σ , the conditions (14) and (15) acquire the following forms

$$(17) \quad \frac{1}{\pi i} \int_0^{+\infty} \frac{b\varphi(\sigma)}{\sigma+1} d\sigma = A_0^*,$$

where A_0^* is an arbitrary number, and

$$(18) \quad \begin{aligned} &\frac{\operatorname{sgn}(\theta)}{\sqrt{a^2-b^2}} \frac{1}{\pi i} \int_0^{+\infty} \frac{f(\sigma)}{\sigma^\beta} \frac{d\sigma}{\sigma+1} \\ &= -\frac{1}{\pi i} \int_0^{+\infty} \varphi(\sigma) \frac{d\sigma}{\sigma+1} + \frac{\operatorname{sgn}(\theta)}{\sqrt{a^2-b^2}} \frac{1}{(\pi i)^2} \int_0^{+\infty} \int_0^{+\infty} \frac{k(\sigma, \sigma_1)}{\sigma^\beta} \frac{\varphi(\sigma)}{\sigma+1} d\sigma_1 d\sigma, \end{aligned}$$

where α, β are specified in (10) or (11) depending on $\theta = \arg \frac{a-b}{a+b}$, respectively.

We have thereby proved the following assertion.

Theorem 2.1. *Suppose that the coefficients a, b occurring in Eq. (6) are given complex numbers such that $a^2 - b^2 \neq 0$, $b \neq 0$, the functions $f^*(t)$, $k^*(t, \tau)$ belong to the Hölder class ($k^*(t, \tau)$ with respect to both variables), and the kernel $k^*(t, \tau)$ can be represented in the form (7) in a neighbourhood of $\tau = 1$.*

If the index of (6) is equal to zero ($\kappa = 0$) and the homogeneous equation (12) is unsolvable, then the problem (6), (14) has a unique solution.

If $\kappa = -1$, then Eq. (6) is equivalent (in the sense of solvability) to the Fredholm equation (12) with $\gamma_0 \equiv 0$, supplemented with the condition (15).

3. Approximate solution

Now, we will find an approximate solution $u_n(t)$ of the problem (6), (14) in $h(1)$ class ($\kappa = 0$). For this purpose, we interpolate the function $f^*(t)$ at Chebyshev nodes

$$(19) \quad t_k = \cos \frac{(2k-1)\pi}{2(n+1)}, \quad k = 1, 2, \dots, n+1,$$

by polynomial $f_n^*(t)$ of degree n of the form (cf. [32])

$$(20) \quad f_n^*(t) = \frac{2}{n+1} \sum_{j=0}^n \delta_j \left(\sum_{k=1}^{n+1} T_j(t_k) f^*(t_k) \right) T_j(t), \quad \delta_j = \begin{cases} 1, & j=0, \\ 2, & j>0, \end{cases}$$

where $T_j(t) = \cos(j \arccos t)$ are Chebyshev polynomials of the first kind. By expressing Chebyshev polynomials $T_j(t)$ in terms of Jacobi polynomials [42], we obtain

$$(21) \quad T_j(t) = \sum_{l=0}^j \rho_{jl} P_l^{(-\alpha, -\beta)}(t),$$

where

$$(22) \quad \begin{aligned} \rho_{jl} &= \frac{1}{h_l^{(-\alpha, -\beta)}} \frac{1}{\pi} \int_{-1}^1 q(t) T_j(t) P_l^{(-\alpha, -\beta)}(t) dt \\ &= (-1) \frac{1}{h_l^{(-\alpha, -\beta)}} \frac{1}{\sin \pi \alpha} \operatorname{Res}_{z=\infty} \left\{ (z-1)^{-\alpha} (z+1)^{-\beta} T_j(z) P_l^{(-\alpha, -\beta)}(z) \right\}, \\ q(t) &= (1-t)^{-\alpha} (1+t)^{-\beta}, \quad \alpha + \beta = 0, \end{aligned}$$

$$(23) \quad h_l^{(-\alpha, -\beta)} = \frac{1}{\pi} \int_{-1}^1 q(t) \left[P_l^{(-\alpha, -\beta)}(t) \right]^2 dt = \frac{2\Gamma(l-\alpha+1)\Gamma(l-\beta+1)}{(2l+1)\pi l! \Gamma(l+1)}.$$

Using (21), the interpolation polynomial (20) takes the form

$$(24) \quad f_n^*(t) = \sum_{k=0}^n f_k^* P_k^{(-\alpha, -\beta)}(t),$$

where

$$f_k^* = \frac{2}{n+1} \sum_{j=k}^n \delta_j \left(\sum_{i=1}^{n+1} T_j(t_i) f^*(t_i) \right) \rho_{jk}, \quad k = 0, 2, \dots, n.$$

Next, we interpolate the regular kernel $k(t, \tau)$ at Chebyshev nodes (19) with the polynomial $k_{nn}^*(x, t)$ of the form

$$(25) \quad k_{nn}^*(t, \tau) = \sum_{j=0}^n \sum_{p=0}^n k_{jp}^* P_j^{(-\alpha, -\beta)}(t) P_p^{(\alpha, \beta)}(\tau),$$

where

$$k_{jp}^* = \sum_{l=j}^n \sum_{m=p}^n \alpha_{lm} \rho_{lj}^{(-\alpha, -\beta)} \rho_{mp}^{(\alpha, \beta)},$$

coefficients $\rho_{lj}^{(-\alpha, -\beta)}$, $\rho_{mp}^{(\alpha, \beta)}$ are given by (22), and α_{lm} are defined as follows

$$\alpha_{lm} = \frac{\delta_l}{n+1} \sum_{k=1}^{n+1} T_l(t_k) \left\{ \frac{\delta_m}{n+1} \sum_{r=1}^{n+1} T_m(\tau_r) k^*(t_k, \tau_r) \right\}.$$

As in this case, by (6) we have

$$(26) \quad \frac{1}{\pi i} \int_{-1}^1 k^*(1, \tau) \frac{Z(\tau)}{a^2 - b^2} u(\tau) d\tau = f^*(1).$$

Therefore the approximate solution

$$(27) \quad u_n(t) = \sum_{k=0}^n c_k P_k^{(\alpha, \beta)}(t),$$

of the problem (6), (14) can be defined as the solution of the following equation

$$(28) \quad \begin{aligned} & AZ(t)u_n(t) + \frac{1}{\pi i} \int_{-1}^1 \frac{BZ(\tau)u_n(\tau)d\tau}{\tau - t} \\ & + \frac{1}{\pi i} \int_{-1}^1 (k_{nn}^*(t, \tau) - k_{nn}^*(1, \tau)) \frac{Z(\tau)u_n(\tau)d\tau}{a^2 - b^2} = f_n^*(t) - f_n^*(1) + A_0^*, \end{aligned}$$

where A_0^* is a given complex number, coefficients α, β are defined in (8) for $0 < \theta < \pi$ and in (9) for $-\pi < \theta < 0$.

Substituting (27), (25), and (24) into (28) and using formula [23, 34]

$$AZ(x)P_k^{(\alpha, \beta)}(x) + \frac{B}{\pi i} \int_{-1}^1 Z(t) \frac{P_k^{(\alpha, \beta)}(t)}{t - x} dt = P_k^{(-\alpha, -\beta)}(x), \quad \alpha + \beta = 0, \quad k \geq 0,$$

for the computation of the singular integrals, we obtain

$$\begin{aligned} & \sum_{k=0}^n c_k P_k^{(-\alpha, -\beta)}(t) + \sum_{l=0}^n \left(\sum_{m=0}^n k_{lm}^* h_m \right) \left(P_l^{(-\alpha, -\beta)}(t) - P_l^{(-\alpha, -\beta)}(1) \right) \\ & = \sum_{k=0}^n f_k^* P_k^{(-\alpha, -\beta)}(t) - f_n^*(1) + A_0^*, \end{aligned}$$

where

$$(29) \quad \begin{aligned} h_m &= \frac{1}{\pi i} \int_{-1}^1 \frac{Z(\tau)}{a^2 - b^2} \left(P_m^{(\alpha, \beta)}(\tau) \right)^2 d\tau = \frac{\operatorname{sgn}(\arg(\theta))}{i\sqrt{a^2 - b^2}} h_l^{(\alpha, \beta)} \\ &= \frac{\operatorname{sgn}(\arg(\theta))}{i\sqrt{a^2 - b^2} \sin \alpha \pi} \operatorname{Res}_{z=\infty} \left((z-1)^\alpha (z+1)^\beta \left(P_m^{(\alpha, \beta)}(z) \right)^2 \right). \end{aligned}$$

Thus, coefficients c_k , $k = 0, 1, \dots, n$, of the solution $u_n(t)$ can be determined from the following system of linear equations

$$(30) \quad \begin{aligned} c_i + \sum_{j=0}^n c_j k_{ij}^* h_j &= f_i^*, \quad i = 1, 2, \dots, n, \\ c_0 + \sum_{j=0}^n c_j k_{0j}^* h_j - \sum_{l=0}^n \sum_{j=0}^n c_j k_{lj}^* h_j P_l^{(-\alpha, -\beta)}(1) &= f_0^* + A_0 - f_n^*(1). \end{aligned}$$

If $\kappa < 0$, then the approximate solution

$$(31) \quad u_{n-1}(t) = \sum_{k=0}^{n-1} c_k P_k^{(\alpha, \beta)}(t),$$

of (16) in the class $h(-1, 1)$ may be found as a solution of the following problem

$$(32) \quad \begin{aligned} AZ(t)u_{n-1}(t) + \frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u_{n-1}(\tau)}{\tau - t} d\tau + \frac{1}{\pi i} \int_{-1}^1 \frac{k_{n,n-1}^*(t, \tau)}{a^2 - b^2} Z(\tau) u_{n-1}(\tau) d\tau \\ = f_n^*(t) + \frac{b}{\pi i} \int_{-1}^1 \frac{f_n^*(\tau)}{Z(\tau)} d\tau - \frac{b}{(\pi i)^2} \int_{-1}^1 \int_{-1}^1 \frac{Z(\tau_1) k_{n,n-1}^*(\tau, \tau_1)}{a^2 - b^2} u_{n-1}(\tau_1) d\tau_1 d\tau, \end{aligned}$$

where $f_n^*(t)$ is defined by the rule (24). Moreover, $k_{n,n-1}^*(t, \tau)$ is the interpolation polynomial of the kernel $k^*(t, \tau)$ with Chebyshev nodes

$$(33) \quad \tau_l' = \cos \frac{(2l-1)\pi}{2n}, \quad l = 1, \dots, n, \quad t_k = \cos \frac{(2k-1)\pi}{2(n+1)}, \quad k = 1, \dots, n+1,$$

and it has the form

$$(34) \quad k_{n,n-1}^*(t, \tau) = \sum_{j=0}^n \sum_{p=0}^{n-1} k_{jp}^* P_j^{(-\alpha, -\beta)}(t) P_p^{(\alpha, \beta)}(\tau),$$

where

$$\begin{aligned} k_{jp}^* &= \sum_{l=j}^n \sum_{m=p}^{n-1} \alpha_{lm} \rho_{lj}^{(-\alpha, -\beta)} \rho_{mp}^{(\alpha, \beta)}, \\ \alpha_{lm} &= \frac{\delta_l}{n+1} \sum_{k=1}^{n+1} T_l(t_k) \left\{ \frac{\delta_m}{n} \sum_{r=1}^n T_m(\tau_r') k^*(t_k, \tau_r') \right\}, \quad \delta_k = \begin{cases} 1, & k = 0, \\ 2, & k = 1, 2, \dots, \end{cases} \end{aligned}$$

and $\rho_{lj}^{(-\alpha, -\beta)}, \rho_{mp}^{(\alpha, \beta)}$ are defined in the same way as in (22). Let us stress that in this case taking into account the relation (26) doesn't have an influence on final algorithm.

Substituting (31), (34) and (24) into (32), and using formula [23, 34]

$$AZ(x)P_k^{(\alpha, \beta)}(x) + \frac{B}{\pi i} \int_{-1}^1 Z(t) \frac{P_k^{(\alpha, \beta)}(t)}{t - x} dt = 2P_{k+1}^{(-\alpha, -\beta)}(x), \quad \alpha + \beta = 1, \quad k \geq 0,$$

we obtain

$$2 \sum_{k=0}^{n-1} c_k P_{k+1}^{(-\alpha, -\beta)}(t) + \sum_{l=1}^n \left(\sum_{m=0}^{n-1} k_{lm}^* c_m h_m \right) P_l^{(-\alpha, -\beta)}(t) = \sum_{k=1}^n f_k^* P_k^{(-\alpha, -\beta)}(t),$$

where h_m are defined in the same way as in (29). The coefficients α , β and the function $Z(t)$ occurring in h_m are given by (10) for $0 < \theta = \arg \frac{a-b}{a+b} < \pi$ and by (11) for $-\pi < \theta < 0$. Comparing the coefficients at $P_k^{(-\alpha, -\beta)}(t)$, $k = 1, \dots, n$, we derive the system

$$(35) \quad 2c_{i-1} + \sum_{j=0}^{n-1} c_j k_{ij}^* h_j = f_i^*, \quad i = 1, \dots, n.$$

4. Numerical examples

Let us find the solution the following problem

$$(36) \quad (1+i)\varphi(x) + \frac{1-i}{\pi i} \int_0^{+\infty} \frac{\varphi(\sigma) d\sigma}{\sigma-x} - \frac{2}{\pi i} \int_0^{+\infty} \frac{\varphi(\sigma) d\sigma}{(x+1)(\sigma+1)(\sigma+2)} = \frac{1}{2x+3},$$

$$\frac{1}{\pi i} \int_0^{+\infty} \frac{(1-i)\varphi(\sigma) d\sigma}{\sigma+1} = -1,$$

in the $h(\infty)$ class ($\kappa = 0$). Using the above substitutions (4) the problem (36) acquires the form

$$\frac{1-i}{4} Z(t)u(t) - \frac{1+i}{4\pi i} \int_{-1}^1 \frac{Z(\tau)u(\tau) d\tau}{\tau-t} = \frac{t-1}{t-5} - \frac{1}{\pi i} \int_{-1}^1 \frac{Z(\tau)}{a^2-b^2} \frac{1-t}{\tau-3} u(\tau) d\tau - 1,$$

where

$$Z(t) = \sqrt{2}(1+i)(1-t)^{1/4}(1+t)^{-1/4}.$$

By solving the above equation in the $h(1)$ class (cf. [34]), we obtain

$$\Gamma(t, \tau) = \frac{1}{\pi i} \frac{1-2t}{2 \left(\frac{5(1+i)}{4\sqrt[4]{2}} - i \right)} \frac{Z(\tau)}{4i} \frac{1}{\tau-3}, \quad Z(t) = (\sqrt{2} + i\sqrt{2}) \sqrt[4]{\frac{1-t}{1+t}}.$$

Since

$$u(t) = \frac{2t-1}{2} \frac{\sqrt[4]{3}-\sqrt{2}}{\sqrt{2}} \frac{(1+i)\sqrt[4]{2}}{5+i(1-4\sqrt[4]{2})} + 4 \left(\frac{3}{2} \right)^{\frac{1}{4}} \frac{1}{t-5},$$

or, which is the same,

$$\varphi(x) = -i \frac{e^{\frac{i\pi}{4}}}{4\sqrt[4]{x}} \frac{x-3}{x+1} \frac{\sqrt[4]{3}-\sqrt{2}}{\sqrt{2}} \frac{(1+i)\sqrt[4]{2}}{5+i(1-4\sqrt[4]{2})} + i \frac{e^{\frac{i\pi}{4}}}{\sqrt[4]{x}} \left(\frac{3}{2} \right)^{\frac{1}{4}} \frac{x+1}{2x+3}.$$

The values of $u(t)$, $u_n(t)$ for $n = 10$ are shown in Table 1. In Table 2 we tabulate the values of the exact and approximated solutions of the problem (36) in the $h(\infty)$ class at the points x corresponding to t from the previous table.

Tab. 1: Comparison of the values of $u(t)$, $u_n(t)$

t	$u(t)$	$u_n(t)$
-0.717	-0.690070196863+0.076314953186i	-0.690070196857+0.076314953184i
0.0016	-0.851205627507+0.031230292033i	-0.851205627521+0.031230292033i
-0.956	-0.642559577943+0.091272157657i	-0.642559577944+0.091272157655i
0.185	-0.897813651028+0.019688117490i	-0.897813651035+0.019688117489i

Tab. 2: Comparison of the values of exact and approximated solutions of (36)

x	$\varphi(x)$	$\varphi_n(x)$
41.37	-0.160791967205+0.152881347804i	-0.160791967212+0.152881347811i
7.916	-0.228530579779+0.221248009243i	-0.228530579772+0.221248009237i
4.787	-0.248345980190+0.243720960213i	-0.248345980195+0.243720960218i
0.020	-0.513057865584+0.683492017265i	-0.513057865587+0.683492017263i

5. Estimation of errors

In this section, we determine the estimates of errors of the approximate solutions of (5). In order to investigate proposed algorithms, we need the following theorem presented in [40].

Theorem 5.1. *Let us consider two Fredholm equations, namely the exact*

$$(37) \quad K(\varphi; x) \equiv \varphi(x) + \int_{-1}^1 k(x, t)\varphi(t)dt = f(x), \quad -1 \leq x \leq 1$$

and the approximate

$$(38) \quad K_n(\varphi_n; x) \equiv \varphi_n(x) + \int_{-1}^1 k_n(x, t)\varphi_n(t)dt = f_n(x), \quad -1 \leq x \leq 1,$$

where $k(x, t)$, $k_n(x, t)$, $f(x)$ and $f_n(x)$ are given continuous functions (the first two can have an integrable singularity at ± 1).

In addition, suppose that the homogeneous equation (37) is not solvable (i.e. it has only the zero solution) and $\max_{-1 \leq x \leq 1} \int_{-1}^1 |\gamma(x, t)| dt \leq \rho$, where $\gamma(x, t)$ is the resolvent of the kernel $k(x, t)$.

If the following condition $\varepsilon_1 B < 1$ is satisfied, where

$$\varepsilon_1 = \max_x \int_{-1}^1 \int_{-1}^1 |k(x, t) - k_n(x, t)| |k_n(t, \tau)| d\tau dt, \quad B = 1 + \rho,$$

then the non homogeneous equation (38) is solvable and

$$\|K_n^{-1}\|_\infty \leq \frac{1+BK_1}{1-\varepsilon_1 B}, \quad \text{where} \quad K_1 = \max_x \int_{-1}^1 |k_{n,n}(x,t)| dt.$$

Besides,

$$\|\varphi - \varphi_n\|_\infty \leq \frac{1+BK_1}{1-\varepsilon_1 B} [\varepsilon_2 B \|f\|_\infty + \varepsilon_3],$$

where

$$\varepsilon_2 = \max_x \int_{-1}^1 |k(x,t) - k_n(x,t)| dt, \quad \varepsilon_3 = \|f - f_n\|_\infty.$$

Theorem 5.2. Let us suppose that the Hölder continuous functions $f^*(t)$, $k^*(t, \tau)$ occurring in Eq. (6) are approximated by the polynomials $f_n^*(t)$ and $k_{n,n}^*(t, \tau)$ defined as in (24) and (25), respectively, and that the homogeneous equation corresponding to (12) is not solvable.

Then the system of linear equations (30) is solvable for sufficiently large n , and the estimate

$$(39) \quad \|(1-t)^\alpha(u(t) - u_n(t))\|_\infty \leq M \frac{\ln^3 n}{n^\mu}, \quad \operatorname{Re} \alpha > 0,$$

holds, where M is a constant independent of n .

Proof. The system (30) and the problem (28) are simultaneously solvable or unsolvable. As follows from the above-performed considerations, the latter is equivalent to the solvability of the equation

$$(40) \quad (1-t)^\alpha u_n(t) + \int_{-1}^1 N_n(t, \tau) (1-\tau)^\alpha u_n(\tau) d\tau = F_n(t), \quad \operatorname{Re} \alpha > 0,$$

where

$$N_n(t, \tau) = \frac{(1-t)^\alpha}{(1-\tau)^\alpha} \frac{1}{\pi i} \frac{Z(\tau)}{(a^2 - b^2)} \left[\frac{ak_{nn}^*(t, \tau)}{Z(t)} - \frac{b}{\pi i} \int_{-1}^1 \frac{k_{nn}^*(\tau_1, \tau) d\tau_1}{Z(\tau_1)(\tau_1 - t)} \right],$$

$$F_n(t) = (1-t)^\alpha \left\{ \frac{a}{Z(t)} f_n^*(t) - \frac{1}{\pi i} \int_{-1}^1 \frac{b}{Z(\tau)} \frac{f_n^*(\tau)}{\tau - t} d\tau + \gamma_0 \right\}.$$

By Theorem 5.1, to establish the solvability of Eq.(40), it suffices to estimate

$$\varepsilon_1 = \max_{-1 < t < 1} \int_{-1}^1 \int_{-1}^1 |N(t, \tau) - N_n(t, \tau)| |N_n(\tau, \tau_1)| d\tau d\tau_1.$$

Since in the $h(1)$ class we have

$$Z(t) = \pm \sqrt{a^2 - b^2} (1-t)^\alpha (1+t)^\beta, \quad 0 < \operatorname{Re} \alpha < 1, \quad -1 < \operatorname{Re} \beta < 0,$$

one can apply the estimates from [39] in order to get

$$\|k(t, \tau) - k_{nn}(t, \tau)\|_\infty \leq M_1 \frac{\ln^2 n}{n^\mu},$$

$$\left| \frac{1}{\pi i} \int_{-1}^1 \frac{b}{Z(\tau_1)} [k(\tau_1, \tau) - k_{nn}(\tau_1, \tau)] \frac{d\tau_1}{\tau_1 - t} \right| \leq \frac{M_2}{(1-t)^{\operatorname{Re} \alpha}} \frac{\ln^3 n}{n^\mu}.$$

Consequently, we have

$$|N(t, \tau) - N_n(t, \tau)| = \left| \frac{(1-t)^\alpha Z(\tau)}{\pi i (a^2 - b^2) Z(t)} \left\{ a [k(t, \tau) - k_{nn}(t, \tau)] \right. \right.$$

$$\left. \left. - \frac{Z(t)}{\pi i} \int_{-1}^1 \frac{b}{Z(\tau_1)} [k(\tau_1, \tau) - k_{nn}(\tau_1, \tau)] \frac{d\tau_1}{\tau_1 - t} \right\} \right| \leq M (1+\tau)^{\operatorname{Re} \beta} \frac{\ln^3 n}{n^\mu},$$

where M, M_1, M_2 are constants independent of n .

Hence it follows that $\varepsilon_1 \leq M \frac{\ln^3 n}{n^\mu}$. Thus, the Eq. (40) is solvable for sufficiently large n .

Using the following estimations

$$K_1 = \max_t \int_{-1}^1 |N_n(t, \tau)| d\tau = \max_t \int_{-1}^1 |[N_n(t, \tau) - N(t, \tau)] + N(t, \tau)| d\tau = O(1),$$

$$\varepsilon_2 = \max_t \int_{-1}^1 |N(t, \tau) - N_n(t, \tau)| d\tau \leq M \frac{\ln^3 n}{n^\mu},$$

$$\varepsilon_3 = \|F(t) - F_n(t)\|_\infty \leq M \frac{\ln^2 n}{n^\mu},$$

we finish the proof of (39). □

Theorem 5.3. *Let the Hölder continuous functions $f^*(t), k^*(t, \tau)$ be approximated by polynomials $f_n^*(t)$ and $k_{n,n-1}^*(t, \tau)$ defined as in (24) and (34), respectively, and let the homogeneous equation (12) with $\gamma_0 = 0$ be not solvable.*

Then the system of linear equations (35) is solvable for sufficiently large n and the estimate

$$(41) \quad \|Z(t)(u(t) - u_{n-1}(t))\|_\infty \leq M \frac{\ln^3 n}{n^\mu},$$

holds, where M is a constant independent of n .

Proof. We have to estimate the value

$$\varepsilon_1^* = \max_{-1 < t < 1} \int_{-1}^1 \int_{-1}^1 |N^*(t, \tau) - N_{n-1}^*(t, \tau)| |N_{n-1}^*(\tau, \tau_1)| d\tau d\tau_1,$$

where

$$N_{n-1}^*(t, \tau) = \frac{1}{\pi i} \frac{1}{(a^2 - b^2)} \left[a k_{n,n-1}^*(t, \tau) - Z(t) \frac{b}{\pi i} \int_{-1}^1 \frac{k_{n,n-1}^*(\tau_1, \tau) d\tau_1}{Z(\tau_1)(\tau_1 - t)} \right].$$

Since $Z(t) = \pm \sqrt{a^2 - b^2} (1-t)^\alpha (1+t)^\beta$, $0 < \operatorname{Re} \alpha, \operatorname{Re} \beta < 1$, in the $h(-1, 1)$ class, it follows from [39] that

$$|N^*(t, \tau) - N_{n-1}^*(t, \tau)| \leq M_2 \frac{\ln^3 n}{n^\mu}.$$

Thus, for sufficiently large n , the equation

$$Z(t)u_{n-1}(t) + \int_{-1}^1 N_{n-1}^*(t, \tau) Z(\tau) u_{n-1}(\tau) d\tau = F_n^*(t), \quad -1 < t < 1,$$

where $F_n^*(t) = a f_n^*(t) - \frac{Z(t)}{\pi i} \int_{-1}^1 \frac{b}{Z(\tau)} \frac{f_n^*(\tau)}{\tau - t} d\tau$, is solvable.

Further, using the estimates

$$\begin{aligned} K_1 &= \max_t \int_{-1}^1 |N_{n-1}^*(t, \tau)| d\tau = O(1), \\ \varepsilon_2 &= \max_t \int_{-1}^1 |N^*(t, \tau) - N_{n-1}^*(t, \tau)| d\tau \leq M \frac{\ln^3 n}{n^\mu}, \\ \varepsilon_3 &= \|F^*(t) - F_n^*(t)\|_\infty \leq M \frac{\ln^2 n}{n^\mu}, \end{aligned}$$

we finish the proof of estimate (41). $\square$

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Presented by Julian Ławrynowicz at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 3, 2016.

ZASTOSOWANIE WIELOMIANÓW JACOBIEGO DO ROZWIĄZANIA ZUPEŁNEGO MOCNO OSOBLIWEGO RÓWNIANIA CAŁKOWEGO Z JĄDREM CAUCHY'EGO NA NIEUJEMNEJ PÓŁOSI RZECZYWISTEJ

S t r e s z c z e n i e

W pracy skonstruowane zostały algorytmy przybliżonego rozwiązania zupełnego mocno osobliwego równania całkowego z jądrem Cauchy'ego na nieujemnej półosi rzeczywistej o stałych współczynnikach zespolonych z użyciem wielomianów Jacobiego. Ponadto wyznaczone zostały oszacowania błędów przedstawionych rozwiązań przybliżonych.

Słowa kluczowe: wielomiany Jacobiego, mocno osobliwe równania całkowe, jądro Cauchy'ego, rozwiązanie przybliżone

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Printed directly from camera-ready materials provided to the Łódź University Press

Published by Łódź University Press

First edition. W.08325.17.0.C

Printing sheets 9.5

Łódź University Press

90-131 Łódź, 8 Lindleya St.

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