Investigating at Least One Solution to a Systems of Multi-Points Boundary Value Problems

Kamal Shah¹ and Samia Bushnaq²

¹Department of Mathematics, University of Malakand, Dir(L), Khyber Pakhtunkhwa, Pakistan

²Department of Science, Princess Sumaya University for Technology, Amman 11941, Jordan

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ABSTRACT

Current article is concerned with the investigating a coupled systems of boundary value problems(BVPs) of fractional order differential equations(FDEs). By means of classical fixed point theorems, we prove the existence and uniqueness of positive solution. We include a numerical problem to verify the investigated results.

1. Introduction

In recent years, the theory on existence and uniqueness of solutions to (FDEs) has got considerable attentions. In this regard, plenty of research papers and books are available in literature dealing various aspects of (FDEs) (Ahmad and Sivasundaran, 2010; Rehman and Khan, 2010; Zhang, 2010). Existence and uniqueness of solutions for multi-point initial / (BVPs) have been studied by many researchers (Yang and Ge, 2009; El-Sayed and Bin-Taher, 2013; Cui et al., 2012; Salem, 2009; Han and Wang, 2011; El-Shahed and Nieto, 2010). On the other hand, coupled systems of (BVPs) for non-linear (FDEs) are also studied by many researchers and large numbers of papers can be found dealing with existence and uniqueness of solutions (Gafiychuk et al., 2009; Haq et al., 2016; Shah and Khan, 2015a; Shah et al., 2015b; Shah et al., 2016; Shoab et al., 2016). The area devoted to the study of coupled systems of both classical and arbitrary order differential equations is an active area of research. Because the concerned systems modeled variety of scientific problems of dynamical systems, psychological and biological phenomenon and chemical process (Benchohra et al., 2008; Perov, 1964; Han and Wang, 2011; Shah et al., 2016). Authors some appropriate conditions for existence of solution to the given coupled systems of four point (BVPs)

$$\begin{cases} {}^c\mathbf{D}^\theta u(t) = \mathbf{H}_1(t, v(t), v(t)), t \in \mathbf{J}, \\ {}^c\mathbf{D}^\theta u(t) = \mathbf{H}_2(t, u(t), u(t)), t \in \mathbf{J}, \\ u(0) = \phi(u), u(1) = \delta_1 u(\alpha), \\ v(0) = \psi(v), v(1) = \delta_2 v(\beta), \end{cases}$$

where $1 < \theta, \theta \leq 2$, $\mathbf{J} = [0, 1]$ and $\mathbf{H}_1, \mathbf{H}_2 : \mathbf{J} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ are continuous and ${}^c\mathbf{D}$ is the standard Caputo derivative (Shah et al., 2016). Some researcher

has obtained enough conditions related to the existence theory of positive solutions to the given coupled systems of nonlinear three-point (BVPs)

$$\begin{cases} {}^c\mathbf{D}^\theta u(t) = \mathbf{H}_1(t, v(t)), t \in \mathbf{J}, \\ {}^c\mathbf{D}^\theta v(t) = \mathbf{H}_2(t, u(t)), t \in \mathbf{J} \\ u(0) = 0, v(0) = 0, u(1) = \delta_1 u(\alpha), v(1) = \delta_2 v(\alpha), \end{cases}$$

where $1 < \theta, \theta \leq 2$, $0 \leq \delta_1, \delta_2 \leq 1$ and $0 < \alpha < 1$ and $\mathbf{H}_1, \mathbf{H}_2 : \mathbf{J} \times \mathbf{R} \cup \{0\} \rightarrow \mathbf{R} \cup \{0\}$ are continuous (Ahmad and Nieto, 2009).

Motivated by the above studies, we develop some new existence and uniqueness results for the following coupled systems of nonlinear (BVPs) of (FDEs)

$$\begin{cases} {}^c\mathbf{D}^\theta u(t) = \mathbf{H}_1(t, v(t), {}^c\mathbf{D}^\alpha v(t)), 0 < t < 1, \\ {}^c\mathbf{D}^\theta v(t) = \mathbf{H}_2(t, u(t), {}^c\mathbf{D}^\alpha u(t)), 0 < t < 1, \\ u(0) = A, u(1) = \sum_{i=1}^m \lambda_i u(\alpha_i), \\ v(0) = B, v(1) = \sum_{i=1}^m \mu_i v(\beta_i), \end{cases} \quad (1.1)$$

where $1 < \theta, \vartheta \leq 2, 0 < \mu, \nu < 1$ and $\mathbf{H}_1, \mathbf{H}_2 : \mathbf{J} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ are continuous and ${}^c \mathbf{D}$ denotes Caputo fractional derivative. where $\lambda_i, \mu_i \in \mathbf{R}^+$ with $\sum_{i=1}^m \lambda_i \alpha_i < 1$ and $\sum_{i=1}^m \mu_i \beta_i < 1$. We apply Banach fixed point theorem and Leray-Schauder fixed point theorem to obtain proper conditions for existence and uniqueness results (Ali et al., 2016). We also provide a numerical problem to demonstrate the establish results.

2. Basic material

We recall some fundamental results and definitions (Benchohra et al., 2008; Ahmad and Sivasundaram., 2010; Zhang, 2010; Shah et al., 2016).

Definition 2.1. The fractional integral of order $\theta \in \mathbf{R}^+$ of a function $\omega : \mathbf{R}^+ \rightarrow \mathbf{R}$ is defined by

$$\mathbf{I}^\theta \omega(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-\zeta)^{\theta-1} \omega(\zeta) d\zeta$$

provided the integral converges.

Definition 2.2. The Caputo fractional order derivative of a function $\omega : \mathbf{R}^+ \rightarrow \mathbf{R}$ is defined by

$${}^c \mathbf{D}^\theta \omega(t) = \frac{1}{\Gamma(n-\theta)} \int_0^t (t-\zeta)^{n-\theta-1} \omega^{(n)}(\zeta) d\zeta,$$

where $n = [\theta] + 1$ and $[\theta]$ represents the integer part of θ , such that

the right side is point wise defined on $\mathbf{R}^+$

Theorem 2.3. For (FDEs)

$${}^c \mathbf{D}^\theta \omega(t) = \varphi(t), \quad \theta \in (n-1, n],$$

the given result holds

$$\omega(t) = \mathbf{I}^\theta \varphi(t) + l_1 + l_2 t + l_3 t^2 + \dots + l_n t^{n-1},$$

for arbitrary $l_i \in \mathbf{R}$, $i = 1, 2, \dots, n$.

Lemma 2.4. (Benchohra et al., 2008) Let $\mathbf{B}$ be a Banach space with closed and convex. Let $\mathbf{U}$ be a relatively open subset of $\mathbf{B}$ with $\mathbf{U}$ be a continuous and compact (completely continuous) mapping. Then either

- 1) The mapping $\mathbf{S}$ has a fixed point in $\overline{\mathbf{B}}$ or
- 2) There exist $u \in \partial \mathbf{B}$ and $\kappa \in (0, 1)$ with $\mathbf{B} = \kappa \mathbf{S} u$.

Theorem 2.5. Let $\omega, \sigma \in C[0, 1]$ and $D = 1 - \sum_{i=1}^m \lambda_i \alpha_i$ and

$E = 1 - \sum_{i=1}^m \mu_i \beta_i$, for a given $u, v \in C(I, \mathbf{R})$, then the solution of

$$\begin{cases} {}^c \mathbf{D}^\theta u(t) = \omega, & 1 < \theta \leq 2, & 0 < \mu < 1, & t \in \mathbf{J}, \\ {}^c \mathbf{D}^\vartheta v(t) = \sigma, & 1 < \vartheta \leq 2, & 0 < \nu < 1, & t \in \mathbf{J}, \\ u(0) = A, & u(1) = \sum_{i=1}^m \lambda_i u(\alpha_i), \\ v(0) = B, & v(1) = \sum_{i=1}^m \mu_i v(\beta_i), \end{cases}$$

(2.1)

is equivalent to the following coupled system of Fredholm integral

equations

$$u(t) = a(t) + \int_0^1 \mathbf{W}_1(t, \zeta) \omega(\zeta) d\zeta, \quad (2.2)$$

$$v(t) = b(t) + \int_0^1 \mathbf{W}_2(t, \zeta) \sigma(\zeta) d\zeta,$$

where $a(t) = A - \frac{At(1 - \sum_{i=1}^m \lambda_i)}{D}$, $b(t) = B - \frac{Bt(1 - \sum_{i=1}^m \mu_i)}{E}$. Where

$\mathbf{W}_1(t, \zeta), \mathbf{W}_2(t, \zeta)$ are the Green's functions given by

$$\mathbf{W}_1(t, \zeta) = \frac{1}{\Gamma(\theta)} \begin{cases} (t-\zeta)^{\theta-1} - \frac{1}{D} \left[t(1-\zeta)^{\theta-1} + \sum_{i=1}^m \lambda_i (\alpha_i - \zeta)^{\theta-1} \right], & \zeta \leq t, \alpha_i < \zeta \leq \alpha_{i+1}, \\ -\frac{1}{D} \left[t(1-\zeta)^{\theta-1} + \sum_{i=1}^m \lambda_i (\alpha_i - \zeta)^{\theta-1} \right], & t \leq \zeta, \alpha_i < \zeta \leq \alpha_{i+1}, \end{cases} \quad (2.3)$$

$$\mathbf{W}_2(t, \zeta) = \frac{1}{\Gamma(\vartheta)} \begin{cases} (t-\zeta)^{\vartheta-1} - \frac{1}{E} \left[t(1-\zeta)^{\vartheta-1} + \sum_{i=1}^m \mu_i (\beta_i - \zeta)^{\vartheta-1} \right], & \zeta \leq t, \beta_i < \zeta \leq \beta_{i+1}, \\ -\frac{1}{E} \left[t(1-\zeta)^{\vartheta-1} + \sum_{i=1}^m \mu_i (\beta_i - \zeta)^{\vartheta-1} \right], & t \leq \zeta, \beta_i < \zeta \leq \beta_{i+1}, \end{cases} \quad (2.4)$$

Proof. By means of Definition 2.2 and Theorem 2.3, the solution of ${}^c \mathbf{D}^\theta u(t) = \omega(t)$ is given by $u(t) = l_1 + l_2 t + \mathbf{I}^\theta \omega(t)$. (2.5)

Now applying the boundary conditions

$u(0) = A$, $u(1) = \sum_{i=1}^m \lambda_i u(\alpha_i)$ to (2.5) we get after simplification

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\theta)} \int_0^t (t-\zeta)^{\theta-1} \omega(\zeta) d\zeta - \frac{t}{D\Gamma(\theta)} \int_0^1 (1-\zeta)^{\theta-1} \omega(\zeta) d\zeta \\ &+ A - \frac{At(1 - \sum_{i=1}^m \lambda_i)}{D} - \frac{t}{D\Gamma(\theta)} \sum_{i=1}^m \lambda_i \int_0^{\alpha_i} (\alpha_i - \zeta)^{\theta-1} \omega(\zeta) d\zeta \\ &= a(t) + \int_0^1 \mathbf{W}_1(t, \zeta) \omega(\zeta) d\zeta. \end{aligned} \quad (2.6)$$

Similarly, we can get second equation of (2.2) by solving second equation of system of BVP (1.1) as

$$\begin{aligned} v(t) &= \frac{1}{\Gamma(\vartheta)} \int_0^t (t-\zeta)^{\vartheta-1} \sigma(\zeta) d\zeta - \frac{t}{E\Gamma(\vartheta)} \int_0^1 (1-\zeta)^{\vartheta-1} \sigma(\zeta) d\zeta \\ &+ B - \frac{Bt(1 - \sum_{i=1}^m \mu_i)}{E} - \frac{t}{E\Gamma(\vartheta)} \sum_{i=1}^m \mu_i \int_0^{\beta_i} (\beta_i - \zeta)^{\vartheta-1} \sigma(\zeta) d\zeta \\ &= b(t) + \int_0^1 \mathbf{W}_2(t, \zeta) \sigma(\zeta) d\zeta. \end{aligned} \quad (2.7)$$

Therefore, in view of Theorem 2.5, the considered coupled system (1.1) is equivalent to the following coupled system of Fredholm integral equations

$$\begin{cases} v(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t-\zeta)^{\vartheta-1} \sigma(\zeta) d\zeta - \frac{t}{E\Gamma(\vartheta)} \int_0^1 (1-\zeta)^{\vartheta-1} \sigma(\zeta) d\zeta \\ + B - \frac{Bt(1 - \sum_{i=1}^m \mu_i)}{E} - \frac{t}{E\Gamma(\vartheta)} \sum_{i=1}^m \mu_i \int_0^{\beta_i} (\beta_i - \zeta)^{\vartheta-1} \sigma(\zeta) d\zeta \\ = b(t) + \int_0^1 \mathbf{W}_2(t, \zeta) \mathbf{H}_2(\zeta, u(\zeta), {}^c \mathbf{D}^\nu u(\zeta)) d\zeta, \\ u(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-\zeta)^{\theta-1} \omega(\zeta) d\zeta - \frac{t}{D\Gamma(\theta)} \int_0^1 (1-\zeta)^{\theta-1} \omega(\zeta) d\zeta \\ + A - \frac{At(1 - \sum_{i=1}^m \lambda_i)}{D} - \frac{t}{D\Gamma(\theta)} \sum_{i=1}^m \lambda_i \int_0^{\alpha_i} (\alpha_i - \zeta)^{\theta-1} \omega(\zeta) d\zeta \\ = a(t) + \int_0^1 \mathbf{W}_1(t, \zeta) \omega(\zeta) d\zeta. \end{cases} \quad (2.8)$$

Remark 2.6. We will use

$\mathbf{w}_1^* = \max_{t \in J} \int_0^1 |\mathbf{w}_1(t, \zeta)| d\zeta$, $\mathbf{w}_2^* = \max_{t \in J} \int_0^1 |\mathbf{w}_2(t, \zeta)| d\zeta$,
throughout this paper.

3. Method and discussion

Define $\mathbf{U} = \{u(t) \mid u(t) \in C^1(J)\}$ endowed with the norm

$\|u\|_{\mathbf{U}} = \max_{t \in J} [|u(t)| + |{}^c \mathbf{D}^\mu u(t)|]$ and similarly,

$\mathbf{V} = \{v(t) \mid v(t) \in C^1(J)\}$ endowed with the norm

$\|v\|_{\mathbf{V}} = \max_{t \in J} [|v(t)| + |{}^c \mathbf{D}^\nu v(t)|]$. Further, define the norm

$\|(u, v)\|_{\mathbf{U} \times \mathbf{V}} = \max \{\|u\|_{\mathbf{U}}, \|v\|_{\mathbf{V}}\}$. Then, the product spaces

$(\mathbf{U} \times \mathbf{V}, \|\cdot\|_{\mathbf{U} \times \mathbf{V}})$ is Banach space. Define a cone $\mathbf{P} \subset \mathbf{U} \times \mathbf{V}$ by

$\mathbf{P} = \{(u, v) \in \mathbf{U} \times \mathbf{V} : u(t), v(t) \geq 0, \text{ for all } t \in J\}$. To proceed further the following assumptions hold:

(A_1) · $a(t), b(t) : J \rightarrow \mathbb{R}$ are continuous;

(A_2) · $\mathbf{H}_1, \mathbf{H}_2 : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous.

Define the operator $\mathbf{S} : \mathbf{U} \times \mathbf{V} \rightarrow \mathbf{U} \times \mathbf{V}$ by

$$\mathbf{S}(u, v)(t) = \left(a(t) + \int_0^1 \mathbf{w}_1(t, \zeta) \mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta)) d\zeta, b(t) + \int_0^1 \mathbf{w}_2(t, \zeta) \mathbf{H}_2(\zeta, u(\zeta), {}^c \mathbf{D}^\nu u(\zeta)) d\zeta \right) \quad (3.1)$$

$$= (\mathbf{S}_1 v(t), \mathbf{S}_2 u(t)),$$

Where

$$\mathbf{S}_1 v(t) = a(t) + \int_0^1 \mathbf{w}_1(t, \zeta) \mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta)) d\zeta, \quad t \in J,$$

$$\mathbf{S}_2 u(t) = b(t) + \int_0^1 \mathbf{w}_2(t, \zeta) \mathbf{H}_2(\zeta, u(\zeta), {}^c \mathbf{D}^\nu u(\zeta)) d\zeta, \quad t \in J.$$

Theorem 3.1. Assume that (A_1) and (A_2) hold true then the coupled system has at least one fixed point $(u, v) \in C(J) \times C(J)$.

Proof. Define

$$\mathbf{C} = \{(u, v)(t) : (u, v) \in \mathbf{U} \times \mathbf{V} : \|(u, v)\|_{\mathbf{U} \times \mathbf{V}} \leq r\} \subset \mathbf{P}$$

and

$$\max_{t \in J} |a(t)| = K_1, \quad \max_{t \in J} |b(t)| = K_2.$$

Also assume that

$\max |\mathbf{H}_1(t, v(t), {}^c \mathbf{D}^\mu v(t))| \leq L$, $\max |\mathbf{H}_2(t, u(t), {}^c \mathbf{D}^\nu u(t))| \leq M$
for all $(u, v) \in \mathbf{C}$.

Then

$$|\mathbf{S}_1 v(t)| \leq \max_{t \in J} \left| \frac{1}{\Gamma(\theta)} \int_0^1 (t-\zeta)^{\theta-1} \mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta)) d\zeta - \frac{t}{D\Gamma(\theta)} \int_0^1 (1-\zeta)^{\theta-1} \mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta)) d\zeta \right|$$

$$+ \max_{t \in J} \left| A - \frac{At(1-\sum_{i=1}^m \lambda_i)}{D} - \frac{t}{D\Gamma(\theta)} \sum_{i=1}^m \lambda_i \int_0^{\alpha_i} (\alpha_i - \zeta)^{\theta-1} \mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta)) d\zeta \right|$$

$$\leq K_1 + \frac{L}{\Gamma(\theta+1)} \left(1 + \frac{2}{D}\right).$$

By similar way we can do

$$|\mathbf{S}_2 u(t)| \leq K_2 + \frac{M}{\Gamma(\theta+1)} \left(1 + \frac{2}{E}\right).$$

Also we can easily show that

$$|{}^c \mathbf{D}^\mu \mathbf{S}_1 v(t)| \leq \max_{t \in J} \left| \frac{1}{\Gamma(\theta-\mu)} \int_0^1 (t-\zeta)^{\theta-\mu-1} \mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta)) d\zeta \right|$$

$$+ \max_{t \in J} \left| \frac{t^{1-\mu}}{D\Gamma(2-\mu)} \int_0^1 (1-\zeta)^{\theta-1} \mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta)) d\zeta \right|$$

$$+ \max_{t \in J} \left| \frac{A(1-\sum_{i=1}^m \lambda_i)}{D\Gamma(2-\mu)} + \frac{t^{1-\mu}}{D\Gamma(2-\mu)\Gamma(\theta)} \sum_{i=1}^m \lambda_i \int_0^{\alpha_i} (\alpha_i - \zeta)^{\theta-1} d\zeta \right|$$

$$\leq \frac{A(1-\sum_{i=1}^m \lambda_i)}{D\Gamma(2-\mu)} + \frac{L}{\Gamma(\theta-\mu+1)} \left(1 + \frac{2}{D\Gamma(2-\mu)}\right).$$

Similarly

$$|{}^c \mathbf{D}^\nu \mathbf{S}_2 u(t)| \leq \frac{B(1-\sum_{i=1}^m \mu_i)}{E\Gamma(2-\nu)} + \frac{M}{\Gamma(\theta-\nu+1)} \left(1 + \frac{2}{E\Gamma(2-\nu)}\right).$$

Thus from the above relations we have

$$\|\mathbf{S}_1 v\|_{\mathbf{U}} \leq [K_1 + \frac{L}{\Gamma(\theta+1)} (1 + \frac{2}{D}) + \frac{A(1-\sum_{i=1}^m \lambda_i)}{D\Gamma(2-\mu)} + \frac{L}{\Gamma(\theta-\mu+1)} (1 + \frac{2}{D\Gamma(2-\mu)})] \quad (3.2)$$

$$\|\mathbf{S}_2 u\|_{\mathbf{V}} \leq [K_2 + \frac{M}{\Gamma(\theta+1)} (1 + \frac{2}{E}) + \frac{B(1-\sum_{i=1}^m \mu_i)}{E\Gamma(2-\nu)} + \frac{M}{\Gamma(\theta-\nu+1)} (1 + \frac{2}{E\Gamma(2-\nu)})].$$

Hence from (3.2) we have

$$\|\mathbf{S}(u, v)\|_{\mathbf{U} \times \mathbf{V}} < \infty. \quad (3.3)$$

Next, we prove that $\mathbf{S}$ is equi-continuous on J . For $(u, v) \in \mathbf{C}$,
and $t_1, t_2 \in J$, such that $t_1 < t_2$. Then we have

$$|\mathbf{S}_1 v(t_1) - \mathbf{S}_1 v(t_2)| \leq \frac{1}{\Gamma(\theta)} \int_0^{t_1} [(t_1 - \zeta)^{\theta-1} - (t_2 - \zeta)^{\theta-1}] |\mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta))| d\zeta \quad (3.4)$$

$$+ \int_{t_1}^{t_2} (t_2 - \zeta)^{\theta-1} |\mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta))| d\zeta$$

$$+ \frac{t_1 - t_2}{D\Gamma(\theta)} \int_0^1 (1-\zeta)^{\theta-1} |\mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta))| d\zeta$$

$$+ \frac{A(1-\sum_{i=1}^m \lambda_i)(t_2 - t_1)}{D} + \frac{t_2 - t_1}{\Gamma(\theta)} \sum_{i=1}^m \lambda_i \int_0^{\alpha_i} (\alpha_i - \zeta)^{\theta-1} |\mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta))| d\zeta$$

$$\leq \frac{L}{\Gamma(\theta+1)} [t_1^\theta - t_2^\theta] + \frac{(t_2 - t_1)}{D\Gamma(\theta+1)} L$$

$$+ \frac{A(1-\sum_{i=1}^m \lambda_i)(t_2 - t_1)}{D} + \frac{(t_2 - t_1)}{D\Gamma(\theta+1)} L.$$

Also we have

$$|{}^c \mathbf{D}^\mu \mathbf{S}_1 v(t_1) - {}^c \mathbf{D}^\mu \mathbf{S}_1 v(t_2)| \leq \frac{L}{\Gamma(\theta-\mu+1)} [t_1^{\theta-\mu} - t_2^{\theta-\mu}] + \frac{A(1-\sum_{i=1}^m \lambda_i)}{D\Gamma(2-\mu)} (t_1^{1-\mu} - t_2^{1-\mu}) \quad (3.5)$$

$$+ \frac{2(t_2^{1-\mu} - t_1^{1-\mu})}{D\Gamma(2-\mu)\Gamma(\theta+1)} L.$$

By similar way we can obtain that

$$|\mathbf{S}_2 u(t_1) - \mathbf{S}_2 u(t_2)| \leq \frac{M}{\Gamma(\theta+1)} [t_1^\theta - t_2^\theta] + \frac{(t_2 - t_1)}{E\Gamma(\theta+1)} M \quad (3.6)$$

$$+ \frac{B(1-\sum_{i=1}^m \mu_i)(t_2 - t_1)}{E} + \frac{(t_2 - t_1)}{E\Gamma(\theta+1)} M,$$

$$|{}^c \mathbf{D}^\nu \mathbf{S}_2 u(t_1) - {}^c \mathbf{D}^\nu \mathbf{S}_2 u(t_2)| \leq \frac{M}{\Gamma(\theta-\nu+1)} [t_1^{\theta-\nu} - t_2^{\theta-\nu}] + \frac{B(1-\sum_{i=1}^m \mu_i)}{E\Gamma(2-\nu)} (t_1^{1-\nu} - t_2^{1-\nu}) \quad (3.7)$$

$$+ \frac{2(t_2^{1-\nu} - t_1^{1-\nu})}{E\Gamma(2-\nu)\Gamma(\theta+1)} M.$$

Now as $t_1 \rightarrow t_2$ in right hand sides of (3.4) (3.7), they tend to zero. Therefore $\|\mathbf{S}(u, v)(t_1) - \mathbf{S}(u, v)(t_2)\|_{\mathbf{U} \times \mathbf{V}} \rightarrow 0$ as $t_1 \rightarrow t_2$.

Hence a theorem, we conclude that $\mathbf{S}$ is completely continuous operator. Finally define

$\Omega = \{(u, v) \in \mathbf{U} \times \mathbf{V} : (u, v) = \lambda \mathbf{S}(u, v), \lambda \in (0, 1)\}$. It is enough to show that Ω is bounded. Let $(u, v) \in \Omega$, then $(u, v) = \lambda \mathbf{S}(u, v)$ for some $0 < \lambda < 1$. Thus, for every $t \in \mathbf{J}$ we have $u(t) = \lambda \mathbf{S}_1 v(t)$ and $v(t) = \lambda \mathbf{S}_2 u(t)$, then it is easy to show that $\|(u, v)\|_{\mathbf{U} \times \mathbf{V}} < \infty$.

Which implies that Ω is bounded. Thus, by Schauder fixed point theorem $\mathbf{S}$ has at least one fixed point which is the solution of (1.1).

Theorem 3.2. If $\mathbf{H}_1, \mathbf{H}_2 : \mathbf{J} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \mathbf{Z}$ are continuous and if $\max(d_1, d_2) < 1$, then Coupled system (1.1) has unique positive solution, where

$$d_1 = \max_{t \in \mathbf{J}} \left[2\bar{K}\mathbf{w}_1^*, \frac{2\bar{K}}{\Gamma(2-\mu)} \left(1 + \frac{1}{2} + \frac{A}{D} + \frac{1}{2D\Gamma(\theta+1)} \right) \right],$$

$$d_2 = \max_{t \in \mathbf{J}} \left[2\bar{K}\mathbf{w}_2^*, \frac{2\bar{L}}{\Gamma(2-\nu)} \left(1 + \frac{1}{2} + \frac{B}{E} + \frac{1}{2E\Gamma(\vartheta+1)} \right) \right].$$

Proof. Let $u, \bar{u}, v, \bar{v} \in \mathbf{R}$, then

$$\begin{aligned} \|\mathbf{S}_1(v) - \mathbf{S}_1(\bar{v})\| &= \left\| \int_0^t \mathbf{w}_1(t, \zeta) [\mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta)) - \mathbf{H}_1(\zeta, \bar{v}(\zeta), {}^c \mathbf{D}^\mu \bar{v}(\zeta))] d\zeta \right\| \\ &\leq \bar{K} \int_0^t \mathbf{w}_1(t, \zeta) [|v - \bar{v}| + |{}^c \mathbf{D}^\mu u - {}^c \mathbf{D}^\mu \bar{u}|] d\zeta \\ &\leq 2\bar{K}\mathbf{w}_1^* \|v - \bar{v}\|_{\mathbf{V}}. \end{aligned}$$

Also, we can write

$$\begin{aligned} |{}^c \mathbf{D}^\mu \mathbf{S}_1(v) - {}^c \mathbf{D}^\mu \mathbf{S}_1(\bar{v})| &\leq \max_{t \in \mathbf{J}} \frac{1}{\Gamma(\theta - \mu)} \int_0^t (t - \zeta)^{\theta - \mu - 1} |\mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta)) - \mathbf{H}_1(\zeta, \bar{v}(\zeta), {}^c \mathbf{D}^\mu \bar{v}(\zeta))| d\zeta \\ &+ \frac{1}{D\Gamma(2 - \mu)} \max_{t \in \mathbf{J}} \int_0^t (t - \zeta)^{\theta - 1} |\mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta)) - \mathbf{H}_1(\zeta, \bar{v}(\zeta), {}^c \mathbf{D}^\mu \bar{v}(\zeta))| d\zeta \\ &+ \left(\frac{A}{D\Gamma(2 - \mu)} + \frac{1}{\Gamma(2 - \mu)\Gamma(\theta)} \sum_{i=1}^n \lambda_i \int_0^{\sigma_i} (\alpha - \zeta)^{\theta - 1} \right) \\ &|\mathbf{H}_1(\zeta, v(\zeta), {}^c \mathbf{D}^\mu v(\zeta)) - \mathbf{H}_1(\zeta, \bar{v}(\zeta), {}^c \mathbf{D}^\mu \bar{v}(\zeta))| d\zeta \\ &\leq \frac{2\bar{K}}{\Gamma(\theta - \mu + 1)} \|v - \bar{v}\|_{\mathbf{V}} + \frac{2\bar{K}}{2\Gamma(2 - \mu)} \|v - \bar{v}\|_{\mathbf{V}} + \left(\frac{A}{D\Gamma(2 - \mu)} + \frac{1}{2D\Gamma(2 - \mu)\Gamma(\theta + 1)} \right) 2\bar{K} \|v - \bar{v}\|_{\mathbf{V}} \end{aligned}$$

implies that $\|\mathbf{D}^\mu \mathbf{S}_1(v) - {}^c \mathbf{D}^\mu \mathbf{S}_1(\bar{v})\|_{\mathbf{V}} \leq \frac{2\bar{K}}{\Gamma(2 - \mu)} \left(1 + \frac{1}{2} + \frac{A}{D} + \frac{1}{2D\Gamma(\theta + 1)} \right) \|v - \bar{v}\|_{\mathbf{V}}.$

Now

$$\|\mathbf{S}_1 v - \mathbf{S}_1 \bar{v}\|_{\mathbf{V}} \leq \max_{t \in \mathbf{J}} \left[2\bar{K}\mathbf{w}_1^*, \frac{2\bar{K}}{\Gamma(2 - \mu)} \left(1 + \frac{1}{2} + \frac{A}{D} + \frac{1}{2D\Gamma(\theta + 1)} \right) \right] \|v - \bar{v}\|_{\mathbf{V}}. \quad (3.8)$$

Similarly

$$\|\mathbf{S}_2 u - \mathbf{S}_2 \bar{u}\|_{\mathbf{U}} \leq \max_{t \in \mathbf{J}} \left[2\bar{K}\mathbf{w}_2^*, \frac{2\bar{L}}{\Gamma(2 - \nu)} \left(1 + \frac{1}{2} + \frac{B}{E} + \frac{1}{2E\Gamma(\vartheta + 1)} \right) \right] \|u - \bar{u}\|_{\mathbf{U}}. \quad (3.9)$$

Hence, from (3.8) and (3.9), we have

$$\|\mathbf{S}(u, v) - \mathbf{S}(\bar{u}, \bar{v})\|_{\mathbf{U} \times \mathbf{V}} \leq \max(d_1, d_2) [\|u - \bar{u}\|_{\mathbf{U}} + \|v - \bar{v}\|_{\mathbf{V}}],$$

where $\max(d_1, d_2) < 1$. Hence $\mathbf{S}$ has unique fixed point which is the corresponding unique solution of BVP (1.1).

4. Example

Example 4.1.

$$\begin{cases} {}^c \mathbf{D}^{\frac{3}{2}} u(t) = \frac{t}{4} + \frac{\sin |v(t)| + \cos |{}^c \mathbf{D}^{\frac{1}{2}} v(t)|}{40 + t^2}, & t \in \mathbf{J} \\ {}^c \mathbf{D}^{\frac{3}{2}} v(t) = \frac{t+1}{10} + \frac{\cos |u(t)| + |{}^c \mathbf{D}^{\frac{1}{2}} u(t)|}{50}, & t \in \mathbf{J} \\ u(0) = 1, u(1) = 2, u(1) = \sum_{i=1}^{10} \frac{1}{2^i} u(\frac{1}{2^i}), v(1) = \sum_{i=1}^{10} \frac{1}{3^i} v(\frac{1}{3^i}), & i = 1, 2, \dots, 10. \end{cases}$$

Now for coupled system (1.1), $\theta = \vartheta = \frac{3}{2}, \mu = \nu = \frac{1}{2}$

$$a(t) = \frac{t}{4}, b(t) = \frac{t+1}{10}, \lambda_i = \frac{1}{2^i}, \mu_i = \frac{1}{3^i}, \alpha_i = \frac{1}{2^i}, \beta_i = \frac{1}{2^i}$$

$$\mathbf{H}_1(t, v(t), {}^c \mathbf{D}^{\frac{1}{2}} v(t)) = \frac{\sin |v(t)| + \cos |{}^c \mathbf{D}^{\frac{1}{2}} v(t)|}{40 + t^2}, \mathbf{H}_2(t, u(t), {}^c \mathbf{D}^{\frac{1}{2}} u(t)) = \frac{\cos |u(t)| + |{}^c \mathbf{D}^{\frac{1}{2}} u(t)|}{50}$$

$$A = 1, B = 2, D = 1 - \sum_{i=1}^{10} \lambda_i \alpha_i = 0.667, \bar{K} = \frac{1}{40}, \bar{L} = \frac{1}{50}, E = 1 - \sum_{i=1}^{10} \mu_i \beta_i = 0.800,$$

$$\mathbf{w}_1^* = 0.591, \mathbf{w}_2^* = 0.508$$

$$\text{hence } 2\bar{K}\mathbf{w}_1^* = 0.02956, 2\bar{K}\mathbf{w}_2^* = 0.02032.$$

Now, we computing the values of

$$d_1 = \max \left\{ 2\bar{K}\mathbf{w}_1^*, \frac{2\bar{K}}{\Gamma(2 - \mu)} \left(1 + \frac{1}{2} + \frac{A}{D} + \frac{1}{2D\Gamma(\theta + 1)} \right) \right\}$$

$$= \max\{0.02956, 0.1811\} = 0.1811$$

and

$$d_2 = \max \left\{ 2\bar{K}\mathbf{w}_2^*, \frac{2\bar{L}}{\Gamma(2 - \nu)} \left(1 + \frac{1}{2} + \frac{B}{E} + \frac{1}{2E\Gamma(\vartheta + 1)} \right) \right\}$$

$$= \max\{0.02030, 0.1061\} = 0.1061.$$

We see that $\max\{d_1, d_2\} = 0.1811 < 1$. Hence Thank to Theorem 3.2, the given coupled system has a unique solution.

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