NONLOCAL INTEGRAL BOUNDARY VALUE PROBLEMS WITH CAUSAL OPERATORS AND FRACTIONAL DERIVATIVES

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Abstract. This paper investigates the existence of solutions for a class of nonlocal integral boundary value problems with causal operators and fractional derivatives. By applying the monotone iterative technique and the method of lower and upper solutions, we provide sufficient conditions under which such problems have the maximal and minimal solutions or quasisolutions in a corresponding sector. Finally, an example illustrating how the theory can be applied in practice is also included.

Key Words. Causal operators; Fractional differential inequalities with positive linear operators; Nonlocal integral boundary conditions; Monotone iterative technique

AMS(MOS) subject classification. 30A10

1. Introduction. In recent years the study of differential equations with causal operators has attracted much attention. A causal operator is a non-anticipative operator. The term causal arises from the engineering and the notion of a causal operator turns out to be a powerful tool for unifying problems in ordinary differential equations, integro-differential equations, differential equations with finite or infinite delay, Volterra integral equations, and neutral functional equations, to name a few. A pioneering work has been reported by Corduneanu ([1],[2]), Drici et al. ([3]–[5]), Jankowski ([6],[7]) and Lupulescu ([8]). For some recent development on the topic, see ([9]–[12]) and the references therein.

Fractional differential equations arise in many engineering and scien-

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tific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. ([13]-[16]). The interest in the study of differential equations of fractional order lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. With this advantage, the fractional-order models are regarded as important mathematical tools for the better understanding of several real world problems in applied sciences. For some recent work on differential equations of fractional order, see ([17]-[30]) and the references therein.

Integral type of boundary conditions have recently been addressed by several researchers. The interest in the study of integral boundary conditions lies in the fact that it has various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, and so forth. For a detailed description of the integral boundary conditions, we refer the reader to some recent papers ([31]-[38]) and the references therein. It has been observed that the limits of integration in the integral part of the boundary conditions are taken to be fixed, for instance, from 0 to T in case the independent variable belongs to the interval $[0, T]$. It is imperative to note that the available literature on nonlocal boundary conditions is confined to the nonlocal parameters involvement in the solution or gradient of the solution of the problem. In our paper, a nonlocal type of integral boundary condition with limits of integration involving the parameters $0 < \tau < T$ has been introduced. It is worth mentioning that, in practical situations, such nonlocal integral boundary conditions may be regarded as a continuous distribution of arbitrary finite length, for instance, see [39].

In this paper, we study the following nonlocal integral boundary value problems for fractional differential equation with causal operators

$$(1) \quad \begin{cases} D^\alpha u(t) = (Qu)(t), \\ t^{1-\alpha} u(t)|_{t=0} = t^{1-\alpha} \int_0^\tau w(s, u(s)) ds + x_0, \end{cases}$$

where $t \in J = [0, T]$ ($T > 0$), $E = C(J, \mathbb{R})$, $Q \in C(E, E)$ is a causal operator, $\tau \in (0, T]$, $w \in C(J \times \mathbb{R}, \mathbb{R})$, $x_0 \in \mathbb{R}$ and D^α is the Riemann-Liouville fractional derivative of u and α is such that $0 < \alpha < 1$.

To obtain approximate solutions of nonlinear differential equations, we also develop the monotone iterative technique. This technique, combined with the method of upper and lower solutions provides an effective mechanism to prove constructive existence results for nonlinear differential equations, the

advantage and importance of the technique needs no special emphasis [40]. To the best of our knowledge, the problem of fractional differential equations with causal operators and nonlocal integral boundary conditions has not been attempted so far. It is also the first paper in which the monotone iterative method is applied to problems of the type (1).

The rest of the paper is organized as follows. In Section 2, we list several lemmas and a new comparison principle, which play an important role in the proof of main results. Further, to study the nonlinear problem (1), we consider the associated linear problem and obtain the uniqueness of the solutions to the associated linear problem. In Section 3, we formulate sufficient conditions which guarantee that problem (1) has maximal and minimal solutions. A one-sided Lipschitz condition (with corresponding linear operators) is imposed on the causal operator Q . In Section 4, by using the notion of coupled lower and upper solutions, we testify the existence of quasisolutions of the problem (1). In Section 5, an example is given to illustrate our main results.

2. Preliminaries.

For the convenience of the readers, in this section we first present some useful definitions and lemmas.

DEFINITION 1. *The Riemann-Liouville fractional derivative of order δ for a function $f(t)$ is defined by*

$$D^\delta f(t) = \frac{1}{\Gamma(n-\delta)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\delta-1} f(s) ds, \quad n = [\delta] + 1,$$

provided the right hand side is defined pointwise on $(0, \infty)$.

DEFINITION 2. *The Riemann-Liouville fractional integral of order δ for a function f is defined as*

$$I^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s) ds, \quad \delta > 0,$$

provided that such integral exists.

Let $C_{1-\alpha}(J, \mathbb{R}) = \{u \in C(0, T]; t^{1-\alpha}u \in C(J, \mathbb{R})\}$.

LEMMA 1. *(Comparison result) Let $w \in C_{1-\alpha}(J, \mathbb{R})$ satisfies*

$$(2) \quad \begin{cases} D^\alpha w(t) \leq -M(t)w(t) - (\mathcal{L}w)(t), \\ t^{1-\alpha}w(t)|_{t=0} \leq 0, \end{cases}$$

where $M(t)$ is bounded nonnegative integral function and $\mathcal{L} \in C(E, E)$ is a positive linear operator. Then, $w(t) \leq 0$, $\forall t \in (0, T]$.

Proof. Suppose $w(t) \leq 0$ is not true, then there exist $t^*, t_* \in (0, T]$ such that

$$w(t^*) = 0, \quad w(t_*) > 0$$

and

$$w(t) \leq 0, \quad t \in (0, t^*), \quad w(t) > 0, \quad t \in (t^*, t_*).$$

Thus, if $t \in (t^*, t_*)$, we have $D^\alpha w(t) \leq 0$. Since

$$D^\alpha w(t) = \frac{d}{dt} I^{1-\alpha} w(t),$$

It follows that $I^{1-\alpha} w(t)$ is nonincreasing on (t^*, t_*) . Therefore, we get

$$(3) \quad I^{1-\alpha} w(t) - I^{1-\alpha} w(t^*) \leq 0, \quad t \in (t^*, t_*).$$

On the other hand,

$$(4) \quad \begin{aligned} I^{1-\alpha} w(t) - I^{1-\alpha} w(t^*) &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} w(s) ds - \int_0^{t^*} \frac{(t^*-s)^{-\alpha}}{\Gamma(1-\alpha)} w(s) ds \\ &= \int_0^t \frac{(t-s)^{-\alpha} - (t^*-s)^{-\alpha}}{\Gamma(1-\alpha)} w(s) ds + \int_{t^*}^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} w(s) ds \\ &> 0, \quad t \in (t^*, t_*), \end{aligned}$$

which contradicts (3). Thus, $w(t) \leq 0$ ($t \in (0, T]$) holds. This completes the proof. $\square$

To study the nonlinear problem (1), we first consider the associated linear problem:

$$(5) \quad \begin{cases} D^\alpha u(t) = \sigma(t) - M(t)u(t) - (\mathcal{L}u)(t), \\ t^{1-\alpha} u(t)|_{t=0} = t^{1-\alpha} \int_0^\tau w(s, \eta(s)) ds + x_0, \end{cases}$$

where $\sigma, \eta \in C(J, \mathbb{R})$ and $M(t), \mathcal{L}(t)$ are as in Lemma 1.

LEMMA 2. *If*

$$(6) \quad I^\alpha M(t) + \frac{\|\mathcal{L}\| t^\alpha}{\Gamma(\alpha+1)} < 1.$$

Then (5) has a unique solution $u \in C_{1-\alpha}(J, \mathbb{R})$.

Proof. Obviously, $u(t) \in C_{1-\alpha}(J, \mathbb{R})$ is a solution of (5) if and only if $u(t) \in C(J, \mathbb{R})$ is a solution of the following integral equation

$$(7) \quad u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[\sigma(s) - M(s)u(s) - (\mathcal{L}u)(s) \right] ds + \int_0^\tau w(s, \eta(s)) ds + x_0 t^{\alpha-1} \\ \triangleq Tu(t).$$

For any $u, v \in C(J, \mathbb{R})$, by (7) we have

$$|Tu(t) - Tv(t)| \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[M(s)|(u-v)(s)| + |(\mathcal{L}u) - (\mathcal{L}v)(s)| \right] ds \\ \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[M(s) + \|\mathcal{L}\| \right] ds \|u - v\|_C.$$

Therefore, it follows that

$$(8) \quad \|Tu - Tv\|_C \leq \left[I^\alpha M(t) + \frac{\|\mathcal{L}\|t^\alpha}{\Gamma(\alpha+1)} \right] \|u - v\|_C < \|u - v\|_C.$$

Hence, T is a contraction operator on $C(J, \mathbb{R})$. Consequently, it has a unique fixed point u , i.e. $u(t)$ is a unique solution of (5). □

3. Maximal and minimal solutions of (1).

In this section we shall use the monotone iterative technique to prove the existence of maximal and minimal solutions of nonlocal integral boundary value problem involving causal operators (1).

DEFINITION 3. We say that $u \in C_{1-\alpha}(J, \mathbb{R})$ is a lower solution of (1) if

$$\begin{cases} D^\alpha u(t) \leq (Qu)(t), \\ t^{1-\alpha}u(t)|_{t=0} \leq t^{1-\alpha} \int_0^\tau w(s, u(s)) ds + x_0, \end{cases}$$

and it is an upper solution of (1), if the above inequalities are reversed.

THEOREM 1. Assume that

(H₁) $u_0, v_0 \in C_{1-\alpha}(J, \mathbb{R})$ are the lower and upper solutions of (1), respectively, and $u_0(t) \leq v_0(t)$.

(H₂) The following conditions

$$(Qu)(t) - (Qv)(t) \geq -M(t)(u-v)(t) - (\mathcal{L}(u-v))(t), \\ \int_0^\tau [w(s, u(s)) - w(s, v(s))] ds \geq 0.$$

hold for $u_0 \leq v \leq u < v_0$, here $M(t), \mathcal{L}(t)$ are as in Lemma 2 and satisfy (6). Then there exist monotone iterative sequences $\{u_n\}, \{v_n\} \subset [u_0, v_0]$ which converge uniformly to the minimal and maximal solutions u^*, v^* of (1) in $[u_0, v_0]$, respectively, where $\{u_n\}, \{v_n\}$ are defined by

$$(9) \quad \begin{aligned} u_n(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[(Qu_{n-1})(s) - M(s)(u_n - u_{n-1})(s) - (\mathcal{L}(u_n - u_{n-1}))(s) \right] ds \\ & + \int_0^\tau w(s, u_{n-1}(s)) ds + x_0 t^{\alpha-1}, \end{aligned}$$

$$(10) \quad \begin{aligned} v_n(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[(Qv_{n-1})(s) - M(s)(v_n - v_{n-1})(s) - (\mathcal{L}(v_n - v_{n-1}))(s) \right] ds \\ & + \int_0^\tau w(s, v_{n-1}(s)) ds + x_0 t^{\alpha-1}, \end{aligned}$$

and

$$(11) \quad u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq u^* \leq v^* \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

Proof. For any $u_{n-1}, v_{n-1} \in C(J, \mathbb{R})$, by Lemma 2, we know that (9) and (10) have unique solutions $u_n, v_n \in C(J, \mathbb{R})$, respectively. To complete the proof we need the following three steps:

Step 1: $u_{n-1} \leq u_n$ and $v_n \leq v_{n-1}, n = 1, 2, \dots$.

By (9) and (10), we have

$$(12) \quad \begin{cases} D^\alpha u_n(t) + M(t)u_n(t) + (\mathcal{L}u_n)(t) = (Qu_{n-1})(t) + M(t)u_{n-1}(t) + (\mathcal{L}u_{n-1})(t), \\ t^{1-\alpha}u_n(t)|_{t=0} = t^{1-\alpha} \int_0^\tau w(s, u_{n-1}(s)) ds + x_0, \end{cases}$$

and

$$(13) \quad \begin{cases} D^\alpha v_n(t) + M(t)v_n(t) + (\mathcal{L}v_n)(t) = (Qv_{n-1})(t) + M(t)v_{n-1}(t) + (\mathcal{L}v_{n-1})(t), \\ t^{1-\alpha}v_n(t)|_{t=0} = t^{1-\alpha} \int_0^\tau w(s, v_{n-1}(s)) ds + x_0. \end{cases}$$

Thus, u_1 satisfies

$$\begin{cases} D^\alpha u_1(t) + M(t)u_1(t) + (\mathcal{L}u_1)(t) = (Qu_0)(t) + M(t)u_0(t) + (\mathcal{L}u_0)(t), \\ t^{1-\alpha}u_1(t)|_{t=0} = t^{1-\alpha} \int_0^\tau w(s, u_0(s)) ds + x_0. \end{cases}$$

Let $p(t) = u_0(t) - u_1(t)$, then by condition (H_1) , we have

$$D^\alpha p(t) + M(t)p(t) + (\mathcal{L}p)(t) = D^\alpha u_0(t) - (Qu_0)(t) \leq 0.$$

In addition,

$$t^{1-\alpha}p(t)|_{t=0} = t^{1-\alpha}u_0(t)|_{t=0} - t^{1-\alpha}u_1(t)|_{t=0} \leq 0.$$

Hence, by Lemma 1, $p(t) \leq 0$, which implies that $u_0(t) \leq u_1(t)$.

Now by mathematical induction it easily follows that

$$(14) \quad u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots, \quad n = 1, 2, \dots.$$

Similarly, we have

$$(15) \quad \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0, \quad n = 1, 2, \dots.$$

Step 2: $u_n \leq v_n$, $n = 1, 2, \dots$.

Let $p(t) = u_1(t) - v_1(t)$, then by condition (H_2) , we have

$$D^\alpha p(t) + M(t)p(t) + (\mathcal{L}p)(t) = (Qu_0)(t) + M(t)u_0(t) + (\mathcal{L}u_0)(t) - (Qv_0)(t) - M(t)v_0(t) - (\mathcal{L}v_0)(t) \leq 0.$$

Moreover,

$$t^{1-\alpha}p(t)|_{t=0} = t^{1-\alpha}u_1(t)|_{t=0} - t^{1-\alpha}v_1(t)|_{t=0} = t^{1-\alpha} \int_0^\tau [w(s, u_0(s)) - w(s, v_0(s))] ds \leq 0.$$

Thus by Lemma 1, we have $p(t) \leq 0$, which implies that $u_1(t) \leq v_1(t)$. Now mathematical induction leads to $u_n \leq v_n$, $n = 1, 2, \dots$.

Combining Steps 1 and 2, we find

$$(16) \quad u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

Thus, using standard arguments, we find that $\lim_{n \rightarrow \infty} u_n(t) = u^*(t)$ and $\lim_{n \rightarrow \infty} v_n(t) = v^*(t)$ uniformly on J , and $u^*(t)$, $v^*(t)$ are solutions of (1). Hence, (11) holds.

Step 3: $u^*(t)$, $v^*(t)$ are the minimal and maximal solutions of (1).

Assume that $\omega(t) \in [u_0(t), v_0(t)]$ be any solution of (1), i.e.,

$$(17) \quad \begin{cases} D^\alpha \omega(t) = (Q\omega)(t), \\ t^{1-\alpha} \omega(t)|_{t=0} = t^{1-\alpha} \int_0^\tau w(s, \omega(s)) ds + x_0. \end{cases}$$

Combining (12), (13) and (H_2) , and using Lemma 1, it follows by induction that

$$(18) \quad u_n(t) \leq \omega(t) \leq v_n(t), \quad n = 1, 2, \dots.$$

Now, letting $n \rightarrow +\infty$ in (18), we find $u^* \leq \omega \leq v^*$. That is, u^*, v^* are the minimal and maximal solutions of (1) in $[u_0, v_0]$. $\square$

4. Quasisolutions of (1).

This section deals with the existence of quasisolutions for nonlocal integral boundary value problem (1).

DEFINITION 4. *We say that $u, v \in C_{1-\alpha}(J, \mathbb{R})$ are coupled lower and upper solutions of (1), if*

$$\begin{cases} D^\alpha u(t) \leq (Qu)(t), \\ t^{1-\alpha}u(t)|_{t=0} \leq t^{1-\alpha} \int_0^\tau w(s, v(s))ds + x_0, \\ D^\alpha v(t) \geq (Qv)(t), \\ t^{1-\alpha}v(t)|_{t=0} \geq t^{1-\alpha} \int_0^\tau w(s, u(s))ds + x_0, \end{cases}$$

DEFINITION 5. *We say that $U, V \in C_{1-\alpha}(J, \mathbb{R})$ are quasisolutions of (1), if*

$$\begin{cases} D^\alpha U(t) = (QU)(t), \\ t^{1-\alpha}U(t)|_{t=0} = t^{1-\alpha} \int_0^\tau w(s, V(s))ds + x_0, \\ D^\alpha V(t) = (QV)(t), \\ t^{1-\alpha}V(t)|_{t=0} = t^{1-\alpha} \int_0^\tau w(s, U(s))ds + x_0, \end{cases}$$

THEOREM 2. *Assume that*

(H'_1) $u_0, v_0 \in C_{1-\alpha}(J, \mathbb{R})$ are the coupled lower and upper solutions of (1), respectively, and $u_0(t) \leq v_0(t)$.

(H'_2) The following conditions

$$\begin{aligned} (Qu)(t) - (Qv)(t) &\geq -M(t)(u - v)(t) - (\mathcal{L}(u - v))(t), \\ \int_0^\tau [w(s, u(s)) - w(s, v(s))]ds &\leq 0. \end{aligned}$$

hold for $u_0 \leq v \leq u < v_0$, here $M(t), \mathcal{L}(t)$ are as in Lemma 2 and satisfy (6).

Then there exist monotone iterative sequences $\{u_n\}, \{v_n\} \subset [u_0, v_0]$ which converge uniformly to the quasisolutions u^, v^* of (1) in $[u_0, v_0]$, respectively, where $\{u_n\}, \{v_n\}$ are defined as*

(19)

$$\begin{aligned} u_n(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[(Qu_{n-1})(s) - M(s)(u_n - u_{n-1})(s) - (\mathcal{L}(u_n - u_{n-1}))(s) \right] ds \\ &\quad + \int_0^\tau w(s, v_{n-1}(s))ds + x_0 t^{\alpha-1}, \end{aligned}$$

$$(20) \quad v_n(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[(Qv_{n-1})(s) - M(s)(v_n - v_{n-1})(s) - (\mathcal{L}(v_n - v_{n-1}))(s) \right] ds \\ + \int_0^\tau w(s, u_{n-1}(s)) ds + x_0 t^{\alpha-1},$$

and

$$(21) \quad u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq u^* \leq v^* \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0.$$

Proof. The proof is similar to that of Theorem 1. $\square$

5. Example. Consider the problem:

$$(22) \quad \begin{cases} D^\alpha u(t) = -M(t)u(t) + M(t) \sin\left(\frac{\pi}{2}u\left(\frac{t^2}{3}\right)\right) - 3 \int_0^t t^2 s^3 u(s) ds, \\ t^{1-\alpha} u(t)|_{t=0} = t^{1-\alpha} \int_0^{\frac{2}{3}} \left(s^2 + \frac{u^3(s)}{27}\right) ds, \end{cases}$$

here $t \in J = [0, 1]$, D^α is the standard Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ and $M(t)$ is bounded nonnegative integrable function.

Clearly,

$$\begin{cases} (Qu)(t) = -M(t)u(t) + M(t) \sin\left(\frac{\pi}{2}u\left(\frac{t^2}{3}\right)\right) - 3 \int_0^t t^2 s^3 u(s) ds, \\ w(t, u(t)) = t^2 + \frac{u^3(t)}{27}. \end{cases}$$

Take $u_0(t) = 0, v_0(t) = 1$. It is not difficult to verify that u_0, v_0 are lower and upper solutions of (22), respectively, and $u_0 \leq v_0$. Thus, condition (H_1) holds. In addition, we have

$$(Qu)(t) - (Qv)(t) \geq -M(t)(u - v)(t) - (\mathcal{L}(u - v))(t), \\ \int_0^\tau [w(s, u(s)) - w(s, v(s))] ds \geq 0,$$

for $u_0 \leq v \leq u < v_0$, where $(\mathcal{L}u)(t) = 3 \int_0^t t^2 s^3 u(s) ds$.

If we assume that

$$(23) \quad I^\alpha M(t) + \frac{\|\mathcal{L}\|t^\alpha}{\Gamma(\alpha + 1)} < 1.$$

Then all conditions of Theorem 1 are satisfied. Therefore, by Theorem 1, there exist monotone iterative sequences $\{u_n\}, \{v_n\} \subset [u_0, v_0]$ which converge uniformly to the minimal and maximal solutions u^*, v^* of (1) in $[u_0, v_0]$, respectively.

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