OSCILLATION OF UNBOUNDED SOLUTIONS OF FOURTH ORDER NEUTRAL DYNAMIC EQUATIONS VIA TAYLOR'S FORMULA

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Abstract. This paper deals with the unbounded oscillation properties of nonlinear fourth order neutral delay dynamic equations of the form

$$\left(p(t) \left(u(t) + r(t)u(\alpha(t))\right)^{\Delta^2}\right)^{\Delta^2} + g(t)G(u(\beta(t))) - h(t)H(u(\gamma(t))) = 0$$

on an arbitrary time scales $\mathbb{T}$, under the assumptions

$$\int_{t_0}^{\infty} \frac{t}{p(t)} \Delta t = \infty \text{ or } \int_{t_0}^{\infty} \frac{t}{p(t)} \Delta t < \infty, \quad t_0 > 0$$

for various ranges of $r(t)$ with $|r(t)| < \infty$.

Key Words. Oscillation; Nonlinear; Delay dynamic equations; Time scales.

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1. Introduction. In [14] and [15], Panigrahi and Reddy have discussed the oscillation and asymptotic properties of solutions of functional dynamic equations of type

$$(1.1) \quad \left(p(t)(u(t) + r(t)u(\alpha(t)))^{\Delta^2}\right)^{\Delta^2} + g(t)G(u(\beta(t))) = 0,$$

where $g, p \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$, $\alpha, \beta \in C_{rd}(\mathbb{T}, \mathbb{T})$ such that $\alpha(t) \leq t$, $\beta(t) \leq t$, and $\lim_{t \rightarrow \infty} \alpha(t) = \infty = \lim_{t \rightarrow \infty} \beta(t)$, $G \in C(\mathbb{R}, \mathbb{R})$ is a continuous function with

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the property $xG(x) > 0$ for $x \neq 0$, and $r \in C_{rd}(\mathbb{T}, \mathbb{R})$ such that $|r(t)| < \infty$, under the assumptions

$$(A_0) \quad \int_{t_0}^{\infty} \frac{t}{p(t)} \Delta t < \infty, \quad t_0 > 0$$

and

$$(A_{00}) \quad \int_{t_0}^{\infty} \frac{t}{p(t)} \Delta t = \infty, \quad t_0 > 0.$$

In time scale calculus, we take the sign convention Δ as the differential operator for right-scattered points on $\mathbb{T}$ with the property $\Delta^k = \Delta \Delta^{k-1}$ and C_{rd} as the continuous for right-scattered and right-dense points on $\mathbb{T}$.

When we are concerned with oscillation of all solutions of (1.1), immediately we go for the works [19] and [20] by Tripathy under two sorts of assumptions (A_{00}) and (A_0) respectively. Similarly, in an another work [8] of Grace and Zafer we find an all solution oscillatory problem for the dynamic equations

$$\left(p(t)((u(t) + r(t)u(\alpha(t)))^{\Delta^3})^{\mu_1} \right)^{\Delta} + g(t)u^{\mu_2}(\beta(t)) = 0$$

under the following restrictions:

- (i) μ_1 and μ_2 are the quotient of odd positive integers such that $\mu_2 \leq \mu_1$,
- (ii) $|r(t)| \leq 1$, (iii) $\int_{t_0}^{\infty} p^{-\frac{1}{\mu_1}}(t) \Delta t = \infty$, $t_0 > 0$.

And they have established the results with the helping tool

$$h_{n+1}(t, s) = \int_s^t h_n(\tau, s) \Delta \tau, \quad n = 1, 2, 3, \dots; \quad h_0(t, s) = 1,$$

where $h_n : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$ which is known as Taylor monomials, but the neutral coefficient is too restrictive. Upon the choice of same problem, it is not hard to review the work for other ranges of $r(t)$ with respect to the works [19] and [20] even if,

$$\int_{t_0}^{\infty} p^{-\frac{1}{\mu_1}}(t) \Delta t < \infty, \quad t_0 > 0$$

also. However, there are no results in case when we consider $g^+(t) = \max\{0, g(t)\}$ and $g^-(t) = \max\{0, -g(t)\}$ for (1.1) which takes the form:

$$(1.2) \quad \left(p(t)(u(t) + r(t)u(\alpha(t)))^{\Delta^2} \right)^{\Delta^2} + g^+(t)G(u(\beta(t))) - g^-(t)G(u(\beta(t))) = 0.$$

The objective of this work is to go for an all solution oscillatory problem for the functional equation

$$(1.3) \quad \left(p(t)(u(t) + r(t)u(\alpha(t)))^{\Delta^2} \right)^{\Delta^2} + \\ + g(t)G(u(\beta(t))) - h(t)H(u(\beta(t))) = 0$$

for different ranges of $r(t)$. In (1.3), we impose conditions upon H as follows:

$$H \in C(\mathbb{R}, \mathbb{R}), \quad xH(x) > 0 \text{ for } x \neq 0, \quad H \text{ is bounded.}$$

Graef et al.[9] have established the sufficient conditions for oscillation and asymptotic behaviour of solutions of (1.3) by means of key assumption (A_{00}) but, with a slight deviation (1.3) has been studied by Panigrahi et al.[16] under the assumption

$$\int_{t_0}^{\infty} \frac{\sigma(t)}{p(t)} \Delta t < \infty, \quad t_0 > 0.$$

We notice that (1.3) can be studied by means of (A_0) also. Unlike the methods of [9] and [16], the purpose of this work is to study (1.3) via Taylor's theorem in the sense of Time scales (see for e.g [4], [5]). For our work, we assume the following assumptions on G, α and β :

- (A_1) $G(xy) \geq G(x)G(y)$, for $x, y \in \mathbb{R}$ and $x, y > 0$,
- (A_2) $G(-x) = -G(x)$, for $x \in \mathbb{R}$,
- (A_3) there exists $\lambda > 0$, such that $G(x) + G(y) \geq \lambda G(x + y)$, for $x, y \in \mathbb{R}$ and $x, y > 0$,
- (A_4) $\alpha(\beta(t)) = \beta(\alpha(t))$ for every right - scattered point $t \in [t_0, \infty)_{\mathbb{T}}, t_0 \geq 0$.

REMARK 1. The prototype of G satisfying $(A_1), (A_2)$ and (A_3) could be of the form

$$G(u) = (a^* + b^*|u|^\gamma)|u|sgnu,$$

$$G(u) = |u|^\eta sgnu,$$

where $a^* \geq l, b^* \geq 1, \gamma \geq 0$ and $\eta > 0$.

Indeed, the credit goes to S. Hilger [11], who has developed the concept of time scales in his Ph.D work. The purpose of the time scales was not only to unify the continuous and discrete aspects of mathematics, but also some cases in between. The study of dynamic equations on time scales allows us to avoid proving results twice, once for differential equations and once for difference equations. Our general idea is to prove a result for the dynamic equation (1.3), where the domain of the unknown function is a time scale $\mathbb{T}$ which is a non-empty closed subset of the real numbers $\mathbb{R}$. In this way, the results of this paper not only apply to $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ for the subsequent equations

$$(1.4) \quad (p(t)((u(t) + r(t)u(\alpha(t)))'' + g(t)G(u(\beta(t))) - \\ - h(t)H(u(\beta(t)))) = 0$$

and

$$(1.5) \quad \Delta^2 (p(n)(\Delta^2(u(n) + r(n)u(\alpha(n)))) + g(n)G(u(\beta(n))) - \\ - h(n)H(u(\beta(n)))) = 0,$$

but also to more general time scales such as $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = q^{\mathbb{N}_0} = \{t : t = q^k, k \in \mathbb{N}_0\}$ with $q > 1$ (which has important applications in quantum theory [12]), $\mathbb{T} = \mathbb{N}_0^2 = \{t^2 : t \in \mathbb{N}_0\}$, $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}_0\}$ etc. For basic notations on time scales and time scale calculus, we refer the survey paper [1] by Agarwal and the monographs [4, 5] by Bohner et al.

In electric networks, we find the application of neutral delay differential equations due to high speed computers concerning the lossless transmission lines (see for e.g. [10]). So, we are interested in the problem to obtain sufficient conditions in order to ensure that the nonlinear neutral delay dynamic equation (1.3) is oscillatory. We may note that, (1.3) includes a class of differential and difference equations with delay argument of neutral/non-neutral type. In comparison with the present work, we refer the reader to some of the works [7, 13, 17, 18, 19, 20, 21] on dynamic equations and the references cited there in. More on oscillation of unbounded solutions when $\mathbb{T} = \mathbb{R}$, we refer the monograph by Bereznansky et al. [3] and some related works [6], [2]. Since we are interested in the oscillatory behaviour of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$.

Let $t_{-1} = \inf_{t \in [t_0, \infty)_{\mathbb{T}}} \{\alpha(t), \beta(t)\}$. By a solution of (1.3) we mean a non-trivial real valued function u on $[T_u, \infty)_{\mathbb{T}}$ such that $(u(t) + r(t)u(\alpha(t))) \in C_{rd}^2(\mathbb{T}, \mathbb{R})$, $(p(t)(u(t) + r(t)u(\alpha(t)))^{\Delta^2}) \in C_{rd}^2(\mathbb{T}, \mathbb{R})$ and satisfies (1.3) for $T_u \geq t_{-1} > t_0 > 0$. In this paper, we do not consider the solutions that eventually vanish identically. A solution u of (1.3) is said to be *oscillatory*, if it is neither eventually positive nor eventually negative and it is *nonoscillatory* otherwise.

2. Some Preparatory Results.

In this section, we present some preparatory results which are useful for our discussion. Throughout our text, we use the quasi-derivative operators on $\mathbb{T}$ as follows:

$$L_0 u = u, L_1 u = L_0 u^{\Delta}, L_2 u = p(t)L_1 u^{\Delta}, L_3 u = L_2 u^{\Delta}, L_4 u = L_3 u^{\Delta}.$$

LEMMA 1. [9] Let (A_{00}) hold. Let u be a real valued function on $[t_0, \infty)_{\mathbb{T}}$ such that $L_4 u(t) \leq 0$ for large t . If $u(t) > 0$ ultimately, then one of cases (a) and (b) holds for large t , and if $u(t) < 0$ ultimately, then one of cases (b), (c), (d) and (e) holds for large t , where
(a) $L_1 u(t) > 0$, $L_2 u(t) > 0$ and $L_3 u(t) > 0$,

- (b) $L_1u(t) > 0$, $L_2u(t) < 0$ and $L_3u(t) > 0$,
- (c) $L_1u(t) < 0$, $L_2u(t) < 0$ and $L_3u(t) > 0$,
- (d) $L_1u(t) < 0$, $L_2u(t) < 0$ and $L_3u(t) < 0$,
- (e) $L_1u(t) < 0$, $L_2u(t) > 0$ and $L_3u(t) > 0$.

LEMMA 2. [20] Let (A_0) hold. Let u be a real valued function on $[t_0, \infty)_{\mathbb{T}}$ such that $L_4u(t) \leq 0$ for large t . If $u(t) > 0$ ultimately, then one of cases (a)-(d) holds for large t , and if $u(t) < 0$ ultimately, then one of cases (b)-(f) holds for large t , where

- (a) $L_1u(t) > 0$, $L_2u(t) > 0$ and $L_3u(t) > 0$,
- (b) $L_1u(t) > 0$, $L_2u(t) < 0$ and $L_3u(t) > 0$,
- (c) $L_1u(t) > 0$, $L_2u(t) < 0$ and $L_3u(t) < 0$,
- (d) $L_1u(t) < 0$, $L_2u(t) > 0$ and $L_3u(t) > 0$,
- (e) $L_1u(t) < 0$, $L_2u(t) < 0$ and $L_3u(t) > 0$,
- (f) $L_1u(t) < 0$, $L_2u(t) < 0$ and $L_3u(t) < 0$.

3. Unbounded Oscillation Results.

In this section, unbounded oscillation criteria for (1.3) are established under the assumptions (A_0) and (A_{00}) . Before stating our main results, we assume the following hypotheses for our use in the sequel:

$$B[s, v] = \int_v^s \frac{(s - \sigma(t))(t - v)}{p(t)} \Delta t, s > \sigma(t) > t > v,$$

$$C[v, u] = \int_u^v \frac{(\sigma(t) - u)(v - t)}{p(t)} \Delta t, v > \sigma(t) > t > u,$$

$$D[v, u] = \int_u^v \frac{(\sigma(t) - u)(t - u)}{p(t)} \Delta t, v > \sigma(t) > t > u.$$

THEOREM 1. Let there exist $0 < a < \infty$ such that $0 \leq r(t) \leq a < \infty$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Assume that $(A_0) - (A_4)$ hold. Furthermore, assume that

$$(A_5) \int_T^\infty \frac{\sigma(s)}{p(s)} \int_s^\infty \sigma(t) h(t) \Delta t \Delta s < \infty, T > 0$$

$$(A_6) \frac{G(x)}{x} \geq M_1 > 0, x \neq 0,$$

$$(A_7) \limsup_{s \rightarrow \infty} \int_{\beta(s)}^s Q(\theta) G[B(\beta(\theta), \beta(s))] \Delta \theta > \frac{1+G(a)}{\lambda M_1},$$

$$(A_8) \limsup_{t \rightarrow \infty} \int_{\beta(t)}^t Q(\theta) G[C(\beta(\theta), \beta(t))] \Delta \theta > \frac{1+G(a)}{\lambda M_1},$$

$$(A_9) \limsup_{\theta \rightarrow \infty} \int_{\beta^2(\theta)}^{\beta(\theta)} Q(v) G[D(\beta(v), \beta(\theta))] \Delta v > \frac{(1+G(a))}{\lambda M_1}$$

and

$$(A_{10}) \limsup_{s \rightarrow \infty} \int_{\beta(s)}^s Q(\theta) G[D(\beta(s), \beta(\theta))] \Delta \theta > \frac{(1+G(a))}{\lambda M_1}$$

hold, where $Q(t) = \min\{g(t), g(\alpha(t))\}$ for $t \geq t_{-1}$. Then every unbounded solution of (1.3) oscillates.

Proof. On the contrary, let $u(t)$ be an unbounded non-oscillatory solution of (1.3) on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality and due to (A_2) , we can find a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large such that $u(t) > 0, u(\alpha(t)), u(\beta(t)), u(\gamma(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Setting

$$z(t) = u(t) + r(t)u(\alpha(t)),$$

$$K(t) = \int_t^\infty \frac{(\sigma(s) - t)}{p(s)} \int_s^\infty (\sigma(\theta) - s)h(\theta)H(u(\gamma(\theta)))\Delta\theta\Delta s,$$

$$m(t) = z(t) - K(t) = u(t) + r(t)u(\alpha(t)) - K(t)$$

in (1.3), we get

$$(3.1) \quad L_4m(t) = -g(t)G(u(\beta(t))) \leq 0.$$

Therefore, there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $L_i m(t), i = 0, 1, 2, 3$ are eventually of one sign on $[t_2, \infty)_{\mathbb{T}}$. Assume that $L_0 m(t) > 0$ and hence $m(t) \leq z(t)$. In what follows, we consider *Cases(a) – (d)* of Lemma 2.

Case(a) For $x \geq y > t_2$,

$$L_2m(x) - L_2m(y) = \int_y^x L_3m(s)\Delta s \geq (x - y)L_3m(x)$$

implies that $L_2m(x) \geq (x - y)L_3m(x)$, that is,

$$(3.2) \quad m^{\Delta^2}(x) \geq \frac{(x - y)}{p(x)} L_3m(x).$$

For $s > \sigma(t) > t > t_2$, we have the Taylor's formula

$$m(s) - m(t_2) - (s - t_2)m^{\Delta}(t_2) = \int_{t_2}^s (s - \sigma(t))m^{\Delta^2}(t)\Delta t.$$

Therefore,

$$\begin{aligned} m(s) &> \int_{t_2}^s (s - \sigma(t))m^{\Delta^2}(t)\Delta t \geq \\ &\geq \int_{t_2}^s \frac{(s - \sigma(t))(t - y)}{p(t)} L_3m(t)\Delta t \geq \\ &\geq L_3m(s) \int_y^s \frac{(s - \sigma(t))(t - y)}{p(t)} \Delta t = \\ &= L_3m(s)B[s, y], \quad s > y \geq t_2 \end{aligned}$$

due to (3.2). Letting $s = \beta(\theta)$ and $y = \beta(s)$ in the preceding relation, we obtain

$$(3.3) \quad m(\beta(\theta)) \geq L_3m(\beta(\theta))B[\beta(\theta), \beta(s)], \quad \beta(\theta) > \beta(s) \geq t_2.$$

In addition to (1.3), it is easy to see that

$$\begin{aligned} 0 &= L_4m(t) + g(t)G(u(\beta(t))) + G(a)L_4(\alpha(t)) + G(a)g(\alpha(t))G(u(\beta(\alpha(t)))) \geq \\ &\geq L_4m(t) + G(a)L_4m(\alpha(t)) + Q(t)[G(u(\beta(t))) + G(a)G(u(\alpha(\beta(t))))] \geq \\ &\geq L_4m(t) + G(a)L_4m(\alpha(t)) + \lambda Q(t)G(z(\beta(t))) \geq \\ &\geq L_4m(t) + G(a)L_4m(\alpha(t)) + \lambda Q(t)G(m(\beta(t))) \end{aligned}$$

due to (A_1) , (A_3) , (A_4) and $z(t) \leq u(t) + au(\alpha(t))$. Applying (3.3) in the last inequality, it follows that

$$\begin{aligned} 0 &\geq L_4 m(\theta) + G(a)L_4 m(\alpha(\theta)) + \lambda Q(\theta)G(L_3 m(\beta(\theta)))B[\beta(\theta), \beta(s)] \geq \\ &\geq L_4 m(\theta) + G(a)L_4 m(\alpha(\theta)) + \lambda Q(\theta)G(L_3 m(\beta(\theta)))G(B[\beta(\theta), \beta(s)]). \end{aligned}$$

Integrating the above inequality from $\beta(s)$ to s , we obtain

$$\lambda \int_{\beta(s)}^s Q(\theta)G(L_3 m(\beta(\theta)))G(B[\beta(\theta), \beta(s)])\Delta\theta \leq L_3 m(\beta(s)) + G(a)L_3 m(\alpha(\beta(s))),$$

that is,

$$\lambda G(L_3 m(\beta(s))) \int_{\beta(s)}^s Q(\theta)G(B[\beta(\theta), \beta(s)])\Delta\theta \leq (1 + G(a))L_3 m(\alpha(\beta(s))),$$

where $\alpha(\beta(s)) \leq \beta(s)$. As a result,

$$\begin{aligned} &\int_{\beta(s)}^s Q(\theta)G(B[\beta(\theta), \beta(s)])\Delta\theta \leq \\ &\leq \frac{(1 + G(a))L_3 m(\alpha(\beta(s)))}{\lambda G(L_3 m(\beta(s)))} \leq \\ &\leq \frac{(1 + G(a))}{\lambda M_1} \end{aligned}$$

due to (A_6) , a contradiction to (A_7) .

Case(b) For $y > \sigma(t) > t > x \geq t_2$, we have the Taylor's formula

$$\begin{aligned} (3.4) \quad -m(y) &= -m(x) - (y - x)m^{\Delta}(y) + \int_x^y (\sigma(t) - x)m^{\Delta^2}(t)\Delta t \leq \\ &\leq \int_x^y (\sigma(t) - x)m^{\Delta^2}(t)\Delta t. \end{aligned}$$

Following to *Case(a)*, it is easy to see that $-L_2 m(x) \geq (y - x)L_3 m(y)$, that is, $-m^{\Delta^2}(x) \geq \frac{(y-x)}{p(x)}L_3 m(y)$. Therefore,

$$\begin{aligned} m(y) &\geq \int_x^y \frac{(\sigma(t) - x)(y - t)}{p(t)}L_3 m(y)\Delta t = \\ &= L_3 m(y)C[y, x], \quad y \geq s > \sigma(t) > t > x \geq t_2. \end{aligned}$$

Letting $y = \beta(\theta)$ and $x = \beta(t)$ in the preceding relation, we obtain

$$m(\beta(\theta)) \geq L_3 m(\beta(\theta))C[\beta(\theta), \beta(t)], \quad \beta(\theta) > \beta(t) \geq t_2.$$

The rest of this case follows from *Case(a)*.

Case(c) From (3.4), we have

$$-m(y) \leq \int_x^y (\sigma(t) - x)m^{\Delta^2}(t)\Delta t.$$

Further,

$$L_2m(y) - L_2m(x) = \int_x^y L_3m(s)\Delta s \leq (y-x)L_3m(x)$$

implies that $L_2m(y) \leq (y-x)L_3m(x)$, that is, $m^{\Delta^2}(y) \leq \frac{(y-x)}{p(y)}L_3m(x)$.
Consequently,

$$\begin{aligned} m(y) &\geq \int_x^y (\sigma(t) - x) \frac{(t-x)}{p(t)} (-L_3m(x)) \Delta t = \\ &= (-L_3m(x))D[y, x] \end{aligned}$$

for $y \geq s > \sigma(t) > t > x \geq t_2$. Letting y and x by $\beta(y)$ and $\beta(\theta)$ respectively in the last inequality, we get

$$(3.5) \quad m(\beta(y)) > (-L_3m(\beta(\theta)))D[\beta(y), \beta(\theta)]$$

for $\beta(y) \geq s > \sigma(t) > t > \beta(\theta) \geq t_2$. Proceeding as in *Case(a)*, we have

$$(3.6) \quad 0 \geq L_4m(t) + G(a)L_4m(\alpha(t)) + \lambda Q(t)G(m(\beta(t)))$$

which on applying (3.5), we find

$$0 \geq L_4m(y) + G(a)L_4m(\alpha(y)) + \lambda Q(y)G(-L_3m(\beta(\theta)))G(D[\beta(y), \beta(\theta)]).$$

Integrating the above inequality from $\beta^2(\theta)$ to $\beta(\theta)$, we obtain

$$\begin{aligned} \lambda \int_{\beta^2(\theta)}^{\beta(\theta)} Q(y)G(-L_3m(\beta(\theta)))G[D(\beta(y), \beta(\theta))]\Delta y &\leq -L_3m(\beta(\theta)) - G(a)L_3m(\alpha(\beta(\theta))) \leq \\ &\leq -(1 + G(a))L_3m(\beta(\theta)). \end{aligned}$$

As a result,

$$\int_{\beta^2(\theta)}^{\beta(\theta)} Q(y)G[D(\beta(y), \beta(\theta))]\Delta y \leq \frac{(1 + G(a))}{\lambda} \frac{(-L_3m(\beta(\theta)))}{G(-L_3m(\beta(\theta)))} \leq \frac{(1 + G(a))}{\lambda M_1},$$

a contradiction to (A_9) .

Case(d) From (3.4), we have

$$z(x) \geq \int_x^y (\sigma(t) - x)m^{\Delta^2}(t)\Delta t.$$

Since

$$L_2m(y) - L_2m(x) = \int_x^y L_3m(s)\Delta s \geq (y-x)L_3m(y),$$

then $L_2m(y) \geq (y-x)L_3m(y)$, that is, $m^{\Delta^2}(y) \geq \frac{(y-x)}{p(y)}L_3m(y)$ and

$$\begin{aligned} m(x) &\geq \int_x^y (\sigma(t) - x) \frac{(t-x)}{p(t)} L_3m(t) \Delta t \geq \\ &\geq L_3m(y) \int_x^y (\sigma(t) - x) \frac{(t-x)}{p(t)} \Delta t = \\ &= L_3m(y) D[y, x]. \end{aligned}$$

Letting $x = \beta(\theta)$ and $y = \beta(s)$, we get

$$(3.7) \quad m(\beta(\theta)) \geq L_3m(\beta(s)) D[\beta(s), \beta(\theta)],$$

for $\beta(s) > \beta(\theta) \geq t_2$. Using (3.7) in (3.6) it follows that

$$0 \geq L_4m(\theta) + G(a)L_4m(\alpha(\theta)) + \lambda Q(\theta)G(L_3m(\beta(s)))G(D[\beta(s), \beta(\theta)]).$$

Integrating the above inequality from $\beta(s)$ to s , we obtain

$$\begin{aligned} \lambda \int_{\beta(s)}^s Q(\theta)G(L_3m(\beta(s)))G[D(\beta(s), \beta(\theta))]\Delta\theta &\leq L_3m(\beta(s)) + G(a)L_3m(\alpha(\beta(s))) \leq \\ &\leq (1 + G(a))L_3m(\alpha(\beta(s))), \end{aligned}$$

Consequently,

$$\int_{\beta(s)}^s Q(\theta)G[D(\beta(s), \beta(\theta))]\Delta\theta \leq \frac{(1 + G(a))L_3m(\alpha(\beta(s)))}{\lambda G(L_3m(\beta(s)))} \leq \frac{(1 + G(a))}{\lambda M_1},$$

a contradiction to our hypothesis (A_{10}) .

Let $m(t) < 0$ for $t \geq t_2$. Then $m(t) = z(t) - K(t) < 0$ implies that

$$u(t) \leq z(t) = u(t) + r(t)u(\alpha(t) < K(t) \text{ for } t \geq t_2.$$

It is noticed that $K(t) > 0$ and $K^{\Delta}(t) < 0$, then $z(t)$ is bounded and so also $u(t)$ which is a contradiction. Therefore, this case doesn't arise when $u(t)$ is unbounded. This completes the proof of the theorem. ■

THEOREM 2. Let $-1 < r(t) \leq 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Assume that $(A_0) - (A_2), (A_5)$ and (A_6) hold. Furthermore, assume that

$$(A_{11}) \quad \limsup_{s \rightarrow \infty} \int_{\beta(s)}^s g(\theta)G[B(\beta(\theta), \beta(s))]\Delta\theta > \frac{1}{M_1},$$

$$(A_{12}) \quad \limsup_{t \rightarrow \infty} \int_{\beta(t)}^t g(\theta)G[C(\beta(\theta), \beta(t))]\Delta\theta > \frac{1}{M_1},$$

$$(A_{13}) \quad \limsup_{\theta \rightarrow \infty} \int_{\beta^2(\theta)}^{\beta(\theta)} g(v)G[D(\beta(v), \beta(\theta))]\Delta v > \frac{1}{M_1},$$

$$(A_{14}) \quad \limsup_{s \rightarrow \infty} \int_{\beta(s)}^s g(\theta)G[D(\beta(s), \beta(\theta))]\Delta\theta > \frac{1}{M_1}$$

and

$$(A_{15}) \quad \alpha^n(t) = \alpha(\alpha^{n-1}(t)), \lim_{n \rightarrow \infty} \alpha^n(t) < \infty$$

hold. Then every unbounded solution of (1.3) oscillates.

Proof. Suppose on the contrary that $u(t)$ is an unbounded nonoscillatory solution of (1.3) such that $u(t) > 0$, $u(\alpha(t)) > 0$, $u(\beta(t)) > 0$, $u(\gamma(t)) > 0$ for $t \geq t_0$. The case $u(t) < 0$ can similarly be dealt with due to (A_2) . Proceeding as in Theorem 1, we get (3.1). So, there exists a $t_1 > t_0$ such that $L_i m(t)$ for $i = 0, 1, 2, 3$ are eventually of one sign on $[t_1, \infty)$. Assume that $m(t) > 0$ for $t \geq t_2$. Then, $m(t) \leq u(t)$ for $t \geq t_2$ and hence (1.3) can be written as

$$(3.8) \quad L_4 m(t) + g(t)G(m(\beta(t))) \leq 0, \quad t \geq t_2.$$

Ultimately, any one of four *Cases(a) – (d)* of Lemma 2 holds for $t \geq t_2$. With (3.8), we can apply the arguments of the cases of Theorem 1 to get contradictions to $(A_{11}) – (A_{14})$. Therefore, $m(t) < 0$ for $t \geq t_2$, that is, $z(t) < K(t)$ for $t \geq t_2$ implies that $z(t)$ is bounded and hence $m(t)$ is bounded for $t \geq t_2$. Since $m(t)$ is monotonic, then $\lim_{t \rightarrow \infty} m(t)$ exists. On the other hand, $u(t)$ is unbounded and therefore it suggests that either $u(t) > u(\alpha(t))$ or $u(t) < u(\alpha(t))$ for $t \geq t_2$. If the former holds, then

$$m(t) = u(t) + r(t)u(\alpha(t)) - K(t) > (1 + r(t))u(\alpha(t)) - K(t) \rightarrow \infty \text{ as } t \rightarrow \infty$$

gives a contradiction. If the later holds, then

$$u(t) < u(\alpha(t)) < u(\alpha^2(t)) < \dots < u(\alpha^n(t)) < \dots,$$

leads to the fact that $u(t)$ is bounded due to (A_{15}) which is absurd. Hence, the theorem is proved. ■

THEOREM 3. Let $-\infty < r(t) < -1$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Assume that all conditions of Theorem 2 hold. If

$$(A_{16}) \quad \int_0^\infty g(t)dt = \infty,$$

then every unbounded solution of (1.3) oscillates.

Proof. On the contrary, we proceed as in the proof of Theorem 2 and the case $m(t) > 0$ is similar. When $m(t) < 0$ for $t \geq t_2$, we have that $z(t) < K(t)$ for $t \geq t_2$, that is, $z(t)$ is bounded and thus $m(t)$ is bounded for $t \geq t_2$. Because $m(t)$ is monotonic, $\lim_{t \rightarrow \infty} m(t)$ exists. Here, we consider *Cases(b) – (f)* of Lemma 2.

Cases (b), (d), (e) Since $u(t)$ is unbounded, then either $u(t) > u(\alpha(t))$ or $u(t) < u(\alpha(t))$ for $t \geq t_2$. Suppose that $u(t) < u(\alpha(t))$ for $t \geq t_2$. Then

$$m(t) = u(t) + r(t)u(\alpha(t)) - K(t) < (1 + r(t))u(\alpha(t)) - K(t) \rightarrow -\infty \text{ as } t \rightarrow \infty$$

which gives a contradiction to the existence of the limit. Hence, $u(t) > u(\alpha(t))$ implies that

$$u(t) > u(\alpha(t)) > u(\alpha^2(t)) > \dots > u(\alpha^n(t)) > \dots,$$

that is, $\liminf_{t \rightarrow \infty} u(t) > 0$. Let there exist a $t_3 > t_2$ and $\eta > 0$ such that $u(\beta(t)) > \eta$ for $t \geq t_3$. Integrating (3.1) from t_3 to ∞ , we get a

contradiction to (A_{16}) . In *Cases(c)* and *(f)*, it is immediate to see that $\lim_{t \rightarrow \infty} L_1 m(t) = -\infty$ and $\lim_{t \rightarrow \infty} L_0 m(t) = -\infty$, that is, these cases are not possible. Hence, the proof of the theorem is complete. ■

THEOREM 4. *Let there exist $0 < a < \infty$ such that $0 \leq r(t) \leq a < \infty$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Assume that (A_{00}) and $(A_1) - (A_8)$ hold. Then every unbounded solution of (1.3) oscillates.*

Proof. The proof of the theorem follows from Lemma 1 and Theorem 1. Hence, the details are omitted. ■

THEOREM 5. *Let $-1 < r(t) \leq 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. If $(A_{00}), (A_1), (A_2), (A_5), (A_6), (A_{11})$ and (A_{12}) hold, then every unbounded solution of (1.3) oscillates.*

Proof. The proof of the theorem follows from Lemma 1 and Theorem 2 and hence the details are omitted. ■

THEOREM 6. *Let $-\infty < r(t) < -1$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Assume that all conditions of Theorem 5 hold. If (A_{16}) hold, then every unbounded solution of (1.3) oscillates.*

Proof. The proof of the theorem follows from Theorem 5 and Theorem 3 and hence the details are omitted. ■

REMARK 2. *In this work, we didn't discuss about the existence of nonoscillatory solutions of (1.3). However, the results of [9] and [16] are taken into account.*

REMARK 3. *It would be interesting to apply the results of this work to the following nonlinear dynamic equations:*

$$\left(p(t) \left((u(t) + r(t)u(\alpha(t)))^{\Delta^2} \right) \right)^{\Delta^2} + \sum_{i=1}^n (-1)^{i+1} g_i(t) G_i(u(\beta_i(t))) = 0.$$

Example. Let $\mathbb{T} = \mathbb{R}$ and consider Theorem 5. In this case,

$$\frac{(s-t)(t-v)}{p(t)} dt, \quad s > t > v,$$

$$C[v, u] = \int_u^v \frac{(t-u)(v-t)}{p(t)} dt, \quad v > t > u,$$

that is, $B[s, v] = C[v, u]$, and $(A_5), (A_{11}), (A_{12})$ reduce to

$$\int_T^\infty \frac{s}{p(s)} \int_s^\infty t h(t) dt ds < \infty, \quad T > 0,$$

$$\limsup_{s \rightarrow \infty} \int_{\beta(s)}^s g(\theta) G[B(\beta(\theta), \beta(s))] d\theta > \frac{1}{M_1}, \quad \limsup_{t \rightarrow \infty} \int_{\beta(t)}^t g(\theta) G[C(\beta(\theta), \beta(t))] d\theta > \frac{1}{M_1},$$

that is, $(A_{11}) = (A_{12})$ respectively. Consider

$$\left(e^{-t} \left(u(t) + \left(-\frac{1}{25} e^\pi \right) u(t - \pi) \right) \right)'' + g(t)G(u(t - \frac{3\pi}{2})) - h(t)H(u(t - \frac{\pi}{2})) = 0$$

for $t \in [\frac{\pi}{2}, \infty)$, where $\alpha(t) = t - \pi, \beta(t) = t - \frac{3\pi}{2}, \gamma(t) = t - \frac{\pi}{2}, G(u) = u, H(u) = \frac{u}{1+u^2}, g(t) = e^{\frac{3\pi}{2}}(\frac{52}{25}e^{-t} - e^{-3t})$ and $h(t) = e^{\frac{\pi}{2}}(e^{-2t} + e^{-\pi} \cos^2 t)e^{-t}$. Indeed,

$$\int_0^\infty \frac{s}{p(s)} \int_s^\infty t h(t) dt ds = \int_0^\infty \frac{s}{e^{-s}} \int_s^\infty t [e^{\frac{\pi}{2}}(e^{-2t} + e^{-\pi} \cos^2 t)e^{-t}] dt ds < \infty$$

and

$$\limsup_{s \rightarrow \infty} \int_{s-\beta}^s B[\theta - \beta, s - \beta] g(\theta) d\theta > 1$$

in which

$$B[s, v] = \int_v^s \frac{(s-t)(t-v)}{e^{-t}} dt.$$

By the generalized Leibnitz formula, we notice that

$$\frac{dB}{ds} = se^s [1 - \frac{1}{s} - \frac{v}{s}] + e^v \rightarrow \infty \text{ as } s \rightarrow \infty.$$

Hence, we can find a constant $C > 0$ such that $B[s, v] \geq Ce^s$. Therefore,

$$\begin{aligned} & \int_{s-\beta}^s B[\theta - \beta, s - \beta] g(\theta) d\theta \\ & \geq C \int_{s-\beta}^s e^\theta e^{\frac{3\pi}{2}} (\frac{52}{25} e^{-\theta} - e^{-3\theta}) d\theta = \\ & = Ce^{\frac{3\pi}{2}} \left[\frac{52\beta}{25} + \frac{e^{-2s}}{2} (1 - e^{2\beta}) \right] > 1 \end{aligned}$$

as $s \rightarrow \infty$. Clearly, all conditions of Theorem 5 are satisfied. Hence (3.9) is oscillatory. In particular, $u(t) = e^t \sin t$ is such an unbounded oscillatory solution of (3.9). **Example.** On $\mathbb{T} = \mathbb{Z}$, consider Theorem 2. In this case

$$\begin{aligned} B[s, v] &= \sum_{t=v}^s \frac{(s-t-1)(t-v)}{p(t)}, s > t+1 > v, \\ C[v, u] &= \sum_{t=u}^v \frac{(t+1-u)(v-t)}{p(t)}, v > t > u, \\ D[v, u] &= \sum_{t=u}^v \frac{(t+1-u)(t-u)}{p(t)}, v > t > u. \end{aligned}$$

Let's consider a nonlinear neutral difference equation

$$\Delta^2(\Delta(u(t) + r(t)u(t-2))) + g(t)u(t-1) - h(t)\frac{u(t-2)}{1+u^2(t-2)} = 0, \quad t \geq 3,$$

where $M_1 = 1, r(t) = -1 < -\frac{e}{e^t} \leq 0, p(t) = e^t, h(t) = e^2 \frac{(e^4 + e^{2t})}{(t+1)e^{4t}}$ and $g(t) = e(e+1)^2(e^2+1)^2e^t + 4e(e^{-1}+1)^2e^{-t} - \frac{e}{(t+1)e^{4t}}$. For (3.10), we have

$$\begin{aligned} (A_5) \quad & \sum_{s=T}^{\infty} \frac{s+1}{p(s)} \sum_{t=s}^{\infty} (t+1)h(t) < \infty, \quad T > 0, \\ (A_{11}) \quad & \limsup_{s \rightarrow \infty} \sum_{\theta=s-1}^s g(\theta)[B(\theta-1, s-1)] > 1, \\ (A_{12}) \quad & \limsup_{t \rightarrow \infty} \sum_{\theta=t-1}^t g(\theta)[C(\theta-1, t-1)] > 1, \\ (A_{13}) \quad & \limsup_{\theta \rightarrow \infty} \sum_{v=\theta-2}^{\theta-1} g(v)[D(v-1, \theta-1)] > 1, \\ (A_{14}) \quad & \limsup_{s \rightarrow \infty} \sum_{\theta=s-1}^s g(\theta)[D(s-1, \theta-1)] > 1. \end{aligned}$$

Clearly,

$$\sum_{s=T}^{\infty} \frac{s+1}{p(s)} \sum_{t=s}^{\infty} (t+1)h(t) = \sum_{s=1}^{\infty} \frac{s+1}{e^s} \sum_{t=s}^{\infty} (t+1)e^2 \frac{(e^4 + e^{2t})}{(t+1)e^{4t}} < \infty.$$

If we denote

$$[B(\theta-1, s-1)] = \sum_{t=s-1}^{\theta-1} \frac{(\theta-t-2)(t-s+1)}{e^t} = \sum_{t=s}^{\theta-1} \frac{(\theta-t-2)(t-s+1)}{e^t} = B(\theta) \text{ (say)}$$

for $\theta-1 > t+1$, then

$$\begin{aligned} \Delta B(\theta) &= \sum_{t=s}^{\theta} \frac{(\theta-t-1)(t-s+1)}{e^t} - \sum_{t=s}^{\theta-1} \frac{(\theta-t-2)(t-s+1)}{e^t} \geq \\ &\geq \sum_{t=s}^{\theta-1} \frac{(t-s+1)}{e^t} > 0. \end{aligned}$$

Therefore, we can find a constant $C_1 > 0$ such that $[B(\theta-1, s-1)] \geq C_1$. Similarly, we can find constants $C_2 > 0$ and $C_3 > 0$ such that $[C(\theta-1, t-1)] \geq C_2$ and $[D(v-1, \theta-1)] \geq C_3$. Consequently,

$$\begin{aligned} \limsup_{s \rightarrow \infty} \sum_{\theta=s-1}^s g(\theta)[B(\theta-1, s-1)] &\geq C_1 \limsup_{s \rightarrow \infty} \sum_{\theta=s-1}^s g(\theta) = \\ &= C_1 \limsup_{s \rightarrow \infty} \sum_{\theta=s-1}^s \left[e(e+1)^2(e^2+1)^2e^{\theta} + 4e(e^{-1}+1)^2e^{-\theta} - \frac{e}{(\theta+1)e^{4\theta}} \right] > 1 \end{aligned}$$

for every $C_1 > 0$. Also, it is easy to see that (A_{12}) , (A_{13}) and (A_{14}) hold true for every $C_2, C_3 > 0$. As a result, (3.10) satisfies all conditions of Theorem 2. In particular, $u(t) = (-e)^t$ is such an unbounded oscillatory solution of (3.10). **Acknowledgement** The author is thankful to the three referees for their careful reading and valuable suggestions about the completion of this work in this format.

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