THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH NONLINEAR NEUTRAL TERMS

G. CHATZARAKIS * AND S. GRACE †

Abstract. This paper deals with the oscillatory and asymptotic behavior of third-order nonlinear differential equations with nonlinear neutral terms. We present new oscillation criteria, which improve, extend and simplify existing ones in the literature. Two examples are provided in order to illustrate the significance of our main results.

Key Words. Oscillation, convergence, nonlinear neutral term, third-order differential equations.

AMS(MOS) subject classification. 34K05, 34K11, 34K25

1. Introduction. This paper is concerned with oscillatory and asymptotic behavior of all solutions of the nonlinear third-order differential equations with nonlinear neutral terms of the form

$$(1.1) \quad (a(t) (y''(t))^\alpha)' + q(t)x^\gamma(\tau(t)) = 0, \quad t \geq t_0 \geq 0,$$

where

$$y(t) = x(t) \pm p(t)x^\beta(\sigma(t)),$$

$\alpha \geq 1$, β and γ are the ratios of positive odd integers, $a(t) : [t_0, \infty) \rightarrow \mathbb{R}^+$ is a nondecreasing function, $p, q : [t_0, \infty) \rightarrow \mathbb{R}^+$ are continuous functions, and $\tau, \sigma : [t_0, \infty) \rightarrow \mathbb{R}$ are continuous functions such that $\tau(t) \leq t$, $\sigma(t) \leq t$ and $\lim_{t \rightarrow \infty} \tau(t) = \infty$. Moreover, $\sigma(t)$ is strictly increasing and $h(t) := \sigma^{-1}(\tau(t)) \leq t$ with $\lim_{t \rightarrow \infty} h(t) = \infty$.

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We study (1.1) under the conditions

$$(1.2) \quad \lim_{t \rightarrow \infty} A(t, t_0) = \infty \quad \text{where} \quad A(v, u) := \int_u^v \frac{1}{a^{1/\alpha}(s)} ds$$

and

$$(1.3) \quad \begin{aligned} & \int_{t_0}^{\infty} q(s) ds = \infty \\ & \text{or} \\ & \int_{t_0}^{\infty} q(s) ds < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \int_{t_0}^v \left(\frac{1}{a(u)} \int_u^{\infty} q(s) ds \right)^{1/\alpha} dudv = \infty. \end{aligned}$$

By a solution of (1.1), we mean a real function $x(t)$ continuous on $[T_x, \infty)$, $T_x \geq t_0$ which satisfies (1.1) for all $t \in [T_x, \infty)$. We consider only those solutions $x(t)$ which satisfy

$$\sup \{|x(t)| : t \geq T \geq T_x\} > 0.$$

Such a solution is called *oscillatory*, if it has arbitrarily large zeros. Otherwise, the solution is said to be *nonoscillatory*. Equation (1.1) is *oscillatory* if all its solutions oscillate.

In recent years, the oscillation theory of functional differential equations has received much attention since it has a great number of applications in engineering and natural sciences. For some related contributions on the oscillatory behavior of various classes of functional differential equations, we refer the reader to [1]–[28] and the references cited therein.

Neutral delay differential equations can be applied in several disciplines/fields of science and mathematics, including circuit theory, bifurcation analysis, population dynamics, stability theory, the dynamics of delayed network systems and others. They also occur in problems dealing with vibrating masses attached to elastic bar and in some variational problems, see [21]. As a result of the wide range of applications, neutral differential equations have attracted a great interest in the literature.

The problem of the oscillation of solutions of differential equations has been widely studied by many authors and by many techniques since the pioneering work of Sturm on second-order linear differential equations. In the past 30 years, the oscillation theory for second-order neutral delay differential equations and third-order retarded delay differential equations have been well developed; see, for example, the monographs [1, 4] and papers [3, 10, 11, 13–18] as well as the references cited therein. Compared to second-order neutral delay differential equations, it seems that not much work has been done concerning with the oscillation and asymptotic of third-order neutral

differential equations [10, 17]. However, to the best of our knowledge, there are no results for third-order differential equations with mixed neutral terms.

The aim of the present paper is to initiate the study of the oscillation problem of (1.1) with via comparison with first-order equations whose oscillatory behavior are known. The obtained results are new, improve and correlate many of the known oscillation criteria appeared in the literature even for the case of equation (1.1) with $p(t) = 0$.

2. Main Results. The proofs of our main results are essentially based on the following lemma.

LEMMA 1. *Let $q : [t_0, \infty) \rightarrow \mathbb{R}^+$, $g : [t_0, \infty) \rightarrow \mathbb{R}$ are continuous functions, and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing continuous function. Moreover, $xf(x) > 0 \forall x \neq 0$ and $g(t) \leq t$ with $\lim_{t \rightarrow \infty} g(t) = \infty$. If the first-order delay differential inequality*

$$y'(t) + q(t)f(y(g(t))) \leq 0, \quad t \geq t_0 \geq 0$$

has an eventually positive solution, then the first-order delay differential equation

$$y'(t) + q(t)f(y(g(t))) = 0, \quad t \geq t_0 \geq 0$$

has an eventually positive solution.

REMARK 1. *This lemma is an extension of known results in [23, 24 and 26] and the proof is immediate.*

2.1. $y(t) = x(t) + p(t)x^\beta(\sigma(t))$.

THEOREM 1. *Assume that (1.2) and (1.3) hold. If $\beta \leq 1$, $\lim_{t \rightarrow \infty} p(t) = 0$, and there exists $\theta \in (0, 1)$ such that the first-order delay differential equation*

$$(2.1) \quad W'(t) + \theta q(t) \left(\int_{t_0}^{\tau(t)} s a^{-1/\alpha}(s) ds \right)^\gamma W^{\gamma/\alpha}(\tau(t)) = 0$$

is oscillatory, then every solution of (1.1) is oscillatory or tends to zero.

Proof. Assume, for the sake of contradiction, that $x(t)$ is a nonoscillatory solution of (1.1). Then it is either eventually positive or eventually negative. As $-x(t)$ is also a solution of (1.1), we may restrict ourselves only to the case where $x(t) > 0$ for all large t . Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1 \geq t_0$. In view of this, Eq.(1.1) becomes

$$(2.2) \quad (a(t)(y''(t))^\alpha)' = -q(t)x^\gamma(\tau(t)) \leq 0, \quad t \geq t_1,$$

which means that $a(t)(y''(t))^\alpha$ is nonincreasing and of one sign. We claim that, there exists a $t_2 \geq t_1$ such that $y''(t) > 0$ for all $t \geq t_2$. If not, there

exists a $t_2 \geq t_1$ such that and a constant $b > 0$ such that $a(t)(y''(t))^\alpha \leq -b$ for $t \geq t_2$. Integrating this inequality from t_2 to t and using (1.2), we have

$$y'(t) \leq y'(t_2) - b \int_{t_2}^t a^{-1/\alpha}(s) ds \rightarrow -\infty \text{ as } t \rightarrow \infty,$$

which is a contradiction. Thus $y''(t) > 0$ for all $t \geq t_2$.

We shall distinguish the following two cases:

$$y(t) > 0 \quad \text{and} \quad y'(t) > 0, \quad \text{Case (I)}$$

or

$$y(t) > 0 \quad \text{and} \quad y'(t) < 0, \quad \text{Case (II).}$$

Case (I): Since $y(t) = x(t) + p(t)x^\beta(\sigma(t))$, we have

$$\begin{aligned} x(t) &= y(t) - p(t)x^\beta(\sigma(t)) = y(t) \left(1 - \frac{p(t)x^\beta(\sigma(t))}{y(t)} \right) \\ &\geq y(t) \left(1 - \frac{p(t)y^\beta(t)}{y(t)} \right) = y(t) \left(1 - \frac{p(t)}{y^{1-\beta}(t)} \right). \end{aligned}$$

Taking into account the fact that $y(t)$ is nondecreasing, there exists a real constant $k > 0$ such that $y(t) \geq k$. In view of this, the last inequality becomes

$$x(t) \geq \left(1 - \frac{p(t)}{k^{1-\beta}} \right) y(t),$$

or

$$(2.3) \quad x(t) \geq cy(t),$$

where $c \in (0, 1)$.

Combining (2.2) and (2.3), we get

$$(2.4) \quad (a(t)(y''(t))^\alpha)' \leq -c^\gamma q(t)y^\gamma(\tau(t)).$$

Since the function $a(t)$ is nondecreasing, we obtain $y'''(t) \leq 0$ for all and so, we get $y''(t) > 0$ and $y'(t) > 0$ for every $t \geq t_2$. Now, one can easily see that there exists a constant $\theta \in (0, 1)$ such that

$$y'(t) \geq \theta ty''(t) = \theta ta^{-1/\alpha}(t) (a^{1/\alpha}(t) y''(t)).$$

Integrating this inequality from t_1 to t , we obtain

$$(2.5) \quad y(t) \geq \left(\theta \int_{t_1}^t sa^{-1/\alpha}(s) ds \right) (a^{1/\alpha}(t) y''(t)).$$

Combining (2.4) and (2.5), and setting $W(t) = a(t)(y''(t))^\alpha$, we take

$$W'(t) + \theta^* q(t) \left(\int_{t_1}^{\tau(t)} s a^{-1/\alpha}(s) ds \right)^\gamma W^{\gamma/\alpha}(\tau(t)) \leq 0,$$

where $\theta^* \in (0, 1)$. Hence, in view of Lemma 1, (2.1) has a positive solution, which contradicts our hypothesis.

Case (II): By (1.3), it is easy to see that any solution converges to zero.

The proof of the theorem is complete. $\square$

REMARK 2. As a consequence of Theorem 1, we derive the following corollary for Eq. (1.1):

COROLLARY 1. Assume that (1.2) and (1.3) hold, $\lim_{t \rightarrow \infty} p(t) = 0$ and $\gamma \leq \alpha$. If

$$(2.6) \quad \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t q(u) \left(\int_{t_0}^{\tau(u)} s a^{-1/\alpha}(s) ds \right)^\gamma du = \infty,$$

then every solution of (1.1) is oscillatory or tends to zero.

EXAMPLE 1. Consider the nonlinear third-order differential equation

$$(2.7) \quad \left(t^3 \left(\left(x(t) + \frac{1}{t} x^{1/3} \left(\frac{t}{2} \right) \right)'' \right)^3 \right)' + x \left(\frac{t}{2} \right) = 0, \quad t \geq 1.$$

Here, $a(t) = t^3$, $p(t) = \frac{1}{t}$, $q(t) = 1$, $\gamma = 1$, $\alpha = 3$, $\beta = \frac{1}{3}$ and $\tau(t) = \sigma(t) = \frac{t}{2}$.

By simple calculations one can deduce that

$$\lim_{t \rightarrow \infty} p(t) = \lim_{t \rightarrow \infty} \frac{1}{t} = 0,$$

$$\lim_{t \rightarrow \infty} A(t, 1) = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{s} ds = \infty,$$

$$\int_1^\infty ds = \infty$$

and

$$\liminf_{t \rightarrow \infty} \int_{t/2}^t \left(\int_1^{u/2} s (s^3)^{-1/3} ds \right) du = \infty.$$

That is, all conditions of Corollary 1 are satisfied, and therefore every solution of (2.7) is oscillatory or tends to zero.

Observe, however, that the results in [17] and [19] are not applicable to equation (2.7) because of the presence of sublinear neutral term.

2.2. $y(t) = x(t) - p(t)x^\beta(\sigma(t))$.

THEOREM 2. *Assume that (1.2) and (1.3) hold, and $\beta \leq 1$. Furthermore, suppose that there exist $\theta \in (0, 1)$ and a continuous function $\xi(t)$ such that $h(t) \leq \xi(t) \leq t$ with $\lim_{t \rightarrow \infty} \xi(t) = \infty$. If the first-order delay differential equations*

$$W'(t) + \theta q(t) \left(\int_{t_0}^{\tau(t)} s a^{-1/\alpha}(s) ds \right)^\gamma W^{\gamma/\alpha}(\tau(t)) = 0$$

and

$$X'(t) + \theta Q(t) \left(\frac{h(t)}{a^{1/\alpha}(\xi(t))} (\xi(t) - h(t)) \right)^{\gamma/\beta} (X(\xi(t)))^{\gamma/\alpha\beta} = 0,$$

where

$$Q(t) = \frac{q(t)}{(p(\sigma^{-1}(\tau(t))))^{\gamma/\beta}}$$

are oscillatory, then every solution of (1.1) is oscillatory or tends to zero.

Proof. Assume, for the sake of contradiction, that $x(t)$ is a nonoscillatory solution of (1.1). Then it is either eventually positive or eventually negative. As $-x(t)$ is also a solution of (1.1), we may restrict ourselves only to the case where $x(t) > 0$ for all large t . Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1 \geq t_0$. In view of this, Eq.(1.1) becomes

$$(2.2) \quad (a(t)(y''(t))^\alpha)' = -q(t)x^\gamma(\tau(t)) \leq 0, \quad t \geq t_1,$$

which means that $a(t)(y''(t))^\alpha$ is nonincreasing and of one sign. As in the proof of Theorem 1, there exists a $t_2 \geq t_1$ such that $y''(t) > 0$ for all $t \geq t_2$.

We shall distinguish the following four cases:

$$y(t) > 0 \quad \text{and} \quad y'(t) > 0, \quad \text{Case (I)}$$

or

$$y(t) > 0 \quad \text{and} \quad y'(t) < 0, \quad \text{Case (II)}$$

or

$$y(t) < 0 \quad \text{and} \quad y'(t) > 0, \quad \text{Case (III)}$$

or

$$y(t) < 0 \quad \text{and} \quad y'(t) < 0, \quad \text{Case (IV)}.$$

Case (I): Since $y(t) = x(t) - p(t)x^\beta(\sigma(t))$, it is obvious that $x(t) \geq y(t)$. Thus

$$(2.8) \quad (a(t)(y''(t))^\alpha)' = -q(t)x^\gamma(\tau(t)) \leq -q(t)y^\gamma(\tau(t)).$$

Since $y''(t) > 0$ and $y'(t) > 0$ for every $t \geq t_2$, one can easily see that there exists a constant $\theta \in (0, 1)$ such that

$$y'(t) \geq \theta t y''(t) = \theta t a^{-1/\alpha}(t) (a^{1/\alpha}(t) y''(t)).$$

Integrating this inequality from t_1 to t , we obtain

$$(2.9) \quad y(t) \geq \left(\theta \int_{t_1}^t s a^{-1/\alpha}(s) ds \right) (a^{1/\alpha}(t) y''(t)).$$

Combining (2.8) and (2.9), and setting $W(t) = a(t)(y''(t))^\alpha$, we take

$$W'(t) + \theta q(t) \left(\int_{t_1}^{\tau(t)} s a^{-1/\alpha}(s) ds \right)^\gamma W^{\gamma/\alpha}(\tau(t)) \leq 0.$$

Hence, in view of Lemma 1, the corresponding equation also has a positive solution, which is a contradiction.

Case (II): By (1.3), it is easy to see that any solution converges to zero.

Case (III): Let $z(t) = -y(t)$. Clearly, $z'(t) = -y'(t) < 0$ and $z''(t) < 0$. This is impossible due to condition (1.2).

Case (IV): It is obvious that

$$z(t) = -y(t) = -x(t) + p(t)x^\beta(\sigma(t)) \leq p(t)x^\beta(\sigma(t)),$$

or

$$x(\sigma(t)) \geq \left(\frac{z(t)}{p(t)} \right)^{1/\beta}.$$

Thus

$$x(t) \geq \left(\frac{z(\sigma^{-1}(t))}{p(\sigma^{-1}(t))} \right)^{1/\beta}.$$

Therefore

$$\begin{aligned} (a(t)(z''(t))^\alpha)' &= q(t)x^\gamma(\tau(t)) \\ &\geq \frac{q(t)}{(p(\sigma^{-1}(\tau(t))))^{\gamma/\beta}} z^{\gamma/\beta}(\sigma^{-1}(\tau(t))), \end{aligned}$$

or

$$(2.10) \quad (a(t)(z''(t))^\alpha)' \geq Q(t)z^{\gamma/\beta}(h(t)).$$

Also, there exists a constant $\theta \in (0, 1)$ such that

$$(2.11) \quad z(h(t)) \geq \theta t z'(h(t)).$$

Combining (2.10) and (2.11), we get

$$(a(t) (z''(t))^\alpha)' \geq Q(t) z^{\gamma/\beta}(h(t)) \geq \theta^{\gamma/\beta} Q(t) h^{\gamma/\beta}(t) (z'(h(t)))^{\gamma/\beta},$$

or

$$(2.12) \quad (a(t) (Z'(t))^\alpha)' \geq \theta^{\gamma/\beta} Q(t) h^{\gamma/\beta}(t) (Z(h(t)))^{\gamma/\beta},$$

where $Z(t) = z'(t) > 0$.

For $v \geq u \geq t_1$, we see that

$$Z(u) - Z(v) \geq (v - u) (-Z'(v)).$$

Setting $u = h(t)$ and $v = \xi(t)$, we have

$$(2.13) \quad Z(h(t)) \geq (\xi(t) - h(t)) (-Z'(\xi(t))).$$

Combining (2.12) and (2.13), we have

$$\begin{aligned} & (a(t) (Z'(t))^\alpha)' + \theta^{\gamma/\beta} Q(t) h^{\gamma/\beta}(t) (Z(h(t)))^{\gamma/\beta} - X'(t) \\ & \geq \theta^{\gamma/\beta} Q(t) \left(\frac{h(t)}{a^{1/\alpha}(\xi(t))} (\xi(t) - h(t)) \right)^{\gamma/\beta} (X(\xi(t)))^{\gamma/\alpha\beta}, \end{aligned}$$

where $X(t) = -a(t) (Z'(t))^\alpha$. The rest of the proof is similar to the above cases and hence is omitted.

The proof of the theorem is complete. $\square$

REMARK 3. *In fact, the conditions of Theorem 2 are very normal and can be applied easily. According to [23] and [24] we replace the two the first-order equations given in the Theorem by*

$$(2.14) \quad \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t q(u) \left(\int_{t_0}^{\tau(u)} s a^{-1/\alpha}(s) ds \right)^\gamma du = \infty, \quad \text{when } \gamma \leq \alpha$$

and

$$(2.15) \quad \liminf_{t \rightarrow \infty} \int_{\xi(t)}^t q(u) Q(u) \left(\frac{h(u)}{a^{1/\alpha}(\xi(u))} (\xi(u) - h(u)) \right)^{\gamma/\beta} du = \infty, \quad \text{when } \gamma \leq \alpha\beta.$$

In this case and instead of Theorem 2 we have:

COROLLARY 2. *Assume that (1.2), (1.3) and (2.1) hold, and $\beta \leq 1$. Furthermore, suppose that there exist $\theta \in (0, 1)$ and a continuous function $\xi(t)$ such that $h(t) \leq \xi(t) \leq t$ with $\lim_{t \rightarrow \infty} \xi(t) = \infty$. If (2.14) and (2.15) hold, then every solution of (1.1) is oscillatory or tends to zero.*

EXAMPLE 2. *Consider the nonlinear third-order differential equation*

$$(2.16) \quad \left(t^3 \left(\left(x(t) - \frac{1}{t} x^{1/3} \left(\frac{t}{2} \right) \right)'' \right)^3 \right)' + x \left(\frac{t}{4} \right) = 0.$$

Here, $a(t) = t^3$, $p(t) = \frac{1}{t}$, $q(t) = 1$, $\alpha = 3$, $\beta = \frac{1}{3}$, $\gamma = 1$ and $\tau(t) = \frac{t}{4}$. So $h(t) = \frac{t}{2}$ and let $\xi(t) = \frac{3t}{4}$. It is easy to check that the conditions of Corollary 2 are satisfied, and therefore every solution of (2.16) is oscillatory or tends to zero.

Observe, however, that the results in [17] and [19] are not applicable to (2.16).

3. Concluding Remarks. The paper is presented in a form which is essentially new. The obtained results improve, extend and simplify existing ones in the literature. See, for example, [17, 19].

It will be of interest to investigate the higher-order equations of the form

$$\left(a(t) \left(y^{(n-1)}(t) \right)^\alpha \right)' + q(t) x^\gamma (\tau(t)) = 0, \quad \mathbb{N} \ni n \geq 4.$$

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