RESEARCH ON SIMILARITY VALUES RELEVANT FOR FUZZY MATRIX CONSISTENCY CHECK

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Abstract

There is a problem of inherent inconsistency when forming pairwise comparison matrices in both classical and fuzzy Analytic Hierarchy Process. The smallest decision making element of fuzzy Analytic Hierarchy Process is square, positive, fuzzy reciprocal matrix 3x3, and this paper focus to the research on fuzzy numbers' similarities regarding the consistency check of fuzzy matrix 3x3. The research is performed using Java application created specifically for this purpose. The application, for the selected fuzzy scale, forms all 3x3 matrices, for each matrix calculates all similarities values relevant for consistency check, average similarity and determines the minimal similarity value. The application calculates CR (consistency ratio) values for corresponding classical matrices. Research results are presented and future research suggested.

Keywords: Pairwise Comparison Matrix; Consistency; Fuzzy Numbers, Fuzzy Scale; Similarity Measure; Fuzzy Analytic Hierarchy Process

1. Introduction

Numerous different methods are developed to solve multicriteria decision making problems, mostly related to the strict numerical valuation of selected criteria - examples of papers using such methods written by the authors relate to multicriteria evaluation of supervisory boards [1], selection of buildings' insulation [2] or selection of best alternative for highway tunnel doors [3].

One of frequently used multicriteria decision making aid methods is Analytic Hierarchy Process (classical, but also fuzzy). Limited research is done so far regarding evaluation of consistency in forming the pairwise comparison matrices, with few papers available dealing with the issue of consistency of fuzzy pairwise comparison matrices (examples are [4, 5, 6, 7]). According to the work of Buckley [8], fuzzy pairwise comparison matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is consistent if $\tilde{a}_{ik} \otimes \tilde{a}_{kj} \approx \tilde{a}_{ij}$ where $i, j, k = 1, 2, \dots, n$, and where " $\otimes$ " represents multiplication of fuzzy numbers, while " $\approx$ " means "fuzzy equal" [9].

The concept of fuzzy equality has the same meaning as the notion of similarity, since the similarity in essence is a generalization of the concept of equivalence [10]. Similarity is a relation which is an extension to the concept of functions

belonging to the set of elements, where, abandoning the idea of exact equality, the elements of the domain are considered to have different degrees of similarity [11].

The application of fuzzy numbers in resolving problems requires the application of different similarity measures, calculating the degree of similarity between two fuzzy number; similarity measures appropriate for solving one type of problem often are not appropriate for solving the other problem or type of problems. Therefore, various similarity measures of fuzzy numbers exist [11]. The similarity measure of triangular fuzzy numbers proposed by Chen and Lin ([7]) is based on distances between the midpoints and the appropriate boundary points of the support interval: if $A = (a^l, a^m, a^u)$ and $B = (b^l, b^m, b^u)$ are triangular fuzzy numbers, similarity can be calculated with

$$S_{MB}(A, B) = 1 - \frac{|a^l - b^l| + |a^m - b^m| + |a^u - b^u|}{3} \quad (1)$$

Hsieh and Chen have presented the utility value concept of triangular fuzzy numbers, and based on these values have also defined the similarity [9]:

$$S_{UV}(A, B) = \frac{U_{UV}(A) \times U_{UV}(B)}{\max((U_{UV}(A))^2, (U_{UV}(B))^2)}. \quad (2)$$

The similarity measure is function fulfilling the following two conditions [9]:

- Reflexivity, i. e. $S(A, A) = 1$ and
- Symmetry, i.e. $S(A, B) = S(B, A)$.

Similarity measure based on the distance is used in this paper (as defined in [9]):

$$S(A, B) = \frac{1}{1 + d(A, B)}, \quad (3)$$

where $d(A, B)$ is a measure of the distance defined with:

$$d(A, B) = |a^l - b^l| + |a^m - b^m| + |a^u - b^u|, \quad (4)$$

so the similarity measure used is

$$S(A, B) = \frac{1}{1 + |a^l - b^l| + |a^m - b^m| + |a^u - b^u|}. \quad (5)$$

This similarity measure fulfils the conditions of reflexivity and symmetry.

2. Research

Pairwise comparison matrices of the fuzzy Analytic Hierarchy Process (fAHP) based decision making models are positive and reciprocal. The smallest matrix in fAHP is 3x3 pairwise comparison matrix, square, positive and reciprocal:

$$\begin{bmatrix} (1 \ 1 \ 1) & a_{12} & a_{13} \\ 1/a_{12} & (1 \ 1 \ 1) & a_{23} \\ 1/a_{13} & 1/a_{23} & (1 \ 1 \ 1) \end{bmatrix}. \quad (6)$$

The elements a_{ij} ($i, j=1, 2, 3$) are obtained using appropriate fuzzification of scale given by Saaty [13]. Original Saaty's scale and fuzzified Saaty's scale are presented with Tables 1 & 2 respectively.

Definition	Importance intensity	Reciprocals
Equal importance	1	1
Moderate importance	3	1/3
Strong importance	5	1/5
Very strong importance	7	1/7
Extreme importance	9	1/9
Intermediate values	2, 4, 6 and 8	1/2, 1/4, 1/6 and 1/8

Table 1. Scale given by Saaty

Definition	Importance intensity	Reciprocals
Equal importance	(1, 1, 1)	(1, 1, 1)
Moderate importance	(2, 3, 4)	(1/4, 1/3, 1/2)
Strong importance	(4, 5, 6)	(1/6, 1/5, 1/4)
Very strong importance	(6, 7, 8)	(1/8, 1/7, 1/6)
Extreme importance	(9, 9, 9)	(1/9, 1/9, 1/9)
Intermediate values	(1, 2, 3), (3, 4, 5), (5, 6, 7) and (7, 8, 9)	(1/3, 1/2, 1), (1/5, 1/4, 1/3), (1/7, 1/6, 1/5) and (1/9, 1/8, 1/7)

Table 2. Fuzzified Saaty's scale

In order to explore relationship between the similarity measures value and consistency of fuzzy matrices, Java application is created and used. The application creates $17^{n(n-1)/2}$ matrices, where n is dimension of the particular matrix explored. For each created matrix application calculates similarity of fuzzy numbers $\tilde{a}_{ik} \otimes \tilde{a}_{kj}$ and $\tilde{a}_{ij}$, for every $i, j, k = 1, 2, \dots, n$. After that, for each matrix the application finds minimum, calculates the average similarity and forms MS Excel file where all the fuzzy matrices with the corresponding minimum and average similarities are given. The application also calculates and presents CR value (consistency ratio) [13] for classical matrices derived from the fuzzy matrices. Matrices are formed using classical matrices' elements a_{ij} representing the midpoint of the fuzzy numbers interval $\tilde{a}_{ij} = (a_{ij}^l, a_{ij}^m, a_{ij}^u)$. Thus, the elements of the classical matrix are $a_{ij} = a_{ij}^m$ for all $i, j = 1, 2, \dots, n$. Similarity values (minimal and average) and CR values for fuzzy and classical matrices 3x3 respectively are explored. It should be noted that these matrices, as the corresponding values of displayed indicators, would be different if fuzzy scales would be different than those presented in the Table 2.

3. Results and Discussion

For the case of selected fuzzy scale, all the values of minimal similarity of matrices 3x3 are rounded to 4 decimal places and they belong to the interval [0.0041,1]. There are total of 162 minimal similarity values, with the smallest pop-up value of 0.0041, for the selected fuzzy scale. Analysis of the output Excel file, containing all matrices so as corresponding minimal and average similarities and CR values of correspondent classic matrices, is performed. It was found that for the fuzzy matrices whose minimal similarities belong to the interval [0.0041,0.0294] all correspondent classical matrices have CR values greater than 10%, what means that they are not consistent. The total number of these matrices is 3576 and indicates the level of the problem of inconsistency check with matrices of dimension three. The total number of different minimal similarities within the range [0.0041,0.0294] is 122. For these matrices, the lowest average similarity is 0.6577, the largest average similarity is 0.8297, and the smallest CR is 10.1587%. The focus of further analysis of the minimal similarities values is at the values belonging to the interval [0.0303,1]. The values for these similarities (a total of 40 values) are presented with Table 3.

Ordinal Number	Minimal Similarity	Ordinal Number	Minimal Similarity	Ordinal Number	Minimal Similarity	Ordinal Number	Minimal Similarity
1.	0.0303	11.	0.0417	21.	0.0667	31.	0.2069
2.	0.0321	12.	0.0435	22.	0.0714	32.	0.2500
3.	0.0323	13.	0.0455	23.	0.0759	33.	0.2727
4.	0.0333	14.	0.0476	24.	0.0769	34.	0.4000
5.	0.0346	15.	0.0500	25.	0.0833	35.	0.4839
6.	0.0355	16.	0.0522	26.	0.0909	36.	0.5455
7.	0.0357	17.	0.0526	27.	0.1000	37.	0.5932
8.	0.0370	18.	0.0556	28.	0.1111	38.	0.6316
9.	0.0397	19.	0.0588	29.	0.125	39.	0.6632
10.	0.0400	20.	0.0625	30.	0.1429	40.	1

Table 3. Values of the minimal similarities from the interval [0.0303 to 1]

Analysis of the entire range of minimal values of similarities [0.0041, 1] has shown that:

- For the minimal similarities lower than 0.0303 there is no consistent classical matrix.
- For some minimal similarities within the range (0.0303, 0.1429) there are corresponding average similarities for which both consistent and inconsistent classical matrices exist.
- For some minimal similarities within the range (0.0303, 0.1] there are corresponding average similarities for which classical matrices are always consistent (Table 4).
- When the minimal similarities are higher than 0.1 (except if the minimal similarity is 0.1429 and corresponding average similarity is 0.7873, when CR of classical matrix is 12.7714%). classical matrices are consistent.,
- For minimal similarities higher than 0.1429 all the classic matrices are consistent.

Minimal Similarity	Average Similarity
0.0333	0.7919
0.0357	0.8154
	0.796
0.037	0.7545
	0.7627
	0.7641
0.0417	0.7583
	0.7652
	0.7962
	0.7969
0.0435	0.7673
	0.7985
0.0476	0.7494
	0.7513
0.05	0.7621
	0.7984
0.0526	0.7731
0.0556	0.7245
	0.7425
	0.7637
	0.8135
0.0588	0.7559
	0.7766
0.0625	0.7699
	0.8017
0.0667	0.7312

Minimal Similarity	Average Similarity
	0.7459
	0.7762
0.0714	0.7316
	0.7485
	0.7647
	0.7658
	0.7999
0.0769	0.7677
	0.7888
	0.7966
	0.8338
0.0833	0.7159
	0.7767
	0.7992
0.0909	0.7105
	0.7403
	0.7492
	0.7872
0.1	0.7126
	0.7586
	0.7734
	0.7808
	0.8015
	0.8166
	0.8509

Table 4. Minimal and average similarities in the range (0.0303 to 0.1]. CR <10%

4. Conclusion

Both for classical and for fuzzy Analytic Hierarchy Process, a problem of inconsistency of decision maker appears when forming the pairwise comparison matrices and this paper focuses on resolving it. Appropriate fuzzy scale, obtained by fuzzification of Saaty's scale, is used in this paper. Research on similarities of fuzzy numbers, relevant in terms of consistency check, resulted with the conclusion that the inconsistency issues appear even with matrices of smaller dimensions. For the case of selected fuzzy scale, all the values of minimal similarity of matrices 3x3, rounded to 4 decimal places, belong to the interval [0.0041, 1]. It is found that for the fuzzy matrices whose minimal similarities are within the interval [0.0041, 0.0294] all correspondent classical matrices have CR values greater than 10%, what means they are not consistent. Total number of these matrices is 3576 (total number of fuzzy matrices is 4913). For some minimal similarities within the range (0.0303, 0.1] there are corresponding average similarities for which classical matrices are always consistent. For minimal similarities higher than 0.1 (except if the minimal similarity is 0.1429 and corresponding average similarity is 0.7873, when CR of classical matrix is 12.7714%). classical matrices are consistent, and for minimal similarities higher than 0.1429 all the related classic matrices are consistent.

Since the matrix 3x3 is the basic element in fuzzy (and classical) Analytic Hierarchy Process, consistency check of fuzzy matrices of dimensions larger than 3 is feasible using decomposition of larger matrices into the matrices 3x3, what will be the subject of the further research.

5. References

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