Recovering current density from electric potential data

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Abstract

In this paper, an inverse problem for galvanic corrosion modelled by the two-dimensional Laplace's equation was studied. The considered problem deals with experimental measurements of electric potential, where due to lack of data, numerical integration is impossible. The problem is reduced to the determination of unknown complex coefficients of approximating functions, which are related to the known potential and unknown current density. Employing continuity of those functions along subdomain interfaces and using condition equations for known data leads to an over-determined system of linear algebraic equations which are subjected to experimental errors. Reconstruction of current density is unique. The reconstruction contains one free additive parameter which does not affect current density. The method is useful in situations where limited data on electric potential are provided.

Keywords: current density, potential measurements, reconstruction, complex variables.

1 Introduction

The evaluation of field of current density is essential in problems of galvanic corrosion. In many cases the direct measurement of current density is not feasible, while the electric potential can be obtained from experimental measurements. This is particularly true in the case of cathodic protection systems in general, where many surveying techniques (for example, DCVG and CIS for underground structures) rely in potential measurements at different points at the electrolyte in order to identify the current distribution along the metallic structures.

In these cases, it is common that the region of interest for collecting data is not accessible to the operator and in order to exploit the available measurements some indirect method is required.

There are many studies that employ numerical methods for the forward modelling of galvanic corrosion problem. In order to obtain or verify results, these techniques are based mainly on boundary value problems (BVP) formulations, such as the finite element method (FEM), finite difference method (FDM) or boundary element method (BEM). These methods are successfully used and shown to be very accurate to solve BVPs. Some of them are also implemented in commercial software.

Situations may arise, where the boundary conditions are unknown and only some experimental data in certain locations are known. In this case, the problem is defined as an inverse one. This situation often occurs in many branches of science and mathematics where only the values of some model parameters can be obtained from observed data or measured data. Data on electric potential can be obtained in galvanic corrosion as a set of discrete data with one free parameter due to measuring potential differences. This situation, where measurements on electric potential can be provided as a set of discrete data within simply connected domain Ω imposes the problem to be inverse.

The theory of complex variables [1, 2] is used in order to connect current density and electric potential with one holomorphic function. The real part of it represents electric potential and the derivative is related to current density. The domain is divided into smaller subdomains where the holomorphic function is approximated with a quadratic function and its derivative with a piecewise linear function. These approximating functions obey continuity across subdomain interfaces.

A similar method was used in Irša and Galybin [3] with heat flow and in Galybin and Mukhamediev [4] and Mukhamediev et al. [5] with stress trajectories. The purpose of this work is to develop an efficient method for identification of current densities within a domain where some data on electric potential are provided.

2 Galvanic current density distribution in electrolyte

Current density is a measure of the density of flow of a conserved charge. The equation governing the distribution of potential and current flow in an electrolyte can be derived from the continuity equation governing charge conservation. The divergence of the current density $\mathbf{I}$ is equal to the negative rate of change of the charge density q [6]:

$$\nabla \cdot \mathbf{I} = -\frac{\partial q}{\partial t} \quad (1)$$

For steady state current distribution, $\partial q / \partial t = 0$ in uniform isotropic electrolyte, where the conductivity of electrolyte k is constant and the electric potential ϕ satisfies Laplace's equation:

$$\Delta\phi = 0 \quad (2)$$

The current density can be defined at any point as:

$$I_j = -k \frac{\partial\phi}{\partial x_j} \quad (3)$$

The fact that the electric potential $\phi(x, y)$ is harmonic, as a consequence of eqn (2), allows the use of the theory of complex variables for 2D potential problems. Introducing the harmonic conjugate function $\psi(x, y)$, where both ϕ and ψ satisfy Cauchy-Riemann equations (CR), one has the holomorphic function:

$$W(z) = \phi(x, y) + i\psi(x, y) \quad (4)$$

where the real part is the electric potential and the imaginary part is a function harmonic conjugated, related to fluxes. The surfaces $\phi(x, y) = \text{constant}$ are equipotential and $\psi(x, y) = \text{constant}$ represent flux lines. Using CR and transformation rules in calculus, one can see that the derivative of holomorphic function W is related to the current density as follows:

$$H(z) \equiv \frac{\partial W}{\partial z} = \frac{\partial\phi}{\partial x} - i \frac{\partial\phi}{\partial y} \quad (5)$$

Comparison of eqns (3) and (5) shows that derivative of holomorphic function, which has a real part equal to the electric potential is related to the conjugate current density $\tilde{I}$ by

$$H(z) = \tilde{I} / (-k).$$

3 Problem description

The problem is to recover the current density, based on experimental data for the electric potential. Let us consider a domain Ω , to be a simply connected domain or a subdomain of a bigger domain which is not necessarily simply connected and bounded. Let potential measurements be known at some discrete points ($j = 1, \dots, N$), located within Ω . The general problem is formulated as follows: *given the discrete data on electric potential $\phi_j = 1 \dots N$, find the field of current density within the domain Ω* . There are no other restrictions, such as data type or boundary conditions. The data can be redistributed uniformly or randomly.

3.1 On the uniqueness of the reconstructed solution

The problem deals with reconstruction of a holomorphic function, where the real and imaginary parts are harmonic and satisfy CR. Therefore having one function known, the second one can be easily obtained by integration of CR. Indeed, assume, for instance, that the real part of the holomorphic function is known analytically $\phi(x, y)$. CR allows finding the imaginary part by integration:

$$\psi(x, y) = \int \frac{\partial \varphi(x, y)}{\partial x} dy + p \quad (6)$$

This formula indicates that the reconstruction can be achieved but it will always have one undetermined parameter as a free parameter. In this particular case, the free parameter is an additive constant in the imaginary part $\psi(x, y)$ of holomorphic function $W(z)$. After performing the differentiation of $W(z)$, the free parameter vanishes. It is therefore obvious that from the electric potential, one can recover the current density without any free parameter. In the case when data are known at discrete points, reconstruction, in general, is non-unique and can have any finite number of free parameters, which is evident from the following. Let an exact solution $f(z) = u(x, y) + iv(x, y)$ of the problem be found by any method. It is obvious, that the modulus must be real; therefore it satisfies the following:

$$\text{Im} \left[f(z_j) e^{-i\omega(z_j)} \right] = 0, j = 1 \dots N \quad (7)$$

where ω is the argument of $f(z)$. Let us consider a function $P(z)$ in the form:

$$P(z) = \prod_{j=1}^m C_j (z - z_j)^{m_j} \quad (8)$$

where C_j are arbitrary complex parameters, m_j are arbitrary positive integers and z_j are the data points. It is evident from eqn (8) that the sum, $f(z) + P(z)$, satisfies condition (7) and therefore it is also the solution of the problem.

In order to be meaningful, the solution has to be sought within a certain class of functions. Hereafter the solution is sought as a set of piecewise linear holomorphic functions. It will be shown that this set of functions allows reconstructing the current density with high accuracy.

4 Mathematical model

A similar mathematical formulation has been described in Irša and Galybin [3], where the required function was related to the modulus of holomorphic function. In this case the required current density is related directly to the derivative of the holomorphic function $W(z)$.

The domain Ω is divided into n smaller subdomains (elements) of an arbitrary shape, as shown in Fig. 1. In order to achieve a piecewise linear function for the current density, the function $W(z)$ must be approximated by a quadratic function. Both these approximations obey continuity along the interfaces (at certain collocation points located on the interfaces).

For function $H(z)$, related to current density, the approximating function is required to be the piece-wise linear function:

$$\chi_H^{(m)}(z) = a^{(m)} + b^{(m)}z, \quad m = 1 \dots n \quad (9)$$

For function $W(z)$, which has real part the electric potential, the approximating function is therefore the integral of $H(z)$:

$$\chi_W^{(m)}(z) = c^{(m)} + a^{(m)}z + \frac{1}{2}b^{(m)}z^2, \quad m = 1 \dots n \quad (10)$$

Here $a^{(m)} = a_0^{(m)} + ia_1^{(m)}$, $b^{(m)} = b_0^{(m)} + ib_1^{(m)}$, $c^{(m)} = c_0^{(m)} + ic_1^{(m)}$ are 3 unknown complex constants; therefore 6 real constants are unknown in every element. The method is not restricted to these two particular approximating functions, however it will be demonstrated that the use of these two particular functions provides results with very high accuracy.

Figure 1: Discretization of the domain with approximating functions.

4.1 Discretization of the domain

In order to obey the continuity of the above mentioned functions, a grid was used consisting of square elements with collocation points (CP) on their interfaces. The CPs are distributed regularly.

The minimum number of nodes strictly depends on the number of unknowns in the approximating functions as well as the element type and number of known data. Should the system consist only from the continuity equations and no data imposed on electric potential, one can find the number of required collocation points as follows. Given that, the square elements are placed in the regular grid with J elements along x -axis and K elements along y -axis and 6 unknown real coefficients in every element, one finds the total number of unknowns N_{unkn} as:

$$N_{unkn} = 6N_{elem} = 6JK \quad (11)$$

The required number of collocation points depends on the number of element interfaces, which is $N_{int} = (J - 1)K + (K - 1)J$. Using the same number of collocation points per each interface, keeping in mind that separation of complex equations into real and imaginary parts presents two real equations at every collocation point, the required number of collocation points per interface is:

$$N_{cp} \geq \frac{N_{unkn}}{2_{int}} = \frac{6JK}{2[(J-1)K + (K-1)J]} = \frac{3JK}{2JK - J - K} \quad (12)$$

For a domain, where J or K is equal to 1, the number of collocation points per interface is $N_{cp} \geq 3$. In the general case, the number of collocation points per

interface $N_{cp} > 1$, therefore $N_{cp} = 2$ is used further on. Given dense data on the electric potential, one could reduce the number of CP with respect to the unknowns and the total number of equations.

Other types of mesh could be used, such as triangular or polygonal with straight or curvilinear sides. The square grid is convenient from computational point of view because the distances between collocation points are regular, which has good influence on the structure of the matrix of the system.

4.2 Equations of continuity and condition equations

The problem consists of continuity equations and two types of condition equations. For the k -th collocation point, lying on the interface between the elements numbered m and $m+1$, the equation of continuity for $W(z)$ is the following:

$$\chi_W^{(m)}(z_k) - \chi_W^{(m+1)}(z_k) = 0, \quad m = 1 \dots n; \quad k = 1 \dots N_{CP} \quad (13)$$

The second equation of continuity for $H(z)$ is expressed following:

$$\chi_H^{(m)}(z_k) - \chi_H^{(m+1)}(z_k) = 0, \quad m = 1 \dots n; \quad k = 1 \dots N_{CP} \quad (14)$$

The condition equation containing data on electric potential ϕ comes from $W(z)$, where the real part of it is equal to the electric potential, therefore:

$$\operatorname{Re}(\chi_W^{(m)}(z_j)) = \phi_j, \quad j = 1 \dots N \quad (15)$$

This condition equation is satisfied in elements, where the data occurs. It is obvious that from the collocation method, one can find the differences between interfering elements coefficients only. In the condition equation (15) all coefficients are presented except the imaginary part of c , which stays undetermined and cannot be obtained uniquely from the continuity equations. This coefficient is the free additive parameter discussed in section 3.1. This would decrease the rank of matrix and would require regularisation procedures. In order to avoid these procedures which contain additional parameters as regularization parameters among others, a second condition equation is used. The second condition equation is due to the free parameter c_1 ; it requires the average of all constants c_1 across whole domain to be zero. The condition is expressed as follows:

$$\sum_{i=1}^N \operatorname{Im}(c^{(i)}) = 0 \quad (16)$$

A system of linear algebraic equations is formed from eqns (13)–(16).

4.3 Solution of the linear system

By extracting the real and imaginary parts of all equations, that is, equations of continuity and condition equations, the real system of linear algebraic equations can be obtained and rewritten in a matrix form:

$$\mathbf{Ax} = \mathbf{b} \quad (17)$$

where $\mathbf{A} \in R^{m \times n}$, $\mathbf{b} \in R^m$ and $m > n$; $\mathbf{x}$ is the vector of the unknown real coefficients, of length n . The vector $\mathbf{x}$ consists of real and imaginary parts of unknown complex coefficients. Vector $\mathbf{b}$ is known exactly, while $\mathbf{A}$, the matrix of the system of linear algebraic equations, depends on CPs as well as the element type and size. Matrix $\mathbf{A}$ is not a square, the system is over-determined and therefore the left-hand side $\mathbf{Ax}$ does not exactly equal to $\mathbf{b}$; thus the system is inconsistent. However an approximate solution, $\mathbf{x}^*$, can be found by means of the least squares method [5] that minimise the residual:

$$\|\mathbf{Ax}^* - \mathbf{b}\|_2 \leq \varepsilon \quad (18)$$

where $\|\cdot\|_2$ stands for the L_2 norm. If the arising system is well-posed and not large, then the inversion of the matrix does not meet any difficulties and the solution takes the form [7]:

$$\mathbf{x}^* = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b} \quad (19)$$

The condition number CN is used in the numerical examples to control well-posedness of the system of linear algebraic equations. In the case of high CN , the methods for ill-posed problems should be used [8].

5 Numerical analysis

For the purpose of testing the methodology, an artificial function was chosen in the form:

$$W(z) = 54 \left(z - \frac{i}{3} \right) - 30 \left(z - \frac{i}{3} \right)^3 \quad (20)$$

Taking the real part and derivative of eqn (20), one has the electric potential and current density for the ideal function, which are used as given electric potential data and for comparison with the numerically obtained results for current density. The conductivity of the electrolyte is a multiplicative parameter to the current density, which was chosen as $k = 1$. Different types and structure of data were tested (ideal data and data subjected to errors).

5.1 Ideal data

This example used data obtained from eqn (20). It used 81 data uniformly redistributed within the domain and 25 elements as shown in Fig. 2.

The reconstructed electric potential and current are shown in Figs. 3 and 4. This configuration revealed accuracy with average error 4.7% by using 25 elements only. From Fig. 5, the improvement with refinement can be seen in residual (r /Equations), maximum and average errors as well as the behaviour of condition number CN .

Figure 2: Discretized domain with given data on electric potential.

Figure 3: Ideal electric potential (left) and reconstructed electric potential (right).

Figure 4: Ideal current density (left) and recovered current density (right).

Figure 5: Refinement results for 81 ideal data points.

5.2 Data subjected to errors

Random error was introduced to the potential data to simulate experimental errors. The error used was $\pm 10\%$ on 81 uniformly redistributed data items and 49 elements. The results for electric potential and current density are shown in Figs. 6 and 7. The average error was 2.9%; the condition number was 571 and $r/\text{Equations}$ was 0.2943.

Figure 6: Ideal electric potential (left) and reconstructed electric potential from data subjected to errors (right).

Figure 7: Ideal current density (left) and recovered current density from data subjected to errors (right).

5.3 Random data subjected to errors

The measurements are usually randomly redistributed. Random data subjected to errors $\pm 10\%$ were used; 81 data items and 49 elements were used as shown in Fig. 8. The results are shown in Fig. 9. The average error was 3.8%; CN was 754 and $r/\text{Equations} = 0.2672$.

Figure 8: 81 random data subjected to errors and discretization.

Figure 9: Reconstructed electric potential (left) and recovered current density (right).

6 Conclusion

The recovering of current density from data on electric potential, satisfying Laplace's equation was studied. In experiments, it is difficult or expensive to obtain many measurements and therefore numerical integration cannot be performed. The recovered results revealed high accuracy with an ideal artificial function W and for ideal data, as well as for data subjected to high errors. The method uses complex variable theory where one can obtain an holomorphic function W related to the electric potential and its derivative related to the current density.

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