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GRAPH MODELS OF COMPLEX NETWORKS

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Abstract. The study of the properties of analytical, social and economic networks marked the beginning of the study of complex networks in general. Typical studies of the network, the models of which are graphs, answered questions of centering and connectivity, finding the shortest paths, cycles or trees, minimal covers, or maximal matching.

As one or another peak, an economic agent or person in the network in question is connected with another peak, an economic agent or another person, which or who of them has the most connections with others. The term “complex networks” appeared at the beginning of this century and refers to networks with a more complex architecture than classical networks with a given number of nodes and connections. Usually in such networks it is communications that largely determine their properties.

Keywords: analytical, social and economic networks, complex networks, weighted location problem, centers, graph model, shortest paths, cycles or trees, minimal covers, or maximal matching

The study of complex networks is concerned with mathematics, economics, sociology, information technology, digital economics, biology, physics, and chemistry, which, in turn, makes it possible to assign complex networks to an interdisciplinary area of science. An example of such tasks is the spread of infection (epidemic), studied in medicine, which is reflected in information technology, as the process of spreading viruses on the Internet, or as the dissemination of fake information in social networks or media.

The situation with the study of complex networks is of general interest also because we are witnessing a “happy for scientists” phenomenon — the practice in this area is far ahead of theory. This means the emergence of an enormous number of new tasks that require generalization and immediate solution, both in practice and new theories.

Historically, the study of complex networks began with their practical study, search for any properties, patterns, confirmation or refutation of the existence of these patterns. One of the main properties of analytical, social and economic networks discovered in this way was the phenomenon of the small world (the small - world phenomenon). Its main idea: often the economic and social world consists of short links of interactions between economic agents (companies, households, power structures) and social objects. Thus, the main task of researchers is to obtain a model that corresponds to certain properties of real networks.

The first model of the small world (graph model) was proposed by Watts and Strogatz (Watts and Strogatz), in which the effect of the small world was explained as follows: large networks tend to have a small diameter and a small average length of the geodesic circuit.

The second important idea arising from the study of economic and social networks was that they tend to have a high clustering coefficient. The idea of clustering is important in analytics, economics, social interactions where they pay attention to regular (homogeneous graphs) interactions that make complex networks resistant to internal and external disturbances (challenges), for example, triangles (three mutually connected vertices) in analytical, social and economic networks.

Many complex networks have the property of homophilia, which reflects the fact that economic agents, social communities (networks) are more inclined to enter into relationships with similar groups.

Another important observation in complex networks is the conclusion that weak links play a connecting role in interactions between groups, while strong links are responsible for cohesion within communities of economic agents, social objects and increase the overall separation in the economy and society.

The first models that tried to isolate the types of graphs that correspond to the properties and structures of real-world networks came from classical graph theory and assumed that networks are formed randomly.

Random graphs can be generated in a variety of ways so that the models built on their basis better correspond to real complex networks. One of such graph generation models is a configuration model that allows you to create networks with a precisely defined distribution of vertex degrees (degree distribution), which in turn allows you to generate different networks with the same distribution as the complex network under study.

In contrast to stationary networks, there is a special class of random graphs in which a new vertex appears over time and, when it appears, forms edges with existing vertices. The main purpose of creating such models was an attempt to describe models of the Internet, where the new page creates links to already existing pages. Similar models can be described and other economic and social networks. For example, establishing new acquaintances when a person comes to a new school, is hired for a job, or gets acquainted with neighbors after moving or creating new supply chains in economic networks.

The next steps in trying to explain the growth phenomena of complex networks were the creation of algorithms for their generation based on the description of the growth of scale-free graphs, including Erdős-Rényi graphs (P. Erdős, A. Rényi) and graphs generated in accordance with the Barabash-Albert model (Albert- László Barabási & Réka Albert, etc. This approach took into account two important general concepts found in real complex networks, the growth of a graph and the principle of preferred connection.

In modern growth models of complex networks, for example, formulated in the works of Ozik and Hunt (Ozik J., Hunt B.), Peng and Zang (Aoyuan Peng, Lianming Zhang), the algorithm of deterministic multidimensional growth (DMG) of the “small world” graph.

At this point in time, there are many deterministic models, like DMG models or geometric assortative growth models, suitable for generating “small world” graphs, which can be used to simulate the growth dynamics of real-world networks, but they are not exhaustive and need further development.

Thus, the existing growth algorithms for small world graphs, despite their deterministic character, make it possible to model the growth of networks that are somewhat similar to real ones, but they take into account the so-called “weak” links to a lesser extent than the latter are actually found in modern analytical, economic and social networks. Note also that not all economic agents and their groups, as well as the communities of modern complex networks fully comply with the definition of a “small world”.

In connection with the rapid development of complex networks, the digital economy, the emergence of new practical problems and the lag of their theoretical support, the development of new models and algorithms that explain and predict the behavior of economic agents (companies, households, power structures) and social communities to understand their goals and predict their future.

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ON SOME CATEGORY OF CHU SPACES OVER CATEGORY OF S-ACTS

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Abstract. This work continues the earlier works of the same authors. We consider a category $Chu(S - Act)$ of Chu spaces over the category of S -acts where S is commutative monoid. Structure of a monoidal category necessary for the determining the Chu spaces is given by the tensor product of S -acts. An important characteristic of the category is the ability to move to the limits and colimits. It is known that for the existence of (co)limits it is necessary and sufficient the existence of (co)equalizers and (co)products. (Co)limits are computed through (co)product and (co)equalizers in a standard way. In category $Chu(S - Act)$ coequalizers and coproducts exist. There is the description of Chu transforms for which equalizer exists in category $Chu(S - Act)$. In this work we give necessary condition for existence of product of Chu spaces in the category $Chu(S - Act)$.

Keywords: Chu spaces, Chu construction, S -Act, monoidal category, product, coproduct.

The construction of Chu that later led to the notion of a Chu space appeared in Po Hsiang-Chu's Master's thesis on category theory and was published in 1979 [1]. It allows for a given symmetric monoidal closed category and fixed object to build a new category. If the original category was $*$ -autonomous, then in the resulting category the selected object becomes dual. In particular, the category $Chu(\mathcal{V})$ is built Chu objects, or Chu spaces, where $\mathcal{V}$ is a category [2]. It is done this way. Let $\otimes: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ is bifunctor, that is the tensor product on the category $\mathcal{V}$ and D is a fixed object of $\mathcal{V}$. An arrow $r: A \otimes X \rightarrow D$ is called Chu spaces of $\mathcal{V}$, where $A, X \in Ob(\mathcal{V})$. An arrow $(f, g): (A_1 \otimes X_1 \rightarrow D) \rightarrow (A_2 \otimes X_2 \rightarrow D)$ of $Chu(\mathcal{V})$ consists of a twice of arrows $f: A_1 \rightarrow A_2$, $g: X_2 \rightarrow X_1$ of $\mathcal{V}$ such that $r_1 \circ (1_{A_1} \otimes g) = r_2 \circ (f \otimes 1_{X_2})$ is called transform of Chu spaces or simply Chu transform.

The categories of Chu spaces are actively studied and studied in connection with by interplication in terms of Chu spaces objects of topology, algebra, computer science, game theory [6, 7. For the case when $\mathcal{V}$ is fixed category general properties of the Chu space category studied in the work [2].

In [8], categories of Chu spaces over the category of S -acts where S is a commutative monoid are studied. Structure of a monoidal category necessary for the determining the Chu spaces is given by the tensor product of S -acts. As a rule the category of Chu spaces over a monoidal category inherits many of its properties, including completeness, that is the existence of limits and colimits. this applies to the category $Chu(S - Act, D)$. However, this is not so in the category $Chu(S - Act)$ of all Chu spaces, that is, when D is not fixed (the same category was first introduced in [8]). This category is complete, but there are examples showing that limits do not always exist [8].

An important characteristic of the category is the ability to move to the limits and colimits. It is known that for the existence of (co)limits it is necessary and sufficient the existence of (co)equalizers and (co)products. (Co)limits are computed through (co)product and (co)equalizers in a standard way. In [8] questions associated with equalizers and co-equalizers for $Chu(S - Act)$ are studied. In [8] there is the description of Chu transforms for which equalizer exists in category $Chu(S - Act)$. In this work we give necessary condition for existence of product of Chu spaces in the category $Chu(S - Act)$. This result is new.

Suppose that $\mathcal{V}$ is a symmetric monoidal closed category with a tensor product $\otimes$.

We define a category $Chu(\mathcal{V})$ [8] that has as objects $r: A \otimes X \rightarrow D$ is an arrow of $\mathcal{V}$ where $A, X, D \in Ob(\mathcal{V})$. Further these arrows will be call Chu spaces. An arrow $(f, g, h): (A_1 \otimes X_1 \rightarrow D_1) \rightarrow (A_2 \otimes X_2 \rightarrow D_2)$ of $Chu(\mathcal{V})$ consists of a triple of arrows $f: A_1 \rightarrow A_2$, $g: X_2 \rightarrow X_1$, $h: D_1 \rightarrow D_2$ of $\mathcal{V}$

(note the direction of the second arrow) such that $h \circ r_1 \circ (1_{A_1} \otimes g) = r_2 \circ (f \otimes 1_{X_2})$. Further arrows of Chu spaces will be call transform of Chu spaces or simply Chu transform.

Recall some concepts from the category theory [5] and S -acts theory [3, 4]. Let $\mathcal{V}$ is the category.

The equalizer of two morphisms f and g between two objects A and B is the monomorphism $e: E \rightarrow A$ such that $f \circ e = g \circ e$ and if $h: F \rightarrow A$ is a morphism of $\mathcal{V}$ such that $f \circ h = g \circ h$ then there is a unique morphism $u: F \rightarrow E$ such that $h = e \circ u$.

The coequalizer of two morphisms f and g between two objects A and B is the epimorphism $q: B \rightarrow Q$ such that $q \circ f = q \circ g$ and if $h: B \rightarrow F$ is a morphism of $\mathcal{V}$ such that $h \circ f = h \circ g$ then there is a unique morphism $v: Q \rightarrow F$ such that $h = v \circ q$.

The product of objects X_j ($j \in J$) is an object X (often denoted $\prod_{j \in J} X_j$) together with a morphisms $\pi_j: X \rightarrow X_j$ ($j \in J$) such that for every object Y and morphisms $f_j: Y \rightarrow X_j$ ($j \in J$) there exists a unique morphism $f: Y \rightarrow X$ such that $\pi_j \circ f = f_j$ for all $j \in J$.

The coproduct of objects X_j ($j \in J$) is an object (often denoted $\coprod_{j \in J} X_j$) together with a morphisms $i_j: X_j \rightarrow X$ ($j \in J$) that satisfy the following universal property: for every object Y and morphisms $f_j: X_j \rightarrow Y$ ($j \in J$) there exists a unique morphism $f: X \rightarrow Y$ such that $f \circ i_j = f_j$ for all $j \in J$.

Throughout, by S we denote a monoid and by 1 we denote unity in S . An algebraic system $\langle A; s \rangle_{s \in S}$ of the language $L_S = \{s | s \in S\}$ is a (left) S -act if $s_1(s_2 a) = (s_1 s_2) a$ and $1a = a$ for all $s_1, s_2 \in S$ and $a \in A$, that is S -act is a set A on which S acts in the left and the unity acts identically. We denote the class of all S -acts by $S - Act$. The class of all S -acts forms a category where objects are S -acts and morphisms are homomorphisms of S -acts.

Note that the product of S -acts A_j ($j \in J$) is S -act $\prod_{j \in J} A_j$ where $(sa)(j) = (sa(j))$ for all $j \in J$, $a \in A$, $s \in S$; the coproduct of S -acts A_j ($j \in J$) is a disjunctive union of this S -acts.

Suppose that S is a commutative monoid. The tensor product $A \otimes B$ of S -acts A и B is defined as a factor act of S -act $A \times B$ by congruence generated by a set $\{(sa, b), (a, sb) | a \in A, b \in B, s \in S\}$. The class of the congruence with representative (a, b) will be denote by $a \otimes b$ where $a \in A$ и $b \in B$. Note that the action of the monoid S on the set $A \otimes B$ has the following property: for any $a \in A, b \in B, s \in S$

$$s(a \otimes b) = sa \otimes b = a \otimes sb.$$

Theorem 1. [8] *Let S is a commutative monoid. Then $S - Act$ is a symmetric monoidal closed category with a tensor product $\otimes$.*

In [8], the Chu construction is applied to the category $S - Act$, more exactly, two categories $Chu(S - Act)$ and $Chu(S - Act, D)$ are introduced, where S is a commutative monoid.

In the category $Chu(S - Act, D)$ for any Chu spaces and Chu transforms there are equalizers, coequalizers, products and coproducts [8]. So for this category there are limits and colimits.

In the category $Chu(S - Act)$ there are similar results for coequalizers and coproducts [8]:

Theorem 2. [8] *Let $r: A \otimes X \rightarrow D_1$, $s: B \otimes Y \rightarrow D_2$ are the Chu spaces and $(f_1, g_1, h_1), (f_2, g_2, h_2): r \rightarrow s$ are the Chu transforms of $Chu(S - Act)$. Then the coequalizer of two Chu transforms $(f_1, g_1, h_1), (f_2, g_2, h_2)$ is the Chu transform $(f, g, h): s \rightarrow t$ where $t: Q \otimes E \rightarrow W$, $Q = B/v(f_1, f_2)$, $W = D/v(h_1, h_2)$, $E = \{y \in Y | g_1(y) = g_2(y)\}$, f, h are canonical epimorphisms, g is a natural embedding.*

Theorem 3. [8] *Let $r_i: A_i \otimes X_i \rightarrow D_i$ are the Chu spaces, where $i \in I$. Then there is the coproduct r of r_i ($i \in I$), where $r: \coprod_{i \in I} A_i \otimes \prod_{i \in I} X_i \rightarrow \prod_{i \in I} D_i$ such that $r(a \otimes x) = r_i(a \otimes x(i))$ for any $i \in I$, $a \in A_i$, $x \in \prod_{i \in I} X_i$.*

Let's look at the issues related to equalizers and products in the category $Chu(S - Act)$. The follow theorem gives the description of Chu transforms for which equalizer exists.

Theorem 4. [9] Let $r: A \otimes X \rightarrow D_1$, $s: B \otimes Y \rightarrow D_2$ are the Chu spaces and $(f_1, g_1, h_1), (f_2, g_2, h_2): r \rightarrow s$ are the Chu transforms. Then the equalizer of two Chu transforms $(f_1, g_1, h_1), (f_2, g_2, h_2)$ exists if and only if there is an element $a \in A$ such that

- 1) $f_1(a) = f_2(a)$;
- 2) $r(a \otimes g_1(y)) = r(a \otimes g_2(y))$ for any $y \in Y$;
- 3) $h_1 \circ r(a \otimes x) = h_2 \circ r(a \otimes x)$ for any $x \in X$.

It gives necessary condition for existence of product of Chu spaces in the category $\text{Chu}(S - \text{Act})$.

Theorem 5. Let $r_1: A_1 \otimes X_1 \rightarrow D_1$, $r_2: A_2 \otimes X_2 \rightarrow D_2$ are the Chu spaces. Suppose that Chu space $r: A \otimes X \rightarrow D$ with Chu transforms $(f_1, g_1, h_1): r \rightarrow r_1$, $(f_2, g_2, h_2): r \rightarrow r_2$ are the product of r_1 and r_2 . Then $A = f_1(A_1) \times f_2(A_2)$, $X = X_1 \coprod X_2$ and $D = D_1 \times D_2$.

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CONTROL APPROACH IN 3D PROBLEMS OF DESIGNING OF BILAYERED MAGNETIC CLOAKS

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Abstract. We consider control problems for the 3D model of magnetic scattering by a permeable isotropic obstacle having the form of a spherical bilayer shell. These problems arise while developing the design technologies of magnetic cloaking devices using the optimization method for solving corresponding inverse problems. The solvability of direct and optimization problems for the magnetic scattering model under study is proved. The optimality system which describes the necessary conditions of extremum is derived. Also, numerical aspects of applying the optimization approach for solving problems of designing magnetic cloaking devices are discussed.

Key words magnetic cloaking, inverse problem, optimization method, existence, optimality system.

1. Introduction

In the last few years, devices for cloaking material objects have attracted the rapid attention in the research fields of “invisibility” and metamaterials. The first works in the mentioned fields were the articles [1, 2, 3]. These were the start of development of different methods, schemes and techniques of solving problems when cloaking electromagnetic waves, acoustic waves, magnetic, electric, thermal and other static fields [4, 5, 6, 7]. It should be noted that the solutions obtained in these papers possess several drawbacks. In particular, some components of spatially dependent parameter tensors of ideal cloak are required to have infinite or zero values at the inner boundary of the cloak [2] which are very difficult to implement.

One of approaches of overcoming these difficulties consists of replacing “exact” singular solutions of cloaking problems under study by approximate non-singular solutions and designing cloaking devices based on these approximations (see., e.g., [8, 9, 10, 11]). Alternative approach is based on using the optimization method of solving inverse problems. This method is based on replacing initial cloaking problem by minimization problem of a suitable tracking-type cost functional which corresponds to inverse problem under consideration. The method to which one should referred to as inverse design method [12, 13] was applied in [14] devoted to the numerical analysis of 2D problems of designing layered cloaking shells. Also, it was applied in [15]–[21] when studying theoretically electromagnetic or acoustic cloaking problems. Paper [22] are devoted to studying invisibility problem in X-ray tomography.

The goal of this paper is theoretical analysis of the cloaking problem for the 3D model of magnetic scattering by bilayer shell using the optimization method. The plan of our paper is as follows. Firstly, we formulate the direct magnetic scattering problem and prove its correct solvability. Then we formulate the corresponding inverse cloaking problem, and reduce it to respective control problem for which we prove the solvability. Finally, we discuss numerical aspects of applying the optimization approach for solving problems of designing magnetic cloaking devices under study.

2. Statement and analysis of direct magnetic scattering problem

We consider the domain in the space R^3 having in spherical coordinates (r, θ, φ) the form of spherical layer $\Omega_1 = \{\mathbf{x} \in R^3 : a < r = |\mathbf{x}| < b\}$ surrounded by another spherical layer $\Omega_2 = \{\mathbf{x} \in R^3 : b < r = |\mathbf{x}| < c\}$, where a , b and c are positive constants. Denote by Ω_i and Ω_e^∞ the interior and exterior of the domain $\Omega_1 \cup \Omega_2$. We will assume that the domains Ω_i and Ω_e^∞ are filled by

homogeneous isotropic media with constant permeabilities μ_i and μ_e , respectively, while Ω_i and Ω_2 are filled by inhomogeneous isotropic media with variable permeabilities μ_1 and μ_2 , respectively (see Fig. 1).

Denote by B_R the ball $|\mathbf{x}| < R$, where $R > c$, containing domains Ω_i , Ω_1 and Ω_2 . We will assume that there are sources outside B_R , which generate externally applied magnetic field $\mathbf{H}_a = -\text{grad}\Phi_a$ corresponding to potential Φ_a which satisfies the Laplace equation $\Delta\Phi_a = 0$ in B_R .

$$\mu_i \Delta\Phi_i = 0 \text{ in } \Omega_i, \quad \text{div}(\mu_1 \nabla\Phi_1) = 0 \text{ in } \Omega_1, \quad \text{div}(\mu_2 \nabla\Phi_2) = 0 \text{ in } \Omega_2, \quad \mu_e \Delta\Phi_e = 0 \text{ in } \Omega_e^\infty, \quad (1)$$

$$\Phi_i = \Phi_1, \quad \mu_i \frac{\partial\Phi_i}{\partial r} = \mu_1 \frac{\partial\Phi_1}{\partial r} \text{ at } r = a, \quad \Phi_1 = \Phi_2, \quad \mu_1 \frac{\partial\Phi_1}{\partial r} = \mu_2 \frac{\partial\Phi_2}{\partial r} \text{ at } r = b, \quad (2)$$

$$\Phi_2 = \Phi_a + \Phi_s, \quad \mu_2 \frac{\partial\Phi_2}{\partial r} = \mu_e \frac{\partial(\Phi_a + \Phi_s)}{\partial r} \text{ at } r = c, \quad \Phi_s(\mathbf{x}) = o(1) \quad \text{as } r = |\mathbf{x}| \rightarrow \infty. \quad (3)$$

In the particular case, when μ_1 and μ_2 are nonnegative constants and, besides, the field $\mathbf{H}_a$ is uniform, the direct problem (1)–(3) admits an exact solution which can be found using Fourier's method (see [9, 10]).

Now, consider the case when the external field $\mathbf{H}_a$ is inhomogeneous and therefore it is impossible to apply Fourier's method for finding an exact solution of the problem (1)–(3). A number of functional spaces will be used while studying the direct problem (1)–(3) and respective control problems. Let $\Omega_e = \Omega_e^\infty \cap B_R$. We will use the space $H^1(D)$, where D is one of domains B_R , Ω_i , Ω_1 , Ω_2 , Ω_e^∞ , and also spaces $L^\infty(\Omega_k)$, $H^s(\Omega_k)$, $s > 0$, $k=1,2$, $L^2(Q)$, $H^{1/2}(\Gamma_R)$ and $H^{-1/2}(\Gamma_R)$. Here $Q \subset B_R$ is an arbitrary open subset of B_R , Γ_R is a boundary of B_R (see Fig. 2).

Figure 1: Layout of external sources and spherical bilayer cloak

Figure 2: Schematic layout of artificial boundary Γ_R

The norms and scalar products in $H^1(D)$, $H^s(\Omega_k)$, $L^2(Q)$, $H^{1/2}(\Gamma_R)$ and $H^{-1/2}(\Gamma_R)$ will be denoted by $\|\cdot\|_{1,D}$, $(\cdot, \cdot)_{1,D}$, $\|\cdot\|_{s,\Omega_k}$, $(\cdot, \cdot)_{s,\Omega_k}$, $\|\cdot\|_Q$, $(\cdot, \cdot)_Q$, $\|\cdot\|_{1/2,\Gamma_R}$ and $\|\cdot\|_{-1/2,\Gamma_R}$. We set $L_{\lambda_0}^\infty(\Omega_k) = \{\lambda \in L^\infty(\Omega_k) : \lambda(\mathbf{x}) \geq \lambda_0\}$, $H_{\lambda_0}^s(\Omega_k) = \{\lambda \in H^s(\Omega_k) : \lambda(\mathbf{x}) \geq \lambda_0\}$, $\lambda_0 = \text{const} > 0$, $k=1,2$. It is well known by the embedding theorem that the continuous and compact embedding $H^s(\Omega_k) \subset L^\infty(\Omega_k)$ at $s > 3/2$, $k=1,2$, holds and the following estimate takes place:

$$\|\lambda\|_{L^\infty(\Omega_k)} \leq C_s \|\lambda\|_{s,\Omega_k} \quad \forall \lambda \in H^s(\Omega_k), \quad s > 3/2, \quad k=1,2. \quad (4)$$

Here C_s is a constant depending on $s > 3/2$ and Ω_1 , Ω_2 . We need also a subspace $H(\Omega_e) = \{\Phi \in H^1(\Omega_e) : \Delta\Phi = 0 \text{ in } \Omega_e\}$, equipped with the norm $\|\cdot\|_{1,\Omega_e} = \|\cdot\|_{H^1(\Omega_e)}$. The space $H(\Omega_e)$ will be served for describing restrictions of externally applied field Φ_a to the domain Ω_e .

It should be noted that by the trace theorem there exists a trace $\Phi|_{\Gamma_R} \in H^{1/2}(\Gamma_R)$ for any function $\Phi \in H^1(B_R)$ while for any function $\Phi^e \in H(\Omega_e)$ there exists a normal trace $\partial\Phi^e/\partial n|_{\Gamma_R}$ and the following estimates hold:

$$\|\Phi\|_{1/2,\Gamma_R} \leq C_R \|\Phi\|_X \quad \forall \Phi \in H^1(B_R), \quad (5)$$

$$\|\partial\Phi^e/\partial n\|_{-1/2,\Gamma_R} \leq C'_R \|\Phi^e\|_{1,\Omega_e} \quad \forall \Phi^e \in H(\Omega_e). \quad (6)$$

Here C_R, C'_R are constants depending on Ω_e and R but are independent of $\Phi \in X$ and $\Phi^e \in H(\Omega_e)$.

We assume below that the following conditions take place:

- (i) $\mu_1 \in L_{\mu_1^0}^\infty(\Omega_1), \mu_2 \in L_{\mu_2^0}^\infty(\Omega_2), \mu_1 \geq \mu_1^0, \mu_2 \geq \mu_2^0, \mu_1^0 = \text{const} > 0, \mu_2^0 = \text{const} > 0$;
- (ii) $\Phi^e \equiv \Phi_a|_{\Omega_e} \in H(\Omega_e)$.

As the potential Φ is determined up to an additive constant we will not distinguish functions of the space $H^1(B_R)$ which differ from each other by an additive constant. Thus the main role below will be played by the following quotient-space $X = H^1(B_R)$ with the norm:

$$\|\Phi\|_X^2 = \|\nabla\Phi\|_{\Omega_i}^2 + \|\nabla\Phi\|_{\Omega_1}^2 + \|\nabla\Phi\|_{\Omega_2}^2 + \|\nabla\Phi\|_{\Omega_e}^2. \quad (7)$$

One can show that the space $X = H^1(B_R)$ is Hilbert for this norm and, besides, the following analogue of Poincare-Friedrichs inequality holds:

$$\|\Phi\|_{B_R} \leq C_P \|\Phi\|_X \quad \forall \Phi \in X. \quad (8)$$

Here C_P is a constant which is independent of $\Phi \in X$.

We begin our analysis with defining weak formulation and weak solution of direct problem (1)–(3). Preliminarily we reduce problem (1)–(3) to an equivalent boundary problem considered in the bounded domain (ball) B_R . To this end we introduce as in [19] the Dirichlet-to-Neumann operator $T: H^{1/2}(\Gamma_R) \rightarrow H^{-1/2}(\Gamma_R)$ which maps every function $h \in H^{1/2}(\Gamma_R)$ to the function $\partial\tilde{\Phi}/\partial\nu \in H^{-1/2}(\Gamma_R)$. Here $\tilde{\Phi}$ is a solution of the external Dirichlet problem for equation $\Delta\tilde{\Phi} = 0$ in $\Omega_e^\infty \setminus B_R$ with the boundary condition $\tilde{\Phi}|_{\Gamma_R} = h$ satisfying the condition $\tilde{\Phi}(\mathbf{x}) = o(1)$ as $r = |\mathbf{x}| \rightarrow \infty$. We note that problem (1)–(3) considered in R^3 is equivalent to boundary value problem (1), (2) considered in the ball B_R under the following additional condition for Φ_s on Γ_R :

$$\partial\Phi_s/\partial n = T\Phi_s \text{ on } \Gamma_R \quad (9)$$

For brevity we will refer below to the problem (1), (2), (9) as Problem 1.

Based on the space X we derive now the weak formulation of Problem 1. Let $S \in X$ be a test function. We multiply every of equations in (1) considered in domains $\Omega_i, \Omega_1, \Omega_2$ and Ω_e , by S , integrate over $\Omega_i, \Omega_1, \Omega_2$ or Ω_e , respectively, and apply Green formulae. Adding the obtained identities and using the boundary conditions in (2) and (9) we arrive at the following identity for the quadruple $\Phi = (\Phi_i, \Phi_1, \Phi_2, \Phi_e) \in X$:

$$a_\mu(\Phi, S) \equiv a_0(\Phi, S) + a(\mu_1, \mu_2; \Phi, S) = \langle F, S \rangle \quad \forall S \in X. \quad (10)$$

Here and below μ denotes the pair (μ_1, μ_2) while $a_0(\cdot, \cdot)$, $a(\mu_1, \mu_2; \cdot, \cdot)$ and F are bilinear and linear forms defined by

$$a_0(\Phi, S) = \mu_i \int_{\Omega_i} \nabla\Phi \cdot \nabla S d\mathbf{x} + \mu_e \int_{\Omega_e} \nabla\Phi \cdot \nabla S d\mathbf{x} - \int_{\Gamma_R} (T\Phi) S d\sigma, \quad (11)$$

$$a(\mu_1, \mu_2; \Phi, S) = a_1(\mu_1; \Phi, S) + a_2(\mu_2; \Phi, S), \quad \langle F, S \rangle = - \int_{\Gamma_R} T\Phi^e S d\sigma + \int_{\Gamma_R} (\partial\Phi^e/\partial n) S d\sigma \quad (12)$$

$$a_1(\mu_1; \Phi, S) = \int_{\Omega_1} \mu_1 \nabla\Phi \cdot \nabla S d\mathbf{x}, \quad a_2(\mu_2; \Phi, S) = \int_{\Omega_2} \mu_2 \nabla\Phi \cdot \nabla S d\mathbf{x}, \quad (13)$$

Identity (10) represents the weak formulation of problem (1), (2), (9) and its solution $\Phi = (\Phi_i, \Phi_1, \Phi_2, \Phi_e) \in X$ will be called a weak solution of Problem 1. Arguing as in [19], one can easily show, that the introducing of the weak solution is admissible in the following sense: it satisfies all equations in (1) in the distribution sense, and also boundary conditions in (2) and (9) in the trace sense.

Using Hölder inequality and definition (7) for the norm $\|\cdot\|_X$ and (9), (10) we have

$$\left| \int_{\Omega_1} \mu_1 \nabla \Phi \cdot \nabla S d\mathbf{x} \right| \leq \|\mu_1\|_{L^\infty(\Omega_1)} \|\nabla \Phi\|_{\Omega_1} \|\nabla S\|_{\Omega_1} \leq \|\mu_1\|_{L^\infty(\Omega_1)} \|\Phi\|_X \|S\|_X,$$

$$\left| \int_{\Omega_2} \mu_2 \nabla \Phi \cdot \nabla S d\mathbf{x} \right| \leq \|\mu_2\|_{L^\infty(\Omega_2)} \|\nabla \Phi\|_{\Omega_2} \|\nabla S\|_{\Omega_2} \leq \|\mu_2\|_{L^\infty(\Omega_2)} \|\Phi\|_X \|S\|_X.$$

Besides, using (5), (6), we derive from (11), (12) and (13) that

$$\left| \mu_i \int_{\Omega_i} \nabla \Phi \cdot \nabla S d\mathbf{x} \right| \leq \mu_i \|\Phi\|_X \|S\|_X, \quad \left| \mu_e \int_{\Omega_e} \nabla \Phi \cdot \nabla S d\mathbf{x} \right| \leq \mu_e \|\Phi\|_X \|S\|_X, \quad (14)$$

$$\left| \int_{\Gamma_R} (T\Phi) S d\sigma \right| \leq \|T\Phi\|_{-1/2, \Gamma_R} \|S\|_{1/2, \Gamma_R} \leq C_T C_R^2 \|\Phi\|_X \|S\|_X, \quad (15)$$

$$|a_0(\Phi, S)| \leq (\mu_0 + C_T C_R^2) \|\Phi\|_X \|S\|_X, \quad \mu_0 = \max(\mu_i, \mu_e), \quad (16)$$

$$|\langle F, S \rangle| \leq \left(\|T\| \|\Phi^e\|_{1/2, \Gamma_R} + \|\partial \Phi^e / \partial n\|_{-1/2, \Gamma_R} \right) \|S\|_{1/2, \Gamma_R} \leq (C_T C_R + C'_R) C \|\Phi^e\|_{1, \Omega_e} \|S\|_X. \quad (17)$$

It follows from these estimates and (12) that forms $a(\mu_1, \mu_2; \cdot, \cdot)$, a_0 and F are continuous on X , and the following estimates hold:

$$|a(\mu_1, \mu_2; \Phi, S)| \leq (\|\mu_1\|_{L^\infty(\Omega_1)} + \|\mu_2\|_{L^\infty(\Omega_2)}) \|\Phi\|_X \|S\|_X, \quad (18)$$

$$\|a_0\| \leq C_0, \|F\|_{X^*} \leq C_0 \|\Phi^e\|_{1, \Omega_e}, \quad C_0 = \max[\mu_0 + C_T C_R^2, (C_T C_R + C'_R) C_R]. \quad (19)$$

Here X^* is a dual of X with respect to space $L^2(B_R)$.

If, moreover, the condition (i) takes place then we have at $\mu_0 = \min(\mu_i, \mu_e)$:

$$a_0(\Phi, \Phi) = \mu_i \int_{\Omega_i} |\nabla \Phi|^2 d\mathbf{x} + \mu_e \int_{\Omega_e} |\nabla \Phi|^2 d\mathbf{x} - \int_{\Gamma_R} (T\Phi) \Phi d\sigma \geq \mu_0 \int_{\Omega_0 \cup \Omega_e} |\nabla \Phi|^2 d\mathbf{x} \quad \forall \Phi \in X,$$

$$\int_{\Omega_1} \mu_1 \nabla \Phi \cdot \nabla \Phi d\mathbf{x} \geq \mu_1^0 \|\nabla \Phi\|_{\Omega_1}^2, \quad \int_{\Omega_2} \mu_2 \nabla \Phi \cdot \nabla \Phi d\mathbf{x} \geq \mu_2^0 \|\nabla \Phi\|_{\Omega_2}^2.$$

It follows from these estimates and the definition of the norm (7) in space X that the bilinear form a_μ defined in (10) is coercive in X under condition (i) and, besides,

$$a_\mu(\Phi, \Phi) \geq \mu_* \|\Phi\|_X^2 \quad \forall \Phi \in X, \quad \mu_* = \min(\mu_i, \mu_e, \mu_1^0, \mu_2^0). \quad (20)$$

We note that the bilinear form a_μ introduced in (10) defines a linear operator $A_\mu : X \rightarrow X^*$ acting by

$$\langle A_\mu \Phi, S \rangle = a_\mu(\Phi, S) \equiv a_0(\Phi, S) + a(\mu_1, \mu_2; \Phi, S), \quad (21)$$

while problem (10) is equivalent to operator equation

$$A_\mu \Phi = F. \quad (22)$$

As the bilinear form a_μ is continuous and coercive on X under condition (i), then from the Lax-Milgram theorem follows that the operator $A_\mu : X \rightarrow X^*$ is an isomorphism and the inverse $A_\mu^{-1} : X^* \rightarrow X$ to operator A_μ is also isomorphism. Let $C_\mu = \|A_\mu^{-1}\|$. It is clear by the Lax-Milgram theorem that $C_\mu \leq C_1 = (1/\mu_*)$. Using this estimate we derive that operator equation (22) has for any element $F \in X^*$ a unique solution $\Phi_\mu = A_\mu^{-1}(F) \in X$, for which the estimate $\|\Phi_\mu\|_X \leq C_1 \|F\|_{X^*}$ holds.

Using this fact and the second estimate in (19) we conclude that for any field $\Phi^e \in H(\Omega_e)$ problem (10) has a unique solution $\Phi_\mu \in X$, and the following estimate holds:

$$\|\Phi_\mu\|_X \leq C_2 \|\Phi^e\|_{1,\Omega_e}, \quad C_2 = C_0 C_1. \quad (23)$$

We emphasize that constants C_1 and C_2 in (23) depend on μ_i , μ_e , μ_1^0 , μ_2^0 and R but do not depend on functions μ_1 and μ_2 satisfying condition (i). Let us formulate the results obtained as the following theorem.

Theorem 1. *Let conditions (i), (ii) take place. Then for any pair $\mu = (\mu_1, \mu_2)$ we have:*

1) *the operator $A_\mu : X \rightarrow X^*$ defined in (21) is an isomorphism and for the inverse operator $A_\mu^{-1} : X \rightarrow X^*$ the estimate $\|A_\mu^{-1}\| \leq C_1 \equiv (1/\mu_*)$ holds where constant μ_* is defined in (20);*

2) *for any external field $\Phi^e \in H(\Omega_e)$ problem (10) has a unique solution $\Phi_\mu \in X$, and the estimate (23) holds where the constant C_2 depends on μ_i , μ_e , μ_1^0 , μ_2^0 and R but is independent of (μ_1, μ_2) .*

3. Formulation of the inverse problem. Applying the optimization method. Derivation of the optimality system

In this section we consider the inverse problem for model (1)–(3) arising while developing the design technologies of magnetic cloaking devices. This problem consists of finding unknown permeabilities μ_1 and μ_2 of inhomogeneous media filling Ω_1 and Ω_2 from the following conditions $\nabla\Phi = 0$ in Ω_i , $\Phi = \Phi_a$ in Ω_e . For solving this inverse problem we apply the optimization method which is based on minimization of certain cost functional. As a cost functional we choose the following:

$$I(\Phi) = [\alpha I_1(\Phi) + \beta I_2(\Phi)] \quad (\alpha + \beta = 1), \quad (24)$$

where

$$I_1(\Phi) = \|\nabla\Phi\|_{\Omega_i}^2 \equiv \int_{\Omega_i} |\nabla\Phi|^2 d\mathbf{x}, \quad I_2(\Phi) = \|\Phi - \Phi_a\|_{\Omega_e}^2 \equiv \int_{\Omega_e} |\Phi - \Phi_a|^2 d\mathbf{x}. \quad (25)$$

Here $\alpha, \beta \in [0, 1]$ are arbitrary parameters satisfying condition $\alpha + \beta = 1$. The choice of functional $I(\Phi)$ responds to the problem of internal (or external) cloaking at $\alpha = 1, \beta = 0$ (or at $\alpha = 0, \beta = 1$). For the remaining values of α and β functional (25) corresponds to the general cloaking problem.

We will assume that the controls μ_1 and μ_2 are changed over certain sets K_1 and K_2 . It is assumed that the following conditions hold.

(j) $K_1 \subset H_{\mu_1^0}^s(\Omega_1)$, $\mu_1^0 = \text{const} > 0$, $K_2 \subset H_{\mu_2^0}^s(\Omega_2)$, $\mu_2^0 = \text{const} > 0$, $s > 3/2$; $\alpha_0 > 0$.

Setting $K = K_1 \times K_2$, $\mu = (\mu_1, \mu_2)$ we define the operator $G : X \times K \rightarrow X^*$ by

$$\langle G(\Phi, \mu), S \rangle = \langle A_\mu \Phi, S \rangle - \langle F, S \rangle \equiv a_0(\Phi, S) + a(\mu_1, \mu_2; \Phi, S) - \langle F, S \rangle \quad \forall S \in X \quad (26)$$

and rewrite the weak formulation (10) of problem 1 as $G(\Phi, \mu) = 0$. Consider the following control problem:

$$J(\Phi, \mu) \equiv \frac{\alpha_0}{2} I(\Phi) + \frac{\alpha_1}{2} \|\mu_1\|_{s,\Omega_1}^2 + \frac{\alpha_2}{2} \|\mu_2\|_{s,\Omega_2}^2 \rightarrow \inf, \quad G(\Phi, \mu) = 0, \quad (\Phi, \mu) \in X \times K. \quad (27)$$

Here $\alpha_0, \alpha_1, \alpha_2$ are non-negative parameters which are used to control the relative importance of terms in (28). Set $Z_{ad} = \{(\Phi, \mu) \in X \times K : G(\Phi, \mu) = 0, J(\Phi, \mu) < \infty\}$.

Theorem 2. *Let, under assumptions (ii) and (j), $\alpha_1 > 0$, $\alpha_2 > 0$ or $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and K_1, K_2 be bounded sets. Then control problem (26) has at least one solution $(\Phi, \mu_1, \mu_2) \in X \times K_1 \times K_2$.*

Proof. Denote by $(\Phi^m, \mu^m) \in Z_{ad}$, $\mu^m = (\mu_1^m, \mu_2^m)$, $m \in N \equiv \{1, 2, \dots\}$ a minimizing sequence for the functional J for which the following relations hold:

$$a_0(\Phi^m, S) + a(\mu_1^m, \mu_2^m; \Phi^m, S) = \langle F, S \rangle \quad \forall S \in X, \quad \lim_{m \rightarrow \infty} J(\Phi^m, \mu_m) = \inf_{(\Phi, \mu) \in Z_{ad}} J(\Phi, \mu) \equiv J^*. \quad (28)$$

By conditions of Theorem 2 we have

$$\|\mu_1^m\|_{s, \Omega_1} \leq c_1, \quad \|\mu_2^m\|_{s, \Omega_2} \leq c_2 \quad \forall m \in N. \quad (29)$$

Here and below c_1, c_2, c_3 are some constants which do not depend on m . It follows from (29) and Theorem 1 that $\|\Phi^m\|_X \leq c_3$. From this estimate and (29) we derive that there are weak limits $\mu_1^* \in K_1, \mu_2^* \in K_2, \Phi^* \in X$ of some subsequences of sequences $\{\mu_1^m\}, \{\mu_2^m\}, \{\Phi^m\}$. Using this fact and compactness of embedding $H^s(\Omega_k) \subset L^\infty(\Omega_k)$ at $s > 3/2, k=1,2$ we conclude that $\mu_1^m \rightarrow \mu_1^*$ strongly in $L^\infty(\Omega_1), \mu_2^m \rightarrow \mu_2^*$ strongly in $L^\infty(\Omega_2)$, while $\Phi^m \rightarrow \Phi^*$ weakly in X .

Let us prove that $G(\Phi^*, \mu_1^*, \mu_2^*) = 0$, i.e. that

$$a_0(\Phi^*, S) + a(\mu_1^*, \mu_2^*; \Phi^*, S) = \langle F, S \rangle \quad \forall S \in X. \quad (30)$$

To this end we pass to the limit in (28) as $m \rightarrow \infty$. It is clear that linear term $a_0(\Phi^m, S)$ passes to $a_0(\Phi^*, S)$ as $m \rightarrow \infty$ while for the difference $a(\mu_1^m, \mu_2^m; \Phi^m, S) - a(\mu_1^*, \mu_2^*; \Phi^*, S)$ we have by (12)

$$|a(\mu_1^m, \mu_2^m; \Phi^m, S) - a(\mu_1^*, \mu_2^*; \Phi^*, S)| \leq |a_1(\mu_1^m; \Phi^m, S) - a_1(\mu_1^*; \Phi^*, S)| + |a_2(\mu_2^m; \Phi^m, S) - a_2(\mu_2^*; \Phi^*, S)|. \quad (31)$$

Let us show that every of terms in the right-hand side of (31) vanishes as $m \rightarrow \infty$. In fact, taking into consideration (13), we have for the first term in (31):

$$|a_1(\mu_1^m; \Phi^m, S) - a_1(\mu_1^*; \Phi^*, S)| \leq \left| \int_{\Omega_1} \mu_1^* (\nabla \Phi^m - \nabla \Phi^*) \cdot \nabla S dx \right| + \left| \int_{\Omega_1} (\mu_1^m - \mu_1^*) \nabla \Phi^m \cdot \nabla S dx \right| \quad \forall S \in X. \quad (32)$$

As $\Phi^m \rightarrow \Phi^*$ weakly in X then the first integral in the right-hand side of (32) tends to zero as $m \rightarrow \infty$ for any $S \in X$. Besides, it follows from the strong convergence $\mu_1^m \rightarrow \mu_1^*$ in $L^\infty(\Omega_1)$ that the second integral in (32) also tends to zero as $m \rightarrow \infty$. This means that the first term in the right-hand side of (31) tends to zero as $m \rightarrow \infty$. In a similar way one can show using (14) that the second term in the right-hand side of (31) tends to zero as $m \rightarrow \infty$. Therefore passing to the limit in (28) as $m \rightarrow \infty$ we arrive at (29). This means that $G(\Phi^*, \mu^*) = 0$ where $\mu^* = (\mu_1^*, \mu_2^*)$. As the functional $J(\Phi)$ is a weakly lower semicontinuous functional on $X \times K$ we derive that $J(\Phi^*, \mu^*) = J^*$. This proves the theorem.

The second stage of analysis of extremum problem (27) consists of derivation an optimality system describing necessary conditions of extremum. For this purpose we make use of the extremum principle in smoothly-convex extremum problems [23]. Preliminarily we find the Fréchet derivative with respect to Φ of the operator $G: X \times K \rightarrow X^*$ defined in (26). It follows from linearity of the operator G with respect to Φ that the Fréchet derivative $G'_\Phi(\hat{\Phi}, \hat{\mu})$ at every point $(\hat{\Phi}, \hat{\mu}) \in X \times K$ where $\hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2)$ is defined by $G'_\Phi(\hat{\Phi}, \hat{\mu}) = \hat{A} \equiv A_{\hat{\mu}}$. Here operator $A_{\hat{\mu}}$ is defined by (21) at $\mu = \hat{\mu}$. Besides, it follows from (24) that

$$\langle (I')_\Phi(\hat{\Phi}), S \rangle = \alpha(\nabla \hat{\Phi}, \nabla S)_{\Omega_1} + \beta(\hat{\Phi} - \Phi_a, S)_{\Omega_2} \quad \forall S \in X. \quad (33)$$

Following [32] we introduce a Lagrange multiplier $\Psi \in X$ which will be referred to as an adjoint stage and consider the Lagrangian $L: X \times K \times X \rightarrow R$ defined by the formula $L(\Phi, \mu, \Psi) \equiv J(\Phi, \mu) + \langle G(\Phi, \mu), \Psi \rangle_{X^* \times X}$. Denote by $\hat{A}^* \equiv G'_\Phi(\hat{\Phi}, \hat{\mu})^*: X \rightarrow X^*$ the operator adjoint for operator $\hat{A} \equiv G'_\Phi(\hat{\Phi}, \hat{\mu}): X \rightarrow X^*$ defined by

$$\langle \hat{A}^* \Psi, S \rangle_{X^* \times X} \equiv \langle G'_\Phi(\hat{\Phi}, \hat{\mu})^* \Psi, S \rangle_{X^* \times X} = \langle G'_\Phi(\hat{\Phi}, \hat{\mu}) S, \Psi \rangle_{X^* \times X} = \langle \hat{A} S, \Psi \rangle_{X^* \times X} \quad \forall \Psi, S \in X.$$

It follows from linearity of operator G with respect to μ_1, μ_2 and from convexity of the set $K = K_1 \times K_2$ that the set $G(\Phi, K) = \{\mathbf{x}^* = G(\Phi, \mu) \in X^*, \mu \in K\}$ is a convex subset of X^* for any

function $\Phi \in X$. As the operator $G'_\Phi(\hat{\Phi}, \hat{\mu}) \equiv \hat{A}$ is an isomorphism by Theorem 1, then from results of [23] the next theorem follows.

Theorem 3. *Let under assumptions (ii) and (j) the pair $(\hat{\Phi}, \hat{\mu}) \in X \times K$ where $\hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2)$ be a solution of problem (27). Then there exists a unique Lagrange multiplier $\hat{\Psi} \in X$ that satisfies the Euler-Lagrange equation*

$$a_{\hat{\mu}}(S, \hat{\Psi}) \equiv a_0(S, \hat{\Psi}) + a(\hat{\mu}_1, \hat{\mu}_2; S, \hat{\Psi}) = -\alpha_0 \left[\alpha (\nabla \hat{\Phi}, \nabla S)_{\Omega_i} + \beta (\hat{\Phi} - \Phi_a, S)_{\Omega_e} \right] \quad \forall S \in X \quad (34)$$

and the minimum principle holds, which is equivalent to the following variational inequalities:

$$\alpha_1(\hat{\mu}_1, \mu_1 - \hat{\mu}_1)_{s, \Omega_1} + a_1(\mu_1 - \hat{\mu}_1, \hat{\Phi}, \hat{\Psi}) \geq 0 \quad \forall \mu_1 \in K_1, \quad (35)$$

$$\alpha_2(\hat{\mu}_2, \mu_2 - \hat{\mu}_2)_{s, \Omega_2} + a_2(\mu_2 - \hat{\mu}_2, \hat{\Phi}, \hat{\Psi}) \geq 0 \quad \forall \mu_2 \in K_2. \quad (36)$$

We note that the direct problem (10), where one should put $\Phi = \hat{\Phi}$, $\mu = \hat{\mu}$, adjoint problem (34) with respect to the adjoint field $\hat{\Psi}$ and the inequalities (35), (36) represent the optimality system describing the necessary extremum conditions for the problem (27). The optimality system plays an important role when solving control problems. Firstly, it can be used to establish sufficient conditions for the data that ensure the uniqueness and stability of the optimal solutions to the considered control problems with respect to small perturbations of the data (see, for example, [24]).

Secondly, the constructed optimality system can be used to develop an efficient numerical algorithm for solving control problem (27). The simplest algorithm can be obtained using the simple iteration method. The m -th iteration of such an algorithm consists in finding unknown values of Φ^m , Ψ^m , μ_r^{m+1} and μ_θ^{m+1} given μ_r^m , μ_θ^m , $m=1, 2, \dots$, starting with the initial data μ_r^0 , μ_θ^0 by successively solving the following problems:

$$\begin{aligned} a_0(\Phi^m, S) + a_1(\mu_r^m; \Phi^m, S) + a_2(\mu_\theta^m; \Phi^m, S) &= \langle F, S \rangle \quad \forall S \in X, \\ a_0(S, \Psi^m) + a_1(\mu_r^m; S, \Psi^m) + a_2(\mu_\theta^m; S, \Psi^m) &= \\ &= \alpha_0 \left[\alpha \int_{\Omega_i} \nabla \Phi^m \cdot \nabla S d\mathbf{x} + \beta \int_{\Omega_e} (\Phi^m - \Phi_a) S d\mathbf{x} \right] \quad \forall S \in X, \\ \alpha_1(\mu_r^{m+1}, \mu_r - \mu_r^m)_{s, \Omega} + a_1((\mu_r - \mu_r^{m+1}) \Phi^m, \Psi^m) &\geq 0 \quad \forall \mu_r \in K_1, \\ \alpha_2(\mu_\theta^{m+1}, \mu_\theta - \mu_\theta^m)_{s, \Omega} + a_2((\mu_\theta - \mu_\theta^{m+1}) \Phi^m, \Psi^m) &\geq 0 \quad \forall \mu_\theta \in K_2. \end{aligned} \quad (37)$$

An alternative algorithm is based on the use of the particle swarm method according to the scheme proposed in [27, 28] for the numerical solution of 2D problem of thermal cloaking.

4. Conclusion

In conclusion, we studied control problems for magnetic scattering model (1)–(3). These problems arise when optimization method is applied for solving cloaking problems for respective scattering model. The magnetic permeabilities μ_1 and μ_2 of the inhomogeneous media filling the layers Ω_1 and Ω_2 of the cloak play the role of controls. We have proved the correctness of direct magnetic scattering problem and proved solvability of general control problem (28). Also, we derived the optimality system describing the necessary conditions of extremum and proposed two numerical algorithms for solving our cloaking problems. Authors plan to devote a forthcoming paper to developing and studying the properties of the mentioned algorithms and to comparative analysis of results of numerical experiments performed using these algorithms.

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OPTIMIZATION METHOD OF SOLVING 2D PROBLEMS OF DESIGNING SHIELDING AND CLOAKING DEVICES

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Abstract. We study inverse problems for the 2D model of electric conductivity that arise when designing circular shielding or cloaking shells and other functional devices used to control static electric fields. It is assumed that the shells consist of a finite number of circular layers filled with isotropic (inhomogeneous or homogeneous) media. By optimization method our inverse problems are reduced to corresponding control problems in which the role of controls is played by shell layer conductivities. The solvability of the direct and control problems is proved. A numerical algorithm based on the particle swarm optimization method is proposed for solving control problems and some simulation results are discussed.

Key words electric conductivity model, inverse problem, optimization method, cloaking shielding, solvability, particle swarm optimization.

1. Introduction

In recent years new direction in mathematical and technical physics has been intensively developed. It concerns developing methods of solving problems of designing special functional devices for controlling physical fields. An important example of such devices is a material shell in the form of the circular ring (or spherical layer in the space R^3) filled with a anisotropic inhomogeneous medium in general case. This shell can be used, for example, to cloak any object placed inside it from detection using a wave or static physical field.

The first works [1, 2] in this field were devoted to designing invisibility cloaking devices with respect to electromagnetic waves on the basis of the transformation optics method proposed in [1]. Then the main results from electromagnetic cloaking were expanded to acoustic cloaking [3] and later to cloaking static fields (see, e.g. [4–8]). However, we emphasize that the solutions obtained in these papers possess several drawbacks. In particular, it is very difficult to implement these solutions technically. One of approaches of overcoming these difficulties consists of using the optimization method of solving inverse problems. The method to which one should referred to as inverse design method [9] was applied firstly in [10]. Also, it was applied in [11–19] when studying theoretically electromagnetic or acoustic cloaking problems.

The paper consists of two parts. In the first part we formulate firstly the direct and inverse problems for electrical conductivity model that are connected with designing circular shielding or cloaking shells. Then we reduce our inverse problems to respective control problems and present two theorems about global solvability both direct and control problems. In the second part we develop a numerical method of solving our control problems based on particle swarm optimization method [20] using the scheme proposed in [21–22] when solving problems of designing thermal shielding or cloaking devices. Finally, we discuss some results of numerical experiments.

2. Statement of direct and inverse problems

We begin with statement of the general direct conductivity problem considered in a rectangle $D = \{\mathbf{x} \equiv (x, y): |x| < x_0, |y| < y_0\}$ with specified numbers $x_0 > 0$ and $y_0 > 0$. We assume that an external electric potential U^e is created by two vertical plates $x = \pm x_0$ which are kept at different values U_1 and U_2 , while the upper and lower plates $y = \pm y_0$ are electrically insulated. Let us assume also that the medium filling D is homogeneous and isotropic and its electroconductive properties are

described by constant conductivity $\sigma_b > 0$. If, besides, $U_1 = \text{const}$, $U_2 = \text{const}$ then the potential U^e describes one-dimensional (depending on coordinate x) field with constant gradient having the form

$$U^e(x) = \frac{U_2 - U_1}{2} \frac{x}{x_0} + \frac{U_1 + U_2}{2}. \quad (1)$$

We assume further that there is a material layered shell $\Omega = \{\mathbf{x}: a < |\mathbf{x}| < b\}$ inside D which consists of M concentric circular layers $\Omega_k = \{r_{k-1} < r = |\mathbf{x}| < r_k\}$, $k = 1, 2, \dots, M$ where $r_0 = a, r_M = b$. Each of these layers is filled with an inhomogeneous in general case isotropic medium, described by variable conductivity $\sigma_k, k = 1, 2, \dots, M$.

We assume also that interior $\Omega_i: |\mathbf{x}| < a$ and exterior $\Omega_e: |\mathbf{x}| > b$ of Ω are filled with the same homogeneous medium having constant electrical conductivity $\sigma_b > 0$ (see Fig. 1).

In this case, the direct electrical conductivity problem consists of finding $M + 2$ functions, namely, u_i in Ω_i, u_k in $\Omega_k, k = 1, 2, \dots, M$ and u_e in Ω_e , which satisfy the equations

$$\sigma_b \Delta u_i = 0 \text{ in } \Omega_i, \quad \sigma_b \Delta u_e = 0 \text{ in } \Omega_e, \quad (2)$$

$$\text{div}(\sigma_k \text{grad } u_k) = 0 \text{ in } \Omega_k, \quad k = 1, 2, \dots, M, \quad (3)$$

obey the following boundary conditions on the boundary ∂D of D :

$$u_e|_{x=-x_0} = u_1, \quad u_e|_{x=x_0} = u_2, \quad \partial u_e / \partial y|_{y=-y_0} = 0, \quad (4)$$

satisfy the matching conditions on internal Γ_i and external Γ_e components of the boundary Γ of the shell Ω having the form

$$u_i = u, \quad \sigma_b \frac{\partial u_i}{\partial n} = (\sigma \nabla u) \cdot \mathbf{n} \text{ on } \Gamma_i, \quad u_e = u, \quad \sigma_b \frac{\partial u_e}{\partial n} = (\sigma \nabla u) \cdot \mathbf{n} \text{ on } \Gamma_e \quad (5)$$

and satisfy the following matching conditions:

$$u_k = u_{k+1}, \quad \sigma_k \frac{\partial u_k}{\partial n} = \sigma_{k+1} \frac{\partial u_{k+1}}{\partial n} \text{ at } |\mathbf{x}| = r_k, k = \overline{1, M-1} \quad (6)$$

on interfaces $|\mathbf{x}| = r_k$. In the particular case when $\sigma_k = \sigma_b, k = 1, 2, \dots, M$, so that the entire medium filling domain D is homogeneous and isotropic and, besides, $U_1 = \text{const}$, $U_2 = \text{const}$, the solution of (2)-(6) is described by formula (1).

Let us assume that the following conditions take place:

- (i) $\sigma_k \in L^\infty(\Omega_k), \sigma_k \geq \sigma_k^0 = \text{const} > 0, k = 1, 2, \dots, M$,
- (ii) $U_1 \in H^{1/2}(\Gamma_1), U_2 \in H^{1/2}(\Gamma_2)$.

Here $H^{1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_2)$ are the well known trace spaces. It should be noted that conditions (i), (ii) ensure the existence and uniqueness of the weak solution to the direct problem (2)–(6). More concretely, the following theorem is valid.

Theorem 1. Let conditions (i), (ii) take place. Then problem (2)–(6) has a unique weak solution $(u_i, u_1, \dots, u_M, u_e) \in H^1(\Omega_i) \times H^1(\Omega_1) \times \dots \times H^1(\Omega_M) \times H^1(\Omega_e)$, which satisfies equations in (2), (3) in the distribution sense and boundary conditions in (4), (5) and (6) in the sense of traces. The following estimate holds for the solution $(u_i, u_1, \dots, u_M, u_e)$:

$$\|u_i\|_{H^1(\Omega_i)} + \sum_{k=1}^M \|u_k\|_{H^1(\Omega_k)} + \|u_e\|_{H^1(\Omega_e)} \leq C \left(\|U_1\|_{H^{1/2}(\Gamma_1)} + \|U_2\|_{H^{1/2}(\Gamma_2)} \right).$$

Here C is a constant independent of U_1, U_2 .

We remind that our goal is to solve inverse problems for the model (2)-(6) associated with designing shielding or cloaking shells and other functional devices for controlling static electric fields. Generally, the inverse problem consists of finding functions $\sigma_1, \sigma_2, \dots, \sigma_M$ from the two conditions

$$\nabla u_i = 0 \text{ in } \Omega_i, \quad u_e = U^e \text{ in } \Omega_e. \quad (8)$$

Here, (u_i, u_e) is the restriction of the solution of problem (2)-(6) to $\Omega_i \times \Omega_e$. The shell $(\Omega, \sigma_1, \sigma_2, \dots, \sigma_M)$ which ensures the exact fulfillment of conditions (8) is called a perfect cloaking shell or simply a cloak. In the case when σ is determined solely from the first (or second) condition in (8) we will refer to the corresponding inverse problem as shielding (or external cloaking) problem.

3. Applying optimization method. Formulation of control problems

For solving our inverse problems we apply the optimization method. By this method the inverse problems are reduced to extremum problems of minimization of certain cost functionals which adequately correspond to the inverse problems of designing corresponding devices [22]. These functionals depend on conductivities $\sigma_1, \dots, \sigma_M$ of separate sublayers $(\Omega_1, \sigma_1), \dots, (\Omega_M, \sigma_M)$, of Ω . It is comfortable to define a variable vector $\mathbf{s} = (\sigma_1, \sigma_2, \dots, \sigma_M)$, to which we will refer to as a conductivity vector of the shell $(\Omega, \mathbf{s})$.

In order to formulate our control problems, we denote by $U[\mathbf{s}] = U[\sigma_1, \sigma_2, \dots, \sigma_M]$ the solution to problem (2)-(6) corresponding to (variable) conductivities σ_k in $\Omega_k, k = 1, \dots, M$, and to constant conductivity σ_b in Ω_i and Ω_e . We assume below that the vector $\mathbf{s} = (\sigma_1, \sigma_2, \dots, \sigma_M)$ belongs to the bounded set

$$S = \{\mathbf{s} : 0 < \sigma_{\min} \leq \sigma_k \leq \sigma_{\max}, k = 1, 2, \dots, M\} \quad (9)$$

to which we refer to as a control set. Here given positive constants $\sigma_{\min}$ and $\sigma_{\max}$ are lower and upper bounds of the control set S . Let us define three cost functionals

$$J_i(\mathbf{s}) = \frac{\|\nabla u_i[\mathbf{s}]\|_{L^2(\Omega_i)}}{\|\nabla U^e\|_{L^2(\Omega_i)}}, \quad J_e(\mathbf{s}) = \frac{\|u_e[\mathbf{s}] - U^e\|_{L^2(\Omega_e)}}{\|U^e\|_{L^2(\Omega_e)}}, \quad J(\mathbf{s}) = \alpha J_i(\mathbf{s}) + \beta J_e(\mathbf{s}), \quad (10)$$

where $\alpha, \beta \in [0, 1]$ are nonnegative numbers satisfying $\alpha + \beta = 1$,

$$\|U[\mathbf{s}] - U^e\|_{L^2(\Omega_e)}^2 = \int_{\Omega_e} |U[\mathbf{s}] - U^e|^2 dx, \quad \|U^e\|_{L^2(\Omega_e)}^2 = \int_{\Omega_e} |U^e|^2 dx, \\ \|\nabla u_i[\mathbf{s}]\|_{L^2(\Omega_i)}^2 = \int_{\Omega_i} |\nabla u_i[\mathbf{s}]|^2 dx,$$

and formulate the following control problem:

$$J(\mathbf{s}) = [\alpha J_i(\mathbf{s}) + \beta J_e(\mathbf{s})] \rightarrow \min, \mathbf{s} \in S. \quad (11)$$

We note that in the particular case $\alpha = 1, \beta = 0$ (or $\alpha = 0, \beta = 1$) problem (11) corresponds to the shielding problem (or to the external cloaking problem). In another case when $\alpha = \beta = 0.5$ it corresponds to the general cloaking problem.

Now we assume additionally that the following condition takes place:

$$(iii) \quad \sigma_k \in H^s(\Omega_k), s > 1, k = 1, 2, \dots, M.$$

For theoretical study of general control problem (11) one can apply the mathematical procedure developed in [17] for solving inverse problems of mathematical physics by using optimization method. Based on this procedure we can prove the following theorem.

Theorem 2. Let conditions (ii), (iii) take place. Then problem (11), where the set S and functionals $J_i(\mathbf{s}), J_e(\mathbf{s})$ are defined in (9), (10), has at least one solution.

Now we consider the important particular case when all layers $\Omega_1, \Omega_2, \dots, \Omega_M$ are filled with different homogeneous media. In this case all conductivities σ_k are constants and therefore problem (11) is finite dimensional. Moreover, one can show arguing as in [23] that in this case a cloaking performance of the cloak $(\Omega, \mathbf{s})$ is connected with a value $J(\mathbf{s})$ by inverse dependence: a smaller value $J(\mathbf{s})$ corresponds to a higher cloaking performance of the cloak $(\Omega, \mathbf{s})$ and vice versa. Therefore, our goal when solving problem (11) will consist of finding a conductivity vector (an optimal solution of

(11)) $\mathbf{s}^{\text{opt}} \in S$ for which functional J takes a minimum value $J^{\text{opt}} = J(\mathbf{s}^{\text{opt}})$ on the set S and therefore the cloak $(\Omega, \mathbf{s}^{\text{opt}})$ possesses a maximum cloaking performance. For numerical solving problem (11) in this case we will use algorithm based on the particle swarm optimization (PSO) method [20].

4. Results of numerical simulations

In this section we discuss numerical results obtained when designing multilayer cloaking shells using PSO algorithm. Numerical simulation was performed for the following problem data: $a = 1\text{m}$, $b = 2\text{m}$, $\sigma_i = \sigma_e = 1.45 \times 10^6 \sigma_0$ (corresponds to stainless steel) where $\sigma_0 = 1\text{ S/m}$, $\sigma_{\min} = 1 \times 10^4\text{ S/m}$ (corresponds to graphite), $\sigma_{\max} = 4.55 \times 10^7\text{ S/m}$ (corresponds to gold). The role of the externally applied field U^e was played by the field (1).

The first group of tests is related to the shielding problem. Our optimization analysis using PSO algorithm for shielding problem showed that optimal values σ_K^{opt} of all parameters σ_K with odd indices $k = 1, 3, 5, \dots, M-1$ coincide with $\sigma_{\max}$ while optimal values $\sigma_2^{\text{opt}}, \dots, \sigma_M^{\text{opt}}$ (with even indices) coincide with $\sigma_{\min}$. This means that the optimization design coincides with so-called alternating design (see details in [23]). The corresponding optimal values $J(\mathbf{s}^{\text{opt}})$ where $\mathbf{s}^{\text{opt}} = (\sigma_1^{\text{opt}}, \sigma_2^{\text{opt}}, \dots, \sigma_M^{\text{opt}})$ of functional $J(\mathbf{s})$ are presented together with $\sigma_M^{\text{opt}}/\sigma_0$ and the values $J_e(\mathbf{s}^{\text{opt}})$, $J_i(\mathbf{s}^{\text{opt}})$ for comparison in Table I for different $M = 2, 4, \dots, 12$ specified in the first column.

It is seen from Table I that values $J_i(\mathbf{s}^{\text{opt}})$ decrease from 8.85×10^{-3} to 1.16×10^{-7} when M varies from 2 to 12. The last value $J_i(\mathbf{s}^{\text{opt}}) = 1.16 \times 10^{-7}$ for $M = 12$ corresponds to very high shielding performance of the optimal shell $(\Omega, \mathbf{s}^{\text{opt}})$. At the same time, the values $J_e(\mathbf{s}^{\text{opt}})$ and $J(\mathbf{s}^{\text{opt}})$ are large enough since we minimize just the functional $J_i(\mathbf{s})$. Presented results confirm the high performance of shielding shells obtained by using the optimization design for shells with a small number of homogeneous isotropic layers.

TABLE I
SIMULATION RESULTS FOR OPTIMIZED MULTILAYERED SHIELDS
($\sigma_{\min} = 1 \times 10^4\text{ S/m}$, $\sigma_{\max} = 4.55 \times 10^7\text{ S/m}$)

M	$\sigma_M^{\text{opt}}/\sigma_0$	$J_i(\mathbf{s}^{\text{opt}})$	$J_e(\mathbf{s}^{\text{opt}})$	$J(\mathbf{s}^{\text{opt}})$
2	4.55×10^7	8.85×10^{-3}	1.43×10^{-1}	1.51×10^{-1}
4	4.55×10^7	2.2×10^{-3}	1.01×10^{-2}	1.03×10^{-1}
6	4.55×10^7	1.52×10^{-5}	7.35×10^{-2}	7.35×10^{-2}
8	4.55×10^7	2.01×10^{-6}	5.32×10^{-2}	5.32×10^{-2}
10	4.55×10^7	4.11×10^{-7}	3.82×10^{-2}	3.82×10^{-2}
12	4.55×10^7	1.16×10^{-7}	2.68×10^{-2}	2.68×10^{-2}

The second group of tests is related to the cloaking problem. Our optimization analysis showed that optimal values σ_k^{opt} obtained by using PSO algorithm for cloaking problem almost coincide with values σ_k^{alt} corresponding to alternating design for all j except $j = M$. The last optimal control σ_M^{opt} takes some intermediate value between $\sigma_{\min}$ and $\sigma_{\max}$. The found values σ_M^{opt} are presented together with the corresponding values $J(\mathbf{s}^{\text{opt}})$, $J_i(\mathbf{s}^{\text{opt}})$ and $J_e(\mathbf{s}^{\text{opt}})$ where $\mathbf{s}^{\text{opt}} = (\sigma_1^{\text{opt}}, \sigma_2^{\text{opt}}, \dots, \sigma_M^{\text{opt}})$ in Table II for different M equaled to 2, 4, 6, 8, 10 and 12.

In particular, it is seen from Table II that values $J(\mathbf{s}^{\text{opt}})$ decrease from 3.1×10^{-2} to 3.08×10^{-7} when M varies from 2 to 12. The last value $J(\mathbf{s}^{\text{opt}}) = 3.08 \times 10^{-7}$ corresponds to very high cloaking performance of the optimal shell $(\Omega, \mathbf{s}^{\text{opt}})$. Thus, presented results confirm the high performance of cloaking shells obtained by using the optimization design for shells with a small number of layers.

5. Conclusion

We studied control problems for electric conductivity model (2)–(6). These problems arise when optimization method is applied for solving static electric field shielding and cloaking problems. The electric conductivities of the shell layers play the role of passive controls. We have proved the

theorem about correct solvability of the direct conductivity problem and proved solvability of general control problem. We also proposed numerical algorithm for solving our control problems which is based on particle swarm optimization method. Optimization analysis showed that a high shielding or cloaking performance of the designed devices can be achieved when using multilayer shell consisting of several isotropic homogeneous layers with optimal constant conductivities. We emphasize that high cloaking and shielding performances can be achieved without use of metamaterials, but using natural materials with high contrast.

TABLE II
SIMULATION RESULTS FOR OPTIMIZED MULTILAYERED CLOAKS
($\sigma_{min} = 1 \times 10^4 \text{ S/m}$, $\sigma_{max} = 4.55 \times 10^7 \text{ S/m}$)

M	σ_M^{opt}/σ_0	$J_i(\mathbf{s}^{opt})$	$J_e(\mathbf{s}^{opt})$	$J(\mathbf{s}^{opt})$
2	5.09×10^6	3.1×10^{-2}	8.66×10^{-6}	3.1×10^{-2}
4	1.04×10^7	4.49×10^{-4}	1.21×10^{-6}	4.51×10^{-4}
6	1.55×10^7	2.41×10^{-5}	3.66×10^{-7}	2.45×10^{-5}
8	2.03×10^7	2.7×10^{-6}	3.85×10^{-6}	6.58×10^{-6}
10	2.49×10^7	5.08×10^{-7}	2.29×10^{-7}	7.38×10^{-7}
12	2.92×10^7	1.34×10^{-7}	1.73×10^{-7}	3.08×10^{-7}

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ON RECOVERING OF DIFFUSION COEFFICIENT FOR GENERALIZED BOUSSINESQ MODEL

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Abstract. Boundary value and inverse coefficient problems for a generalized Oberbeck-Boussinesq model are considered under the assumption that the reaction coefficient depends nonlinearly on the substance's concentration. A diffusion coefficient is considered as the unknown coefficient and is recovered with the help of the additional information about the boundary value problem's solution. The inverse coefficient problem is reduced to multiplicative control problem, the solvability of which is proved in general form.

Key words nonlinear mass transfer model, generalized Oberbeck-Boussinesq model, inverse coefficient problem, multiplicative control problem

1. Introduction

For a long time an interest for the studying of inverse problems for linear and nonlinear models of heat-and-mass transfer doesn't fade. These mentioned problems consist in recovering of unknown densities of boundary or distributed sources of coefficients in differential equations of the model or in boundary conditions with the help of additional information about the system's state, which is described by a model. One of such methods of considering inverse problems is the optimisation method which implies the reduction of inverse or identification problems to extremum ones (see more in [1]).

The papers [2,3] are dedicated to the study of the inverse problems for a linear model of heat-and-mass transfer which consists of a convection-diffusion-reaction equation with boundary conditions. Further, let us also note the articles [4-12], in which inverse problems for nonlinear heat-and-mass transfer models in a classical approximation of Oberbeck-Boussinesq were considered. The papers [13-18] are focused on boundary value and extremum problems for convection-diffusion-reaction equation, in which reaction coefficient depends nonlinearly on substance's concentration. In [19,20] similar models of complex heat transfer were considered. From a range of papers dedicated to the study of boundary value and extremum problems for various models which generalize a classical approximation of Oberbeck-Boussinesq let us note [21-23]. About the research of more complicated rheological models and of models of multi-component viscous compressible fluids see, respectively, [24,25] and [26,27].

2. Boundary value problem

In a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary Γ we consider the following boundary value problem:

$$-\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} + \beta \mathbf{G} \varphi, \quad \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \quad (2.1)$$

$$-\lambda \Delta \varphi + \mathbf{u} \cdot \nabla \varphi + k(\varphi, \mathbf{x}) = f \text{ in } \Omega \quad (2.2)$$

$$\mathbf{u} = \mathbf{0}, \quad \varphi = 0 \text{ on } \Gamma. \quad (2.3)$$

Here $\mathbf{u}$ is a velocity vector, function φ presents the concentration of the pollutant, $p = P/\rho$, where P is pressure, $\rho = \text{const}$ is the fluid density, $\nu = \text{const} > 0$ is the constant kinematic viscosity, $\lambda = \lambda(\mathbf{x})$ is the diffusion coefficient, β is the coefficient of mass expansion, $\mathbf{G} = (0, 0, G)$ is the acceleration of gravity, $\mathbf{f}$ and f are volume densities of external forces or external sources of the substance, the function $k = k(\varphi, \mathbf{x})$ is the reaction coefficient, where $\mathbf{x} \in \Omega$. This problem (2.1)–(2.3) for given functions $\mathbf{f}$, f , λ , β and k will be called Problem 1 below.

In this paper the global solvability of Problem 1 is proved, sufficient condition for its solution's uniqueness are stated. Furthermore, with the help of the optimisation approach the problem of restoration of diffusion coefficient λ using the concentration φ , which was measured in a subdomain $Q \subset \Omega$, is reduced to the multiplicative control problem. Its solvability is proved in a general case. When

the reaction coefficient and cost functionals are Frechet differentiable we get optimality systems for the extremum problem.

While studying the considered problems, we will use Sobolev functional spaces $H^s(D)$, $s \in \mathbf{R}$. Here D means either the domain Ω or some subset $Q \in \Omega$, or the boundary Γ . By $\|\cdot\|_{s,Q}$, $|\cdot|_{s,Q}$ and $(\cdot, \cdot)_{s,Q}$ we will denote the norm, seminorm and the scalar product in $H^s(Q)$. The norms and scalar products in $L^2(Q)$ and $L^2(\Omega)$ will be denoted correspondly by $\|\cdot\|_Q$ and $(\cdot, \cdot)_Q$, $\|\cdot\|_\Omega$ and $(\cdot, \cdot)_\Omega$.

Let

$$L_+^p(\Omega) = \{k \in L^p(\Omega): k \geq 0\}, p \geq \frac{3}{2}, H_{\lambda_0}^{3/2}(\Omega) = \{h \in H^3(\Omega): h \geq \lambda_0 > 0\},$$

by

$$V = \{\mathbf{v} \in H_0^1(\Omega)^3: \text{div } \mathbf{v} = 0 \text{ in } \Omega\}$$

we introduce the main functional space for velocity vector.

Let the following conditions hold:

(i) Ω is a bounded domain in the space $\mathbf{R}^3$ with boundary $\Gamma \in C^{0,1}$;

(ii) $\mathbf{f} \in L^2(\Omega)^3, \mathbf{f} \in L^2(\Omega), \mathbf{b} = \beta \mathbf{G} \in L^2(\Omega)^3, \lambda \in H_{\lambda_0}^{3/2}(\Omega)$;

(iii) for any function $w \in H^1(\Omega)$ the embedding $k(w, \cdot) \in L_+^p(\Omega)$ is true for some $p \geq 3/2$, which does not depend on w , and on any sphere $B_r = \{w \in H_0^1(\Omega): \|w\|_{1,\Omega} \leq r\}$ of radius r the following inequality takes place:

$$\|k(w_1, \cdot) - k(w_2, \cdot)\|_{L^p(\Omega)} \leq L \|w_1 - w_2\|_{L^4(\Omega)} \quad \forall w_1, w_2 \in B_r.$$

Here L is the constant which depends on r , but does not depend on $w_1, w_2 \in B_r$.

Let us note that the condition (iii) describes an operator, acting from $H^1(\Omega)$ to $L^p(\Omega)$, where $p \geq 3/2$, which gives us an opportunity to take into consideration the dependence of the reaction coefficient on either the component of solution the $(\mathbf{u}, \varphi, p)$ of Problem 1 or on the spatial variable $\mathbf{x} \in \Omega$.

For example,

$$k_1 = \varphi^2 \text{ (or } k_1 = \varphi^2 |\varphi|) \text{ in subdomain } Q \subset \Omega \text{ or } k_1 = k_0(x) \text{ in } \Omega \setminus Q, \\ \text{where } k_0(x) \in L_+^2(\Omega \setminus Q).$$

From a physical point of view, the coefficient k_1 corresponds to the situation, when the substance's decomposition rate is proportional to the square (or cube) of substance's concentration in a subdomain $Q \subset \Omega$ and outside Q , and the rate of the chemical reaction depends only on a spatial variable [15,17].

The following theorem holds

Theorem 1. *If the conditions (i)–(iii) hold then there exists a weak solution $(\mathbf{u}, \varphi, p) \in V \times H_0^1(\Omega) \times L_0^2(\Omega)$ of Problem 1 and the following estimates hold*

$$\|\varphi\|_1 \leq M_\varphi, \quad \|\mathbf{u}\|_1 \leq M_u, \quad \|p\| \leq M_p,$$

where M_φ, M_u and M_p are non-decreasing function on initial date of Problem 1.

3. Multiplicative control problem

In the framework of the optimisation approach the problem of the restoration of diffusion coefficient using an additional information about the solution of the Problem 1 can be reduced to the multiplicative control problem (see for example [12,18]). To state the control problem let us divide the whole set of initial data of the Problem 1 into two groups: group of fixed data which includes functions $\mathbf{f}, \mathbf{f}, \mathbf{b}$ and $k(\varphi)$, and group of controls consisting of function λ . Here we assume that it can be changed in some set K .

We set $X = H_0^1(\Omega)^3 \times H_0^1(\Omega) \times L_0^2(\Omega)$, $\mathbf{x} = (\mathbf{u}, \varphi, p) \in X$ and introduce an operator $F = (F_1, F_2): X \times K \rightarrow X^*$ by formula

$$\langle F_1(\mathbf{x}, \lambda), (\mathbf{v}, h) \rangle = \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + (\lambda \nabla \varphi, \nabla h) + ((\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v}) - (p, \text{div } \mathbf{v}) + \\ + (k(\varphi) \varphi, h) + (\mathbf{u} \cdot \nabla \varphi, h) - (\mathbf{f}, \mathbf{v}) - (\mathbf{b} \varphi, \mathbf{v}) - (\mathbf{f}, h),$$

$$\langle F_2(\mathbf{x}, \lambda), r \rangle = -(\operatorname{div} \mathbf{u}, r).$$

The equation $F(\mathbf{x}, \lambda) = 0$ is the operator form of weak formulation of Problem 1.

Let I is a weakly lower semicontinuous functional. Consider the following control problem:

$$J(\mathbf{x}, \lambda) \equiv \frac{\mu_0}{2} I(\varphi) + \frac{\mu_1}{2} \|\lambda\|_{3/2, \Omega}^2 \rightarrow \inf, F(\mathbf{x}, \lambda) = 0, (\mathbf{x}, \lambda) \in X \times K. \quad (3.1)$$

Here μ_0 and μ_1 are non-negative parameters which are used to control the relative importance of terms in (28). Denote by

$$Z_{ad} = \{(\mathbf{x}, \lambda) \in X \times K : F(\mathbf{x}, \lambda) = 0, J(\mathbf{x}, \lambda) < \infty\}$$

The set of feasible pairs for the problem (3.1) and assume that the following conditions hold

(j) $K \subset H_{\lambda_0}^{3/2}(\Omega)$ is nonempty convex closed set,

(jj) $\mu_0 > 0, \mu_1 \geq 0$ and K is a bounded set or $\mu_0 > 0, \mu_1 > 0$ and the functional I is bounded below.

The following cost functional can be used in the capacity of the possible one:

$$\begin{aligned} I_1(\varphi) &= \|\varphi - \varphi^d\|_Q^2, \quad I_2(\varphi) = \|\varphi - \varphi^d\|_{1,Q}^2, \\ I_3(\mathbf{u}) &= \|\mathbf{u} - \mathbf{u}^d\|_Q^2, \quad I_4(p) = \|p - p^d\|_Q^2, \end{aligned} \quad (3.2)$$

where function $\varphi^d \in L^2(Q)$ denote some desired concentration field given in a subdomain $Q \subset \Omega$. Functions $\mathbf{u}^d$ and p^d have similar sense for the velocity field ore pressure.

The following theorem holds

Theorem 2. *Let, under assumptions (i)-(ii) and (j), (jj) $I: X \rightarrow \mathbb{R}$ be a weakly lower semicontinuous functional and a set Z_{ad} is nonempty. Then the control problem (28) has at least one solution $(\mathbf{x}, \lambda) \in X \times K$.*

4. Conclusion

In conclusion, we studied boundary value and control problems for the generalized Oberbeck-Boussinesq model. In future papers we will obtain the optimality systems for the control problems (28). Using obtained systems, we will derive local stability estimates of optimal solutions.

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ON ALGEBRAIC ATTACKS ON SOME KEYSTREAM GENERATORS

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Abstract. Many stream ciphers are used to encrypting the data stream at high speed. These stream ciphers are based on keystream generators. The keystream generator produces an output stream of arbitrary length, called a keystream, after initialization with a secret value, called a key. The legal receiver can produce the same keystream with the secret key and decrypt the encrypted data stream. The main problem of any algebraic attack is the need to solve a nonlinear system of algebraic equations. We used two approaches for solving these systems a linearization method and a method based on the Gröbner basis [1, 2]. All explored in the work generators that are based on linear feedback shift registers (LFSR). We supposed that an adversary knows everything about the generator beside the secret key. As result where build up a few generators and algebraic attacks on them.

Key words: keystream generators, algebraic attacks, Gröbner basis.

Let $K = (k_0, \dots, k_{n-1})$ be the unknown initial state and L be the known feedback matrix (see [1]). The secret key K and matrix L would produce a keystream $k_0, k_1, k_2, \dots$ where $(k_t, \dots, k_{t+n-1}) = KL^t$. If an adversary knows the values $k_{t_1}, \dots, k_{t_m}$ of the keystream elements at the m clocks $t_1, \dots, t_m$ he can set up the following system of linear equations:

$$\begin{aligned} k_{t_1} &= KL^{t_1}P, \\ &\dots \\ k_{t_m} &= KL^{t_m}P, \end{aligned} \tag{1}$$

where $P = (1, 0, \dots, 0)^t$. The linearity of the system is a real treat for a generator of keystream. Thus, to strengthen LFSR-based keystream generators, one has to incorporate some kind of non-linearity. To apply, for example, a non-linear function to the outputs from several LFSRs and to output the result.

Definition 1. Let F be a finite field. A (l, m) –combiner consists of the following components:

- 1) s LFRSs with lengths $n_1, \dots, n_s$ and feedback matrices $L_1, \dots, L_s$;
- 2) an internal state $S \in F^m \times F^n$ where $n = n_1 + \dots + n_s$;
- 3) a matrix L over F of size $n \times n$, defined by

$$L := \begin{pmatrix} L_1 & & 0 \\ & \ddots & \\ 0 & & L_s \end{pmatrix};$$

- 4) a (projection) matrix P over F of size $n \times l$;
- 5) a non-linear next memory state function $\Psi: F^m \times F^l \rightarrow F^m$;
- 6) an output function $f: F^m \times F^l \rightarrow F$.

The definition supposes the combiner consist of s LFSRs and use outputs of them and the memory for generation the final outputs.

Let start with simple generator over field F_2 . It uses three LFSRs lengths three, four and five, respectively. Their initial states are denoted by $A_0 = (a_0, a_1, a_2), B_0 = (b_0, b_1, b_2, b_3), C_0 = (c_0, c_1, c_2, c_3, c_4)$. The minimal polynomial of LFSR A is $m_a(x) = x^3 + x + 1$ and the sequence (a_t) produced by LFRS A fulfills the recursion $a_{t+3} = a_{t+1} + a_t$. The minimal polynomial of B is $m_b(x) = x^4 + x^3 + 1$ and the recursion $b_{t+4} = b_{t+3} + b_t$. The third minimal polynomial is $m_c(x) = x^5 + x^2 + 1$ with the recursion $c_{t+5} = c_{t+2} + c_t$. In this case $K := (a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_0, c_1, c_2, c_3, c_4)$. The feedback matrix L and the projection matrix are defined as

$$L := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

At each clock t , the keystream element z_t is generated by the function $z_t = f(K_t) = f(a_t, b_t, c_t) = a_t b_t + a_t c_t$. Hereby, a_t, b_t, c_t, z_t are the outputs from LFRSs A, B, C and the keystream at the clock t , respectively.

Let $A_0 = (0, 0, 1), B_0 = (0, 0, 0, 1), C_0 = (1, 0, 1, 1, 1)$ than the generated keystream we may see in the table 1.

Table 1

											0	1	2	3	4	5	6	7	8	9
a_t																				
b_t																				
c_t																				
z_t																				

On the base of the Table 1 we can construct the system of non-linear equations. For a successful algebraic attack need to solve the system.

- 1) $a_0 b_0 + a_0 c_0 = 0,$
- 2) $a_1 b_1 + a_1 c_1 = 0,$
- 3) $a_2 b_2 + a_2 c_2 + 1 = 0$
- 4) $a_0 b_3 + a_1 b_3 + a_0 c_3 + a_1 c_3 = 0$
- 5) $a_1 b_0 + a_1 b_3 + a_2 b_0 + a_1 c_4 + a_2 c_4 = 0$
- 6) $a_2 b_1 + a_2 b_0 + a_2 b_3 + a_0 b_1 + a_0 b_0 + a_0 b_3 + a_1 b_1 + a_1 b_0 + a_1 b_3 + a_2 c_0 + a_2 c_2 + a_0 c_0 + a_0 c_2 + a_1 c_0 + a_1 c_2 + 1 = 0$
- 7) $a_0 b_2 + a_0 b_1 + a_0 b_0 + a_0 b_3 + a_2 b_2 + a_2 b_1 + a_2 b_0 + a_2 b_3 + a_0 c_1 + a_0 c_3 + a_2 c_1 + a_2 c_3 = 0$
- 8) $a_0 b_2 + a_0 b_1 + a_0 b_0 + a_0 c_2 + a_0 c_4 = 0$
- 9) $a_1 b_2 + a_1 b_1 + a_1 b_3 + a_1 c_3 + a_1 c_0 + a_1 c_2 = 0$
- 10) $a_2 b_2 + a_2 b_0 + a_2 c_4 + a_2 c_1 + a_2 c_3 = 0$
- 11) $a_0 b_1 + a_0 b_3 + a_1 b_1 + a_1 b_3 + a_0 c_0 + a_0 c_4 + a_1 c_0 + a_1 c_4 = 0$
- 12) $a_2 b_2 + a_2 b_0 + a_2 b_3 + a_1 b_2 + a_1 b_0 + a_1 b_3 + a_2 c_1 + a_2 c_1 + a_2 c_0 + a_2 c_2 + a_1 c_1 + a_1 c_0 + a_1 c_2 + 1 = 0$
- 13) $a_0 b_1 + a_0 b_0 + a_1 b_1 + a_1 b_0 + a_2 b_1 + a_2 b_0 + a_0 c_2 + a_0 c_1 + a_0 c_3 + a_1 c_2 + a_1 c_1 + a_1 c_3 + a_2 c_2 + a_2 c_1 + a_2 c_3 = 0$
- 14) $a_2 b_2 + a_2 b_1 + a_0 b_2 + a_0 b_1 + a_2 c_3 + a_2 c_2 + a_2 c_4 + a_0 c_3 + a_0 c_2 + a_0 c_4 + 1 = 0$
- 15) $a_0 b_2 + a_0 b_3 + a_0 c_4 + a_0 c_3 + a_0 c_0 + a_0 c_2 = 0$
- 16) $a_1 b_0 + a_1 c_0 + a_1 c_2 + a_1 c_4 + a_1 c_1 + a_1 c_3 = 0$
- 17) $a_2 b_1 + a_2 c_1 + a_2 c_3 + a_2 c_0 + a_2 c_4 + 1 = 0$
- 18) $a_0 b_2 + a_1 b_2 + a_0 c_4 + a_0 c_1 + a_0 c_0 + a_1 c_4 + a_1 c_1 + a_1 c_0 = 0$
- 19) $a_1 b_3 + a_2 b_3 + a_1 c_0 + a_1 c_1 + a_2 c_0 + a_2 c_1 = 0$
- 20) $a_2 b_0 + a_2 b_3 + a_0 b_0 + a_0 b_3 + a_1 b_0 + a_1 b_3 + a_2 c_1 + a_2 c_2 + a_0 c_1 + a_0 c_2 + a_1 c_1 + a_1 c_2 = 0$
- 21) $a_0 b_1 + a_0 b_0 + a_0 b_3 + a_2 b_1 + a_2 b_0 + a_2 b_3 + a_0 c_3 + a_0 c_2 + a_2 c_3 + a_2 c_2 + 1 = 0$

- 22) $a_0b_2 + a_0b_1 + a_0b_0 + a_0b_3 + a_0c_4 + a_0c_3 = 0$
- 23) $a_1b_2 + a_1b_1 + a_1b_0 + a_1c_0 + a_1c_2 + a_1c_4 = 0$
- 24) $a_2b_2 + a_2b_1 + a_2b_3 + a_2c_1 + a_2c_3 + a_2c_0 + a_2c_2 = 0$
- 25) $a_0b_2 + a_0b_0 + a_1b_2 + a_1b_0 + a_0c_2 + a_0c_4 + a_0c_1 + a_0c_3 + a_1c_2 + a_1c_4 + a_1c_1 + a_1c_3 = 0$
- 26) $a_2b_1 + a_2b_3 + a_1b_1 + a_1b_3 + a_2c_3 + a_2c_0 + a_2c_4 + a_1c_3 + a_1c_0 + a_1c_4 = 0$
- 27) $a_0b_2 + a_0b_0 + a_0b_3 + a_1b_2 + a_1b_0 + a_1b_3 + a_2b_2 + a_2b_0 + a_2b_3 + a_0c_4 + a_0c_1 + a_0c_0 + a_0c_2 + a_1c_4 + a_1c_1 + a_1c_0 + a_1c_2 + a_2c_4 + a_2c_1 + a_2c_0 + a_2c_2 = 0$
- 28) $a_2b_1 + a_2b_0 + a_0b_1 + a_0b_0 + a_2c_0 + a_2c_1 + a_2c_3 + a_0c_0 + a_0c_1 + a_0c_3 = 0$
- 29) $a_0b_2 + a_0b_1 + a_0c_1 + a_0c_2 + a_0c_4 = 0$

It has been proven that finding a solution of a system quadratic equations is an NP-hard problem. This means that there is probably no polynomial time algorithm for solving general systems of non-linear equations over finite fields. We used two approaches for solving non-linear systems: a linearization method and the Gröbner bases method.

The theory of Gröbner bases has been initiated by Bruno Buchberger as a tool to solve the ideal membership problem. The main useful result is the following theorem.

Theorem. Consider a system of equations $f_1 = 0, \dots, f_n = 0$ where $f_1, \dots, f_n \in F_q[x_1, \dots, x_n]$ and F_q is a finite field. We define the ideal $I = \langle f_1, \dots, f_n, x_1^q - x_1, \dots, x_n^q - x_n \rangle$. Then, for any term ordering, the reduced Gröbner basis of I is equal to

1. $\{x_1 - x_1^0, \dots, x_n - x_n^0\}$ if $(x_1^0, \dots, x_n^0)$ is the unique solution of the system of equations.
2. $\{1\}$ if the system has no solution.

We used the program WOLFRAM MATHEMATIC for construct a Gröbner basis.

The key idea of linearization method is to re-write the system of non-linear equation from n unknowns as a new system of linear equations with increased number of unknowns. This allows us to solve a system of linear equations, which is a much simpler task.

For future computing experiments it was written the program “Generator” in C++ language for generation a key stream generator. Where tested a few generators and attacked by both linearization and Gröbner basis methods.

The most difficult and interesting problem is to estimate the complexity of Gröbner basis method, especially to get a theoretical border. It is obviously that the linearization method is more effective in a small dimension tasks, if the dimension of the task grows, it is hard to say which approach is preferable.

Obviously, any experiment is useful and productive in some way, it is much more important to get a general theoretical base of a stream ciphers resistance.

Definition 2. Let $f(x_1, \dots, x_n)$ be Boolean function. Then algebraic immunity is defined as $AI(f) = \min\{deg(g) \mid fg \equiv 0 \text{ or } (f \oplus 1)g \equiv 0\}$.

We intend to explore the relationship between the cipher resistance and the algebraic immunity of Boolean functions used.

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REPRESENTING POLYNOMIALS BY OTHER POLYNOMIALS II

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Abstract. In this talk, we will consider the problem of representing sums of finite products of various polynomials in terms of the sequences of Bernoulli and several orthogonal polynomials. More specifically, we will represent sums of finite products of Chebyshev polynomials of the first, second, third and fourth kinds, Legendre polynomials, Laguerre polynomials, Fibonacci polynomials, Lucas polynomials, Frobenius polynomials by Bernoulli, Hermite, Legendre, generalized Laguerre, Gegenbauer and Jacobi polynomials. The representations by Bernoulli polynomials will be done by considering the Fourier series expansions of the periodic functions with period 1 closely related to those sums of finite products. While the representations by several orthogonal polynomials will be done by using formulas obtained by the orthogonalities of those polynomials, and Rodrigues' and Rodrigues-type formulas.

Keywords. Fourier series expansion, representation, Bernoulli polynomials, Chebyshev polynomials of the first, second, third and fourth kinds, Legendre polynomials, Laguerre polynomials, Fibonacci polynomials, Lucas polynomials, Frobenius polynomials, Hermite polynomials, generalized Laguerre polynomials, Gegenbauer polynomials, Jacobi polynomials.

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CONVERGENCE AND STABILITY OF A SEMI-LAGRANGIAN METHOD FOR BURGERS' EQUATION WITH DIRICHLET BOUNDARY CONDITIONS

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Abstract. Convergence and stability of a semi-Lagrangian method for Burgers' equation with Dirichlet boundary conditions is considered. Using mathematical induction hypotheses under the mesh restriction $\Delta t = O(\Delta x^{1/2})$ between the temporal step size Δt and the spatial grid size Δx , we prove that the proposed method has the convergence order $O(\Delta t^2 + \Delta x^2 + \Delta x^p/\Delta t)$ in the sense of the discrete H_1 -norm, where p is the degree of an interpolation polynomial.

Keywords. Burgers' equation; Semi-Lagrangian method; Non-linear advection–diffusion equation; Convergence analysis; Stability analysis.

It is well known that the semi-Lagrangian method is used to simulate advection-type flow problems in computational fluid dynamics (CFD) (see, for example, [1, 2, 3, 4, 6, 7, 8, 9], etc.). The movement of particles is described in a flow field along their characteristic curves as a solution of nonlinear Cauchy problem in a fixed framework for each discretized time unit. One of the reasons contributing to the popularity of this method is that it can avoid bad representation problems of fluid features occurring in the Lagrangian method such as the accumulation of particles along characteristic curves or vacancy in a spatial domain for a long-time simulation [5]. Especially, it is known that the backward semi-Lagrangian method (BSLM) has a good stability property such that it is free from the Courant-Friedrichs-Lewy (CFL) condition between the temporal step size and the spatial grid size. In this presentation, we give a talk about a concrete convergence analysis with stability for a backward semi-Lagrangian method in a non-linear Burgers' equation with the Dirichlet boundary conditions. The material time derivative and the diffusion term along characteristic curves are discretized by backward difference formula of type 2 (BDF2) and the second order central finite difference respectively, together with the local Lagrangian interpolation. The Cauchy problem for characteristic curves is resolved by an iteration-free method, error correction method based on a modified quadratic polygon. Using mathematical induction hypotheses under the mesh restriction $\Delta t = O(\Delta x^{1/2})$ between the temporal step size Δt and the spatial grid size Δx , we prove that the proposed method has the convergence order $O(\Delta t^2 + \Delta x^2 + \Delta x^p/\Delta t)$ in the sense of the discrete H_1 -norm, where p is the degree of an interpolation polynomial. Further, we establish the unconditional stability of the method and present numerical tests to support the theoretical analyses.

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ON ONE PROBLEM OF OPTIMAL CONTROL

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Abstract. One important problem of optimal control problem is considered. The efficient procedure, combining the maximum principle and Galerkin method for its solving, is introduced.

Keywords. Optimal control problem, Pontryagin maximum principle, Galerkin method, residual function, trial function.

We consider the problem of optimal control in the form

$$J = \int_{t_0}^{t_1} (c_0 + c_1 x_1 + c_2 x_2 + \dots + c_n x_n + \alpha u^2) dt \rightarrow \min ,$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ \dots \\ B_n \end{pmatrix} u , \quad (1)$$

$$\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \dots \\ x_n(t_0) \end{pmatrix} = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \dots \\ x_n^0 \end{pmatrix}, \quad \begin{pmatrix} x_1(t_1) \\ x_2(t_1) \\ \dots \\ x_n(t_1) \end{pmatrix} = \begin{pmatrix} x_1^1 \\ x_2^1 \\ \dots \\ x_n^1 \end{pmatrix}, \quad u \in U \subseteq R ,$$

where $u(t)$ is control variable, $x(t) \in R^n$ is state variable, $x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \dots \\ x_n^0 \end{pmatrix}$, $x^1 = \begin{pmatrix} x_1^1 \\ x_2^1 \\ \dots \\ x_n^1 \end{pmatrix}$ are fixed ends of

trajectory, $c_1, c_2, \dots, c_n$ and α are constants, t_0, t_1 are fixed moments of time

If $c = 0$ minimum of functional in problem (1) corresponds to minimum of energy spending on control. Problem (1) is its natural generalization allowing to form and solve the variety of real problems.

According to classification [1] problem (1) is the General problem of optimal control and its solution – process $x(t)$, $u(t)$, satisfies Pontryagin maximum principle developed by L.Aschepkov, D. Dolgy, etc. for this type of optimal control problem. In general case of mathematical model of the General problem of optimal control the procedure of obtaining the exact solution on the base of maximum principle is very complicated or impossible. The most of numerical methods - Newton's method, Gradient method and others, can give an approximate solution with some accuracy depending of many factors. And convergence of these methods plays very important role. In particular, there is a bad convergence of initial approximation for the values of conjugate variables to the values that put

zeros for residual functions because of permanent getting by them their local minimum. This leads to that Newton's method or Gradient method cannot give a good result.

In this research, we introduce a new approach for solution of the problem (1) based on the successive using Pontryagin maximum principle and Galerkin method. We show that proper choice of trial functions in Galerkin method can give an exact solution of the problem (1). This significant advantage of Galerkin method can be generalized on the other types of optimal control problems.

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**GENERALIZED COMPOSITE HURWITZ RINGS
AS PF-RINGS AND PP-RING**

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Abstract. Let $(R_n)_{n \geq 0}$ be an ascending chain of commutative ring with identity and D be either the generalized composite Hurwitz series ring or generalized composite Hurwitz polynomial ring. In this talk, we give equivalent conditions for the ring D to be PF-rings or PP-ring.

Keywords. PP-rings, PF-rings, generalized composite Hurwitz series ring, generalized composite Hurwitz polynomial ring.

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SCALAR MULTIPLICATION ON CERTAIN MODELS OF ELLIPTIC CURVES USING THE FROBENIUS MAP

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Abstract. Elliptic curve cryptography was independently proposed by Koblitz [9] and Miller [10] in 1985. The elliptic curve cryptosystem is a public key cryptosystem based on the discrete logarithm problem in the group of points on a curve. In elliptic curve cryptosystems, the efficiency depends essentially on the fundamental operation of the scalar multiplication $[n]P$ for a given point P on an elliptic curve E and an integer n . In general, the computational speed of scalar multiplication $[n]P$ depends on finite field operations, curve point operations, and representation of the scalar n [11,5].

There is vast literature on efficient methods for computational speeding up scalar multiplication. For elliptic curves, the scalar multiplication can be done with various methods (a good reference is [1]). If an elliptic curve admits an efficient endomorphism, its use can speed up scalar multiplication. In [7], Iijima, Matsuo, Chao and Tsujii presented an efficiently computable homomorphism on elliptic curves using the Frobenius map on the quadratic twists of an elliptic curve. The Gallant-Lambert-Vanstone (GLV) gave suitable efficiently computable endomorphisms on elliptic curves for speeding up point multiplication [3, 4]. There are several models of elliptic curves to provide the efficient computation and implement for cryptography in recent year [2, 6, 8].

In this presentation, we consider efficient scalar multiplication on certain models of elliptic curves over a finite field using the Frobenius expansion. The Frobenius expansion is known as an efficient method to implement of point addition on elliptic curves. It has been applied to fast elliptic curve cryptosystem. Applying the Frobenius endomorphism on certain models of elliptic curves, we construct Frobenius maps defined on certain curves. To speed up the scalar multiplication on curves, we use the GLV method combined with the Frobenius endomorphism over the curves.

Key words. Elliptic curve, scalar multiplication, Frobenius map.

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ON THE KRULL DIMENSION OF COMPOSITE POWER SERIES RINGS

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Abstract. In this talk, we discuss some recent results on the Krull dimension of power series rings over one-dimensional nondiscrete valuation domains.

Keywords. One-dimensional nondiscrete valuation domain, Krull dimension, composite power series.

FOX-WRIGHT FUNCTION ON THE BOUNDARY OF ITS CONVERGENCE DOMAIN

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Abstract. This paper summarizes some of our recent results regarding the behavior of Fox-Wright function in the neighborhood of the singular point on the boundary of the disk of convergence. The main result is that the expansion contains two components - singular and regular as is the case for the generalized hypergeometric function. Recursive formulas for the coefficients of each of the two component series are also found.

Keywords: Fox-Wright function, Fox's H function, generalized hypergeometric function, singular point

Fox-Wright function ${}_p\Psi_q(z)$ is a further extension of the generalized hypergeometric function attained by introducing arbitrary positive scaling factors into the gamma functions in the summand:

$${}_p\Psi_q(z) = \sum_{k=0}^{\infty} \frac{\Gamma(A_1 k + a_1) \cdots \Gamma(A_p k + a_p)}{\Gamma(B_1 k + b_1) \cdots \Gamma(B_q k + b_q)} \frac{z^k}{k!}.$$

The importance of the Fox-Wright function comes mostly from its role in fractional calculus, see (Gorenflo et.al, 1999), (Kilbas, 2005), (Kilbas et.al., 2006), (Mainardi, Pagnini, 2007). Other interesting applications also exist. In particular, (Miller, 1991) expressed a solution of a general trinomial equation in terms of ${}_1\Psi_1(z)$. See, also (Miller, Moskowitz, 1995) for an application in information theory. It has been a recent surge of interest in the Fox-Wright function as witnessed by the articles (Chu, Wang, 2008), (Mehrez, 2018), (Mehrez, Sitnik, 2018). In particular, Mehrez (Mehrez, 2018) extended our approach to generalized hypergeometric functions to the Fox-Wright function and presented the Laplace and the generalized Stieltjes transform representations for some cases of ${}_p\Psi_q(z)$ invoking certain facts from our previous works. He further obtained some of the inequalities and monotonicity results for certain cases of ${}_p\Psi_q(z)$, which we established previously for the generalized hypergeometric functions. In another recent work (Mehrez, Sitnik, 2018) the authors obtained further inequalities for the Fox-Wright functions and their ratios.

If the sums of the scaling factors in the denominator and numerator are equal the series defining this function has a positive and finite radius of convergence which we call ρ (otherwise if the sum of scaling factors in the denominator is greater than that if the numerator the series converges for all finite complex values of the variable and defines an entire function). In this report we are only interested in this case. The Fox-Wright function is then holomorphic inside the disk of convergence. The problem of its analytic continuation was completely solved by Braaksma in 1964 who proved that the Mellin-Barnes contour integral representation furnishes the analytic continuation to the entire complex plane cut along the ray $[1/\rho, \infty)$. It follows that the only singularity of the Fox-Wright function on the circle of radius ρ is at the point $z=1/\rho$. The character of this singularity is the main topic of this research. Moreover, when $z=1/\rho$ the series may converge or diverge depending on the values of parameters. Hence, we are encountered with the problem of analytic continuation of ${}_p\Psi_q(1/\rho)$ as the function of parameters $A_1, \dots, A_p, a_1, \dots, a_p, B_1, \dots, B_q, b_1, \dots, b_q$. Both problems (the behavior in the neighborhood of the singular point and analytic continuation in parameters) have a long history. Particular cases of these problems can be traced back to Gauss who found the celebrated summation formula for ${}_2F_1(a, b; c; 1)$ providing analytic continuation in a , b , c , and described the behavior of $z \rightarrow {}_2F_1(a, b; c; z)$ in the neighborhood of $z=1$. Similar problems for ${}_3F_2$ were partly solved by Thomae in 1870 (analytic continuation in parameters) and partly by Ramanujan around the second decade of the 20th century (asymptotic approximation as $z \rightarrow 1$ in the logarithmic case) with further contributions by Evans and Stanton, Wimp and Bühring, see (Bühring, 1987) and references therein. For the general Gauss type

hypergeometric function ${}_pF_p$ with $p \geq 3$ these problems were first solved by (Nørlund, 1955) with further contributions by (Olsson, 1966), (Saigo and Srivastava, 1990), (Bühning, 1992) and (Bühning and Srivastava, 1998) and other authors.

Our main results can be summarized as follows. First, we express the Fox-Wright function as the generalized Stieltjes transform of the delta-neutral H function of Fox, studied by us in (Karp, Prilepkina, 2017). Under certain restrictions on parameters this representation takes the form

$${}_p\Psi_q(z) = \int_0^1 \frac{H(\rho u) du}{u(1-\rho uz)^\sigma}.$$

Next, according to Theorem 1 from (Karp, Prilepkina, 2017) the H function in the integrand can be expanded as follows

$$H(\rho u) = u^{\theta+1}(1-u)^{\mu-1} \sum_{n=0}^{\infty} V_n (1-u)^n,$$

where the parameter θ is arbitrary real number, while μ is expressed by the parameters of the original function ${}_p\Psi_q(z)$. Its precise definition as well as formulas for computing the coefficients V_n can be found in (Karp, Prilepkina, 2017). Substituting this expansion into the integral representation above and performing term-wise integration (to be justified elsewhere) we arrive at the formula:

$${}_p\Psi_q(z) = \sum_{n=0}^{\infty} \frac{V_n}{\mu+n} ({}_2F_1(1, \sigma; \mu+1+n; \rho z))$$

We can now use the representation of the hypergeometric function ${}_2F_1$ in the neighborhood of the singular point $z=1$:

$${}_2F_1(1, \sigma; \mu+1+n; \rho z) = (1-\rho z)^{\mu-\sigma} \sum_{m=0}^{\infty} G_{m,n}(1-\rho z)^{m+n} + \sum_{m=0}^{\infty} D_{m,n}(1-\rho z)^m$$

After substituting this expansion into the above expression for the Fox-Wright function ${}_p\Psi_q(z)$ we can rearrange the summations to obtain:

$${}_p\Psi_q(z) = (1-\rho z)^{\mu-\sigma} \sum_{m=0}^{\infty} R_m(1-\rho z)^m + \sum_{m=0}^{\infty} W_m(1-\rho z)^m$$

The coefficients in this expansion can be computed easily from $G_{m,n}$ and $D_{m,n}$. Explicit formulas will be presented during the talk and published elsewhere. Here we just mention that the form of the above expansion with one "singular" series in fractional powers of $(1-\rho z)$ and one "regular" series in integer powers repeats the structure of the corresponding expansion of the generalized hypergeometric function. This result might be not very surprising, but rigorous confirmation of this expected form is given, to the best of our knowledge, for the first time here. Second important aspect of the above expansion is that assuming $\mu - \sigma > 0$ and setting $z=1/\rho$ we obtain ${}_p\Psi_q(1/\rho)=W_0$, where W_0 is written as a convergent series in term of coefficients V_n . Hence, we get a formula for analytic continuation of ${}_p\Psi_q(1/\rho)$ in parameters $A_1, \dots, A_p, a_1, \dots, a_p, B_1, \dots, B_q, b_1, \dots, b_q$. This formula can be viewed as far reaching generalization of the Gauss formula for ${}_2F_1(a, b; c; 1)$. Further consequences of the expansion around the singular point and integral representation via generalized Stieltjes transform of the delta-neutral H function will be presented during the talk.

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UPPER SEMILATTICES S FOR WHICH THE CONGRUENCE LATTICE OF S -ACT SS IS MODULAR

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Abstract. The works of such authors as Ptakhov D.O., Stepanova A.A., Haliullina A.R., Kazak M.S. are devoted to studying of the S -acts over monoids with the given conditions on their congruence lattices. It is clear, that upper semilattice with the smallest element can be considered as monoid. Stepanova A.A. and Kazak M.S. described S -acts over upper semilattice with linear congruence lattice. More exactly they prove that S -act A over upper semilattice S has linear ordered congruence lattice if and only if A has at most 2 elements. In this research we study the structure of upper semilattices S for which the congruence lattice of S -act S is modular.

Key words: lattice, modular lattice, congruence lattice of algebra, S -act.

The subject of our research in this article are S -acts over monoids, that is the sets on which monoids act. S -acts can be found in different sections of mathematics and its applications. In particular, every unary algebra is S -act over monoid of its endomorphisms. The same holds true for a graph, partially ordered set, etc. The main ideas of the theory of S -acts were reflected in the monograph [6].

The congruences of algebra play an important role in a structural theory since the congruences are the same as kernel of the homomorphism of this algebra into the others. The congruences of algebra forms the lattice according to set-theoretic inclusion. This lattice is sublattice of the lattice of all equivalence relations. The lattice carries much information about the structure of algebra.

Many works have been devoted to studying of the S -acts over monoids with the given conditions on their congruence lattices. From the definition of S -act it follows that S -act is a unary algebra. Studies related to the description of S -acts with some conditions on their congruence lattice, began with the study of unary algebras with linear ordered, distributive and modular congruence lattice. The congruence lattice of unary algebras, in particularly S -acts over monoids, has been studied in the works [1-4,6,7].

In [1] Egorova D.P. described the structure of the unary with linear, distributive and modular congruence lattice.

In [7] Ptakhov D.O. and Stepanova A.A. studied the unconnected S -acts with distributive and modular congruence lattice. In particularly, in this paper it is noticed that S -act with a distributive congruence lattice has at most three connected components. In addition, it is noticed that S -act with a modular congruence lattice has at most four connected components.

In [2] Haliullina A.R. got the characterization of S -acts over semigroups S of right and left zeros with distributive lattice of congruence.

In [4] Kazak M.S. and Stepanova A.A. described the S -acts over a well-ordered monoid such that their congruence lattice is distributive, or modular, or linear ordered. In the theorems below $(S; \leq)$ is an linear ordered set with the smallest element 1 considered as a (linear ordered) monoid $(S; \cdot)$ relative to the operation $ab = \max\{a, b\}$, where $a, b \in S$.

Theorem 1. [4] Let S be a linear ordered monoid. Then S -act A has linear ordered congruence lattice if and only if A has at most 2 elements.

Theorem 2. [4] Let S be a well-ordered monoid. Then S -act A has distributive congruence lattice if and only if

- 1) S -act A has at most two connected components;
- 2) if $a, b \in A$, $s \in S$, s is not 1 and $Sa_1 \cap Sa_2 = Ssa_1 \cap Ssa_2$ then $sa_1 = ra_1$ or $sa_2 = ra_2$ for some $s \in S$, $r < s$.

Theorem 3. [4] Let S be a well-ordered monoid. Then S -act A has modular congruence lattice if and only if

- 1) S -act A has at most three connected components;
- 2) if $a, b \in A$, $s \in S$, s is not 1 and $Sa_1 \cap Sa_2 = Ssa_1 \cap Ssa_2$ then $sa_1 = ra_1$ or $sa_2 = ra_2$ or $sa_1 = sa_2$ for some $s \in S$, $r < s$;
- 3) if $a_1, a_2, a_3 \in A$ and $s \in S$ such that s is not 1, S -act Sa_i is not subact of S -act Sa_j , $sa_i = sa_j$ and $Sa_i \cap Sa_j = Ssa_i$ for any different $i, j \in \{1, 2, 3\}$ then $sa_i = ra_i$ for some $s \in S$, $r < s$ and $i, j \in \{1, 2, 3\}$.

In [5] Kazak M.S. and Stepanova A.A. studied the structure S-act with linear ordered congruence lattice where S is an upper semilattice with the smallest element 1. In the theorem below $(S; \leq)$ is an upper semilattice with the smallest element 1 considered as a monoid $(S; \cdot)$ relative to the operation $ab = \max\{a, b\}$, where $a, b \in S$.

Theorem 4. [5] Let S be an upper semilattice. Then S-act A has linear ordered congruence lattice if and only if A has at most 2 elements.

In this work we study the structure of upper semilattices S for which the congruence lattice of S-act S is modular.

Below we recall some definitions from the theory of S-acts and universal algebra [6,8].

A subalgebra θ of a direct product of algebra A is called a *congruence* if

- 1) $(a, a) \in \theta$ for all $a \in A$;
- 2) if $(a, b) \in \theta$ then $(b, a) \in \theta$ for all $a, b \in A$;
- 3) if $(a, b) \in \theta$ and $(b, c) \in \theta$ then $(a, c) \in \theta$ for all $a, b, c \in A$.

The set of all congruences of algebra A forms the lattice according to following operations:

$$\theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2,$$

$\theta_1 \vee \theta_2$ is the smallest congruence included $\theta_1 \cup \theta_2$,

where θ_1, θ_2 are the congruences of algebra A .

A partially ordered set $(L; \leq)$ is called *upper semilattice (lattice)* if every 2-element subset $\{a, b\} \subseteq L$ has the least upper bound $a \vee b$ (and the greatest lower $a \wedge b$). A lattice $(L; \leq)$ is called *modular lattice* if

$$(a \vee b) \wedge c = a \vee (b \wedge c)$$

for all $a, b, c \in L$ such as $a \leq c$.

A *monoid* is a semigroup with identity 1. Everywhere below the S will denote a monoid with identity 1. *Left S-act* is a non-empty set A on which S acts in the left, that is there is a map $S \times A \rightarrow A$, where $(s, a) \mapsto sa$, such that for all $s, t \in S$ and $a \in A$, $s(t(a)) = (st)a$ and $1a = a$.

Note that monoid S is S-act over S .

Suppose $(S; \leq)$ be an upper semilattice with the smallest element 1 considered as a monoid $(S; \cdot)$ relative to the operation $ab = a \vee b$ where $a, b \in S$.

Theorem 5. Let S be upper semilattice with element 1. A lattice of congruence of S-act S is modular if and only if

$$\neg((b < a) \wedge (b < c))$$

for all different $a, b, c \in S$.

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SOME IDENTITIES OF SPECIAL NUMBERS AND POLYNOMIALS ARISING FROM p -ADIC INTEGRALS ON $\mathbb{Z}_p$

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Abstract. In recent years, studying degenerate versions of various special polynomials and numbers have attracted many mathematicians. Here we introduce degenerate type 2 Bernoulli polynomials, fully degenerate type 2 Bernoulli polynomials and degenerate type 2 Euler polynomials, and their corresponding numbers, as degenerate and type 2 versions of Bernoulli and Euler numbers.

Regarding to those polynomials and numbers, we derive some identities, distribution relations, Witt type formulas and analogues for the Bernoulli's interpretation of powers of the first m positive integers in terms of Bernoulli polynomials. The present study was done by using the bosonic and fermionic p -adic integrals on $\mathbb{Z}_p$.

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AN INVERSE PROBLEM OF OXYGEN TRANSPORT IN BRAIN

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Abstract. A continuum steady-state model of oxygen transport in brain is studied. The problem is considered as an inverse problem with unknown sources. The unique solvability of the inverse problem is proven, and an algorithm to find solutions is proposed.

Key words Boundary-value problem, inverse problem, unique solvability, oxygen transport in brain.

1. Introduction

Delivery of a sufficient amount of oxygen to cerebral cells is the most important requirement for functioning of the brain. The lack of oxygen can lead to irreversible neuronal damage within a few minutes. Therefore, mathematical models of oxygen transport in brain substance are important for simulation of dangerous situations arising from impaired cerebral circulation.

The most promising approach is that the brain material is considered as a two-compartment (blood and tissue) structure, and coupled equations describe convection of oxygen in blood as well as its diffusion and consumption in tissue (Secomb et. al., 2000; Payne, Lucas, 2018; Su, 2011).

A perspective approach to modeling oxygen transport is associated with spatial homogenization of state variables. There, the homogenization is being performed under the assumption that the tissue and blood fractions occupy the same spatial region. In (Khaled, Vafai, 2003), this approach was applied to a model of heat transfer in tissue comprising a network of blood vessels. A similar approach to studying oxygen transport process in brain was used in (Su, 2011), where a two-phase continuum model describing coupled oxygen transport in tissue and blood is considered. A coupled system of equations arises here as the result of homogenization of a two-phase model of oxygen transport in a media contained blood and tissue fractions. Numerical simulations for a simplified domain were conducted, and the comparison with simulations based on the original (non-homogenized) model was done.

In (Kovtanyuk et. al., 2018), a theoretical analysis of steady-state oxygen transport model is fulfilled. A priori estimates of solutions, implying the unique solvability of the problem under some conditions, are obtained. An iterative numerical procedure for finding solutions is proposed, along with the proof of its convergence. In (Kovtanyuk et. al., 2019), the investigation of the steady-state continuum model was continued, the existence and uniqueness of solutions for the boundary-value problem were established, and numerical examples illustrated the theoretical analysis were presented.

In the present work, the boundary-value problem of oxygen transport is considered as an inverse problem with unknown sources. The unique solvability of the inverse problem is proven and numerical algorithm to find a solution is proposed.

2. Problem formulation

A two-compartment (blood and tissue) model of oxygen transport is considered. It is assumed that the both compartments occupy the same spatial region $\Omega \subset \mathbb{R}^3$ and have different volume fractions for the blood and tissue compartments, σ and $1 - \sigma$, respectively. Following (Su, 2011; Kovtanyuk et. al., 2018; Kovtanyuk et. al., 2019), oxygen transport can be described by the following coupled equations:

$$-\alpha \Delta \varphi + \mathbf{v} \cdot \nabla \varphi = G, \quad -\beta \Delta \theta = -\kappa G - \mu, \quad x \in \Omega.$$

Here, φ is the oxygen concentration in blood; θ the oxygen concentration in tissue; μ the tissue oxygen metabolic (consumption) rate associated with brain functioning; $G = c(\theta - \psi)$ the local exchange at the blood-tissue interface, where ψ is the plasma oxygen concentration; $\kappa = \sigma(1 - \sigma)^{-1}$, where σ is the volume fraction of blood; v a given velocity field in the domain Ω ; and α and β are diffusion coefficients for blood and tissue, respectively.

There is a nonlinear monotonic dependence of the tissue oxygen metabolic rate, μ , on the tissue oxygen concentration, θ , and of the plasma oxygen concentration on the blood oxygen concentration, φ . To simplify the model, the following linear approximations: $\mu = p\theta + s$ and $\psi = a\varphi + b$, where $a, p > 0$, will be used.

In (Kovtanyuk et. al., 2018), a perforated model domain was considered to simulate the ends of arterioles and venules. The concentrations corresponding to the ends of arterioles and venules were set at the boundaries of the holes.

In the present work, we will use another approach to simulate the ends of arterioles and venules. Let $\Omega_j \subset \Omega$, $j = 1, \dots, m$, be disjoint subdomains. The oxygen transport will be described by the following system of elliptic equations:

$$-\alpha\Delta\varphi + v \cdot \nabla\varphi = G + \sum_{j=1}^m q_j f_j, \quad -\beta\Delta\theta = -\kappa G - \mu + \sum_{j=1}^m p_j f_j, \quad x \in \Omega. \quad (1)$$

Here, f_j is the characteristic function of the subdomain Ω_j .

Equations (1) are supplemented by the following boundary conditions imposed on $\Gamma := \partial\Omega$:

$$\alpha\partial_n\varphi + \gamma(\varphi - \varphi_b)|_\Gamma = 0, \quad \beta\partial_n\theta + \delta(\theta - \theta_b)|_\Gamma = 0. \quad (2)$$

The inverse problem consists in finding intensities $r = (q_1, \dots, q_m, p_1, \dots, p_m) \in \mathbb{R}^{2m}$ and the corresponding solution $y = \{\varphi, \theta\}$ of the boundary-value problem (1) and (2) with the following integral overdetermination:

$$\int_{\Omega_j} \varphi dx = Q_j, \quad \int_{\Omega_j} \theta dx = P_j, \quad j = 1, 2, \dots, m. \quad (3)$$

Here, Q_j and P_j are the prescribed averaged values of the functions φ, θ with respect to subdomains Ω_j .

3. Problem formalization.

Let $H = L^2(\Omega)$, $V = H^1(\Omega)$, $V \subset H = H' \subset V'$; $W = V \times V$, and $W' = V' \times V'$. By (f, v) (respectively $((F, z))$) we denote a value of the functional $f \in V'$ at the element $v \in V$ (respectively $F \in W'$ at the element $z \in W$) which coincide with the inner product in H (respectively in $H \times H$) if $f \in H$ (respectively $F \in H \times H$). Denote $\|f\|^2 = (f, f)$.

Let the following conditions hold:

(i) $\gamma, \delta \in L^\infty(\Gamma)$, $\gamma \geq \gamma_0 > 0$, $\delta \geq \delta_0 > 0$, $\varphi_b, \theta_b \in L^\infty(\Gamma)$;

(ii) $v \in H^1(\Omega)$, $\nabla \cdot v = 0$, $\gamma + 0.5 v \cdot n \geq 0$ on Γ .

Define the following operators and functionals:

$$A: W \rightarrow W', \quad B: W \rightarrow W', \quad F_j \in W', \quad j = 0, 1, \dots, 2m;$$

$$((Ay, z)) = \alpha(\nabla\varphi, \nabla\psi) + \int_\Gamma \gamma\varphi\psi d\Gamma + \beta(\nabla\theta, \nabla\zeta) + \int_\Gamma \delta\theta\zeta d\Gamma;$$

$$((By, z)) = (v \cdot \nabla\varphi - c(\theta - a\varphi), \psi) + (p\theta + \kappa c(\theta - a\varphi), \zeta);$$

$$((F_0, z)) = \int_{\Gamma} \gamma \varphi_b \psi d\Gamma + \int_{\Gamma} \delta \theta_b \zeta d\Gamma - bc(1, \psi) + (\kappa bc - s)(1, \zeta);$$

$$((F_j, z)) = (f_j, \psi), \quad j = 1, 2, \dots, m;$$

$$((F_j, z)) = (f_{j-m}, \zeta), \quad j = m+1, \dots, 2m.$$

These equalities are valid for any $y = \{\varphi, \theta\}$, $z = \{\psi, \zeta\} \in W$. Using these operators, the inverse problem (1)–(3) can be rewritten in the following form.

Problem (P). Find $y = \{\varphi, \theta\} \in W$ and a vector $r = (r_1, r_2, \dots, r_{2m}) := (q_1, \dots, q_m, p_1, \dots, p_m)$ such that

$$Ay + By = F_0 + \sum_{j=1}^{2m} r_j F_j, \quad (4)$$

$$((F_j, y)) = R_j, \quad \text{where} \quad R_j = \begin{cases} Q_j, & j = 1, 2, \dots, m, \\ P_j, & j = m+1, \dots, 2m. \end{cases} \quad (5)$$

4. Reducing the Problem (P) to a system of linear algebraic equations.

The following auxiliary results are true.

Lemma 1. For any $F \in W'$, there exists a unique solution of the problem

$$Az + Bz = F, \quad z \in W. \quad (6)$$

Let $y_j \in W$, $j = 0, 1, \dots, 2m$, and moreover

$$Ay_j + By_j = F_j, \quad j = 0, 1, \dots, 2m. \quad (7)$$

Then for a solution of the equation (4) the following presentation is true:

$$y = y_0 + \sum_{j=1}^{2m} r_j y_j. \quad (8)$$

From the overdetermination conditions (5), we conclude that the intensities r_j , $j = 1, 2, \dots, 2m$, are solutions of the following system of linear algebraic equations:

$$\sum_{j=1}^{2m} ((F_k, y_j)) r_j = R_k - ((F_k, y_0)), \quad k = 1, 2, \dots, 2m. \quad (9)$$

Lemma 2. The system of linear algebraic equations (9) is uniquely solvable.

Thus, accounting for Lemmas 1 and 2, the following final result with respect to unique solvability of Problem (P) is true.

Theorem. Problem (P) is uniquely solvable. Moreover, its solution $y \in W$, $r \in \mathbb{R}^{2m}$ is defined by formula (8), where $r = (r_1, r_2, \dots, r_{2m})$ are solutions of the system of linear algebraic equations (9).

5. Conclusion.

A new approach to find a solution of the boundary-value problem of oxygen transport in brain is proposed. The problem is considered as an inverse problem with finite overdetermination. The inverse problem is reduced to finding a solution of a system of linear algebraic equations.

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ON SOME PROPERTIES OF GENERALIZED NARAYANA NUMBERS

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Abstract. In this paper, we consider a three-dimensional generalization of the Narayana numbers. We find a generating function for these generalized Narayana numbers and show another way to get explicit formula. Also we obtain new properties of these numbers for the composition of generating functions.

Key words: Narayana numbers, generalized Narayana numbers, explicit formula, generating function, composition.

The Narayana triangle is a classic combinatorial number triangle, which has many different applications (Narayana 1979; Petersen 2015; Stanley 2015). In this paper, we use the definition of this number triangle given in OEIS (Sloane). The Narayana triangle is defined by the following generating function:

$$F(x, y) = \frac{1 - x(y + 1) - \sqrt{(1 - x(y + 1))^2 - 4x^2y}}{2x} = \sum_{n>0} \sum_{m>0} N(n, m)x^n y^m,$$

where the Narayana numbers are defined by

$$N(n, m) = \frac{1}{n} \binom{n}{m-1} \binom{n}{m}.$$

The generating function $F(x, y)$ satisfies the functional equation
 $-xF(x, y)^2 + (-xy - x + 1)F(x, y) - xy = 0.$

The Narayana triangle describes a large number of combinatorial sets. For example, the Narayana numbers are related to the Catalan numbers by the known formula

$$N(n, 1) + N(n, 2) + \dots + N(n, n) = C_n.$$

From this formula, we can conclude that the Narayana numbers describe classes of subsets for the combinatorial sets described by the Catalan numbers. There are more than 200 combinatorial sets that are described by the Catalan numbers (Stanley 2015). For instance, it can be some classes of permutations, lattice paths, bracket expressions, Dyck words, trees, etc. One of the combinatorial interpretations for the Narayana numbers is the set of Dick n -paths with m peaks (Petersen 2015). Also there are a large number of generalizations of the Narayana numbers (Barry 2011; Barry, Hennessy 2012; Mansour, Sun 2009; Alexeev, Tikhomirov 2017). In this paper, we consider the following three-dimensional generalization of the Narayana numbers:

$$N_e(n, m, k) = \frac{k}{n} \binom{n}{m-k} \binom{n}{m}. \quad (1)$$

The study of these numbers was first given in (Guy 2000), where lattice paths were investigated. However, an explicit formula and a generating function were not obtained. The presented explicit formula (1) is given in (Callan; Defant). Also in these papers the connection of the formula (1) with the power of the generating function of the Narayana numbers is described.

In this paper, we present a new proof for the explicit formula of the generalized Narayana numbers and we obtain a multivariate generating function with three variables for these numbers. In addition, we obtain new properties of the generalized Narayana numbers for the composition of generating functions.

Theorem 1. The generalized Narayana numbers $N_e(n, m, k)$ are defined by the generating function

$$F_e(x, y, z) = \frac{1}{1 - zF(x, y)} = \sum_{n>0} \sum_{m>0} \sum_{k>0} N_e(n, m, k)x^n y^m z^k$$

The use of the composition of generating functions with one variable is described in detail in (Stanley 2011; Kruchinin D.V., Kruchinin V.V. 2013; Kruchinin V.V., Kruchinin D.V. 2013). In the next theorem we give another important property of the generalized Narayana numbers, which is associated with the operation of the composition of generating functions.

Theorem 2. Suppose the composition of generating functions

$$A(x) = G(F(x, y)) = \sum_{n \geq 0} \sum_{m \geq 0} a(n, m) x^n y^m,$$

where

$$G(x) = \sum_{n \geq 0} g(n) x^n$$

and $F(x, y)$ is the generating function for the Narayana numbers. Then the coefficients $a(n, m)$ of the composition are

$$a(n, m) = \sum_{k=0}^{n+m} g(k) N_e(n, m, k)$$

or

$$a(n, m) = \frac{1}{n} \binom{n}{m} \sum_{k=1}^m k \binom{n}{m-k} g(k).$$

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SOME IDENTITIES OF TYPE 2 Q-DAEHEE AND TYPE 2 Q-CHANGHEE POLYNOMIAL AND NUMBERS

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Abstract. The purpose of this paper is to introduce and study type 2 q -Daehee and type 2 q -Changhee polynomials and numbers by virtue of the p -adic q -integral on $\mathbb{Z}_p$. Kim-Kim introduced type 2 q -Bernoulli and type 2 q -Euler polynomial and obtained the sum of consecutive odd positive q -integers and alternating sums of consecutive odd q -integers. In this paper, we obtained the results are among other special polynomial and numbers, identities regarding Carlitz's q -Bernoulli polynomials and numbers and Carlitz's q -Euler polynomials and numbers.

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A NOTE ON SOME PROPERTIES OF FULLY DEGENERATE BERNOULLI POLYNOMIALS ASSOCIATED WITH DEGENERATE BERNSTEIN POLYNOMIALS

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Abstract. In this paper, we investigate some properties and identities for fully degenerate Bernoulli polynomials in connection with degenerate Bernstein polynomials by means of bosonic p -adic integrals on $\mathbb{Z}_p$ and generating functions. Furthermore, we study two variable degenerate Bernstein polynomials and the degenerate Bernstein operators.

Key words. Degenerate Bernoulli polynomials; degenerate Bernstein operators.

Introduction and main results

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$, will denote the ring of rational integers, the ring of p -adic integers, the field of p -adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The p -adic norm $|\cdot|_p$ is normalized as $|p|_p = 1/p$.

t^n

For $\lambda, t \in \mathbb{C}_p$ with $|\lambda|_p < p^{-1/p-1}$ and $|t|_p < 1$, the degenerate Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}-1}} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=1}^{\infty} \beta_n(x|\lambda) \frac{t^n}{n!}$$

When $x=0$, $\beta_n(\lambda) = \beta_n(0|\lambda)$ are called the degenerate Bernoulli numbers.

Note that $\lim_{\lambda \rightarrow 0} \beta_n(x|\lambda) = B_n(x)$, where $B_n(x)$ are the ordinary Bernoulli polynomials defined by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=1}^{\infty} B_n(x) \frac{t^n}{n!}$$

and $B_n = B_n(0)$ are called the Bernoulli numbers.

Recently, Kim-Kim introduced the degenerate Bernstein polynomials given by

$$\frac{(x)_{k,\lambda}}{k!} (1+\lambda t)^{\frac{1-x}{\lambda}} = \sum_{n=1}^{\infty} B_{k,n}(x|\lambda) \frac{t^n}{n!}$$

Thus, we introduce the following main results.

Theorem 1 For $n \in \mathbb{N} \cup \{0\}$, we have

$$B_n(1-x|\lambda) = (1-x)_{n,\lambda} + \sum_{m=1}^n B_{m,n}(x|\lambda) B_m(\lambda) \frac{(-\lambda)^{1-m}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} \frac{1}{x-k\lambda}$$

Corollary 1 For $n \in \mathbb{N} \cup \{0\}$, we have

$$B_n(2|\lambda) = (2)_{n,\lambda} + \sum_{m=1}^n B_{m,n}(-1|\lambda) B_m(\lambda) \frac{(-\lambda)^{1-m}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^{k+1} \binom{m-1}{k} B_{k,n} \frac{1}{1+k\lambda}$$

Theorem 2 For $k, n \in \mathbb{N}$, we have

$$(x)_{n,\lambda} \sum_{m=0}^k \binom{k}{m} (-1)^{m-k} B_n(1-x+m|\lambda) = \begin{cases} \sum_{l=0}^n D_l^{(k)}(x) B_{k,n-1}(x|\lambda), & \text{if } n \geq k \\ 0, & \text{if } n < k. \end{cases}$$

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OCEAN BATHYMETRY RECONSTRUCTION USING RADIATIVE TRANSFER THEORY

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Abstract: Based on the mathematical model of the propagation of an acoustic signal in a fluctuating medium, the inverse problem is formulated for determination a function that describes the deviation of the seabottom level from the average specified horizontal plane. In the double scattering approximation and the narrow directivity pattern of the receiving antenna, the solution of the direct problem is obtained. As a solution of the inverse problem, a nonlinear differential equation is obtained for the function describing the deviation of the seabottom relief. A numerical analysis of the solution is carried out and the influence of a double-scattered signal on the retrieval of the bathymetric function is investigated.

Key words. Radiative transfer equation, scattering, bathymetry, remote sensing, sonar.

The study of the ocean is still a priority for the world community. A lot of research complexes are being developed to solve bathymetry problems (see e.g. [1-4]). Nowadays, the problem of mapping the ocean bottom using side scan sonars (SSS) which were equipped of an autonomous unmanned underwater vehicle, is very relevant and promising. The study of the ocean properties by using SSS generates a very interesting problem of determining the relief of the sea bottom. The sonar operation is based on the periodic emission of pulsed sound parcels and the detection of reflected echo signal from remote seabed domains. When the sonar antenna is moved, an acoustic image is formed on the starboard and portside of the underwater vehicle.

The process of propagation of acoustic radiation is described by the radiative transfer equation [4, 5]:

$$\frac{1}{c} \frac{\partial I}{\partial t} + \mathbf{k} \cdot \nabla_{\mathbf{r}} I(\mathbf{r}, \mathbf{k}, t) + \mu I(\mathbf{r}, \mathbf{k}, t) = \frac{\sigma}{2\pi} \int_{\Omega} I(\mathbf{r}, \mathbf{k}', t) d\mathbf{k}' + J(\mathbf{r}, \mathbf{k}, t), \quad (1)$$

where $\mathbf{r} \in \mathbb{R}^2$, $t \in [0, T]$ and wave vector $\mathbf{k}$ belongs to the unique sphere $\Omega = \{\mathbf{k} \in \mathbb{R}^2: |\mathbf{k}| = 1\}$. The function $I(\mathbf{r}, \mathbf{k}, t)$ is interpreted as radiation intensity in moment t in point $\mathbf{r}$, propagated in the direction $\mathbf{k}$ with constant velocity c . μ and σ denote the attenuation and the scattering coefficients, correspondingly. $J(\mathbf{r}, \mathbf{k}, t)$ describes the density of inner sources.

The process of echo signal propagation occurs in the domain $G := \{\mathbf{r} \in \mathbb{R}^2: r_2 > -l + u(r_1)\}$, which is the upper half-space bounded from below by the curve, $\partial G = \gamma = \{\mathbf{y} \in \mathbb{R}^2: y_2 = -l + u(y_1)\}$, where the function $u(y_1)$ describes the change of the ocean bottom relief.

We assume that the function $J(\mathbf{r}, \mathbf{k}, t)$ describes a point isotropic sound source [6]:

$$J(\mathbf{r}, \mathbf{k}, t) = J_0 \delta(\mathbf{r}) \delta(t), \quad (2)$$

where δ denotes the Dirac delta function and J_0 is the source power.

Initial and boundary conditions for (1):

$$I|_{t < 0} = 0, \quad (3)$$

$$I(\mathbf{y}, \mathbf{k}, t) = 2 \sigma_d \int_{\Omega_+(\mathbf{y})} |\mathbf{n}(\mathbf{y}) \cdot \mathbf{k}'| I(\mathbf{y}, \mathbf{k}', t) d\mathbf{k}', \quad \mathbf{y} \in \gamma, \mathbf{k} \in \Omega_-(\mathbf{y}), \quad (4)$$

where $\Omega_{\pm}(\mathbf{y}) = \{\mathbf{k} \in \Omega, \text{sgn}(\mathbf{k} \cdot \mathbf{n}(\mathbf{y})) = \pm 1\}$, σ_d denotes the constant seabottom reflection coefficient, $\mathbf{n}(\mathbf{y})$ denotes the exterior normal to ∂G .

We introduce an additional condition on the carrier of the receiving antenna, which is concentrated at the point $\mathbf{O} = (0, 0)$:

$$\int_{\Omega_+(y)} S^\pm(\mathbf{k}) I|_{\Gamma^\pm}(\mathbf{O}, \mathbf{k}, t) d\mathbf{k} = I^\pm(t).$$

Here, the function $S^\pm(\mathbf{k})$, when $\mathbf{k} \in \{\mathbf{k} \in \Omega : \text{sgn}(k_1) = \pm 1\}$, are characterizes the receiving antenna pattern on the starboard and the portside of the carrier, respectively; $I^\pm(t)$ determines the measured total intensity on starboard and portside along the direction of movement of the antenna carrier; $I|_{\Gamma^\pm}(\mathbf{r}, \mathbf{k}, t) = \lim_{\epsilon \rightarrow 0} I(\mathbf{r} \pm \epsilon \mathbf{k}, \mathbf{k}, t \pm \epsilon)$.

The solution of the initial-boundary problem (1), (3), (4) is deduced to the integral equation [5, 9]:

$$\begin{aligned} I(\mathbf{r}, \mathbf{k}, t) = & \int_0^{d(\mathbf{r}, -\mathbf{k})} \exp(-\mu t') J_0 \left(\mathbf{r} - t' \mathbf{k}, \mathbf{k}, t - \frac{t'}{c} \right) dt' + \\ & + 2\sigma_d \exp(-\mu d(\mathbf{r}, -\mathbf{k})) \int_{\Omega_+(r-d(\mathbf{r}, -\mathbf{k})\mathbf{k})} |\mathbf{n} \cdot \mathbf{k}'| I_{\Gamma^+} \left(\mathbf{r} - d(\mathbf{r}, -\mathbf{k})\mathbf{k}, \mathbf{k}', t - \frac{d(\mathbf{r}, -\mathbf{k})}{c} \right) d\mathbf{k}' \\ & + \\ & + \int_0^{d(\mathbf{r}, -\mathbf{k})} \exp(-\mu t') \frac{\sigma}{4\pi} \int_{\Omega} I \left(\mathbf{r} - t' \mathbf{k}, \mathbf{k}, t - \frac{t'}{c} \right) d\mathbf{k}' dt'. \end{aligned} \quad (5)$$

Here, $d(\mathbf{r}, -\mathbf{k})$ denotes the distance from the point $\mathbf{r} \in G$ in the direction $-\mathbf{k}$ to the boundary of the region G .

For solving (5) authors construct an iteration method. Denote the initial approximation as

$$I_0 = \int_0^{d(\mathbf{r}, -\mathbf{k})} \exp(-\mu t') J_0 \left(\mathbf{r} - t' \mathbf{k}, \mathbf{k}, t - \frac{t'}{c} \right) dt'.$$

Thus, the solution of the direct problem in the double scattering approximation and in the assumption that G is non-scattering ($\sigma = 0$), can be represented in the following form:

$$\begin{aligned} I^\pm(t) = & X_{[0, \pm\infty]} \frac{8 \sigma_d J_0 \exp(-\mu c t) \left(y_1 u'_{y_1} + l - u(y_1) \right)^2}{c^2 t^3 |u'_{y_1} (l - u(y_1)) - y_1| \sqrt{1 + (u'_{y_1})^2}} + \\ & + 4\sigma_d^2 J_0 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S^\pm(\mathbf{k}) \frac{\exp(-\mu |\mathbf{y}|)}{|\mathbf{y}|} \exp(-\mu |\mathbf{y} - \mathbf{z}|) \frac{\exp(-\mu |\mathbf{z}|)}{|\mathbf{z}|} \left| \mathbf{n}(\mathbf{y}) \cdot \frac{\mathbf{y}}{|\mathbf{y}|} \right| \times \\ & \times \left| \mathbf{n}(\mathbf{y}) \cdot \frac{\mathbf{z}}{|\mathbf{z}|} \right| \delta \left(t - \frac{|\mathbf{y} - \mathbf{z}|}{c} - \frac{|\mathbf{z}|}{c} - \frac{|\mathbf{y}|}{c} \right) \sqrt{1 + (u'_{y_1})^2} \sqrt{1 + (u'_{z_1})^2}, \end{aligned} \quad (6)$$

where $\mathbf{y}, \mathbf{z} \in \gamma$.

Further, we found the solution of the inverse problem for the function which described relief changes from the middle-level. It was received by using a single-scattering approximation [6, 7]. A nonlinear differential equation is obtained for the numerical solution of which the following scheme was constructed:

$$u'_i = \frac{1}{y_{1,i}} \left(u_{i-1} - l + \sqrt[4]{1 + v_{0,i-1}^2} \sqrt{\frac{I(t_i) c^2 t_i^3 |v_{0,i-1} (l - u_{i-1}) - y_{1,i}|}{8 \sigma_d J_0 \exp(-\mu c t_i)}} \right), \quad (7)$$

where $t_i = 2c^{-1}(y_{1,i}^2 + (l - u_{i-1})^2)^{1/2}$.

In the numerical algorithm, the function v_{0i} is represented as: $v_{0i} = u'_{i-1}$. For constructing a numerical algorithm, we set two initial conditions: $u(0) = u_0$, $u'(0) = v_0$. The modified Euler's method is used for computational experiments.

In the case of a narrow directivity pattern of the receiving antenna, the problem of remote sensing of the side-scan sonar, moving with a constant velocity V along the axis r_3 , is reduced to solving the problem (1) - (4) and is solved independently at each probing interval.

The sounding parameters for the computational experiments are presented in Table 1. [8,9]. The purpose of the experiment is to determine the effect of double scattering on the restoration of the seabed relief.

Table 1

Probing parameters [1].						
μ, M^{-1}	σ_d	$c, \text{M/C}$	J_0	l, M	y_1, M	y_3, M
0.018	1	1500	1	20	[0, 300]	[0, 80]

Fig. 1. The exact solution u .

Fig. 2. The double scattering effect on recovery I_2/I .

Fig. 3. The retrieval function u in single scattering approximation.

Fig. 4. Error Δu for the single scattering approximation.

Fig. 5. The retrieval function u in double scattering approximation.

Fig. 6. Error Δu for the double scattering approximation.

Figure 1 presents a graph of the exact solution of the function u , based on which the received signal $I(t)$ was calculated using formula (6).

Figures 3, 4 present a graph of the recovered function u and the error Δu , respectively. As a received signal for plotting 3, the values of signal using only single scattered signal I_1 were taken, i.e. only the first term in the formula (6). As a numerical method for solving the differential equation (7), an iterative predictor-corrector method was chosen with an accuracy of $1.0E-15$. Numerical scheme (7) contains the linearization of the derivative on the right-hand side $v_{0i} = u'_{i-1}$, as can be seen from Figure 4, the maximum error reaches 0.01%, which indicates high accuracy of the numerical method.

Figures 5, 6 show the graph of the recovered function u and the error Δu in the case, when the measured signal $I(t)$ was calculated by formula (6) with allowance for double scattering. To calculate the integral in formula (6), an algorithm was developed for determining the propagation paths of the signal corresponding to double reflection from the bottom.

Figure 2 shows a graph of the ratio of signals I_2 / I , showing the contribution of the double-scattered signal to the total measured signal, which does not exceed 0.1%. Despite such a small contribution, the double-scattered signal significantly affects the recovery of the bottom surface u . The recovery error Δu increases with the sensing range and reaches 0.3 m.

Thus, the solution of the bathymetry problem regarding the double scattered signal was investigated. In the proposed experiments, the double-scattered signal did not exceed 0.1% in the total measured SSS signal, but when solving the bathymetry problem, the recovery error reached 20%. It is worth noting that this result was obtained with a 100% reflection of the signal from the seabed. In the case of partial reflection ($\sim 10\%$), the error decreases by a factor of 10 and is $\sim 2\%$.

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SIGN K-POTENT SIGN PATTERN MATRICES THAT ALLOW K-POTENCE

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Abstract. For a real number a , the sign of a is defined $+$, $-$ or 0 if a is a positive, negative or 0 , respectively. A matrix whose entries consist of just symbols $+$, $-$ and 0 is called a sign pattern matrix. If P is a property of a real matrix, then a sign pattern matrix $A=[a_{ij}]$ allow P if there is a matrix B , the signs of whose entries equal to a_{ij} , with the property P .

By the Frobenius theorem, we can prove easily that all the irreducible sign k -potent sign pattern matrices are allows k -potence. But, not all reducible sign k -potent sign pattern matrices allow k -potence.

In this paper, we provide a necessary and sufficient condition for a sign k -potent sign pattern matrix to allow k -potence. Furthermore, we show that if A is a sign k -potent sign pattern matrix, then A allows k -potence if and only if $|A|=[|a_{ij}|]$ allows k -potence.

By the definition of the multiplication of sign pattern matrix, the entries in the block submatrix of A which are induced by the Frobenius normal form does not have the opposite sign. Hence we can construct the reduced block matrix of A , denote $\text{red}(A)$. We prove that for a sign k -potent sign pattern matrix A , A allows the k -potence if and only if $\text{red}(A)$ also allows k -potence.

Key words. Sign pattern matrices, sign k -potent sign pattern matrix, k -potence, allow.

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ON THE GENERALIZED HYPERGEOMETRIC FUNCTIONS WITH INTEGRAL PARAMETER DIFFERENCES

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Abstract. Miller-Paris transformations are extensions of Euler's transformation for the Gauss hypergeometric functions to higher-order generalized hypergeometric functions with integral parameter differences (IPD). Some generalizations of Miller-Paris transformations were proved by us to obtain new results for the generalized hypergeometric function with IPD. In particular, we give several extensions of the Karlsson-Minton summation formula.

Keywords. Generalized hypergeometric function, Karlsson-Minton summation, Miller-Paris transformation, hypergeometric summation, integral parameter differences

Suppose $\mathbf{a}$ and $\mathbf{b}$ are complex vectors, $(a)_n$ is the Pochhammer's symbol. We will use the following convenient abbreviated notation:

$$\Gamma(\mathbf{a}) = \Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p), \quad (\mathbf{a})_n = (a_1)_n(a_2)_n\cdots(a_p)_n, \quad \mathbf{a} + \mu = (a_1 + \mu, a_2 + \mu, \dots, a_p + \mu)$$

Inequalities like $\operatorname{Re}(\mathbf{a}) > 0$ will be understood element-wise.

The generalized hypergeometric function is defined by the series

$${}_pF_q\left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \middle| x\right) = \sum_{n=0}^{\infty} \frac{(\mathbf{a})_n x^n}{(\mathbf{b})_n n!} \quad (1)$$

for the values of x making the series convergent and by analytic continuation elsewhere. The size of a vector appearing in the parameters of the generalized hypergeometric function can usually be read off its indices. More detailed information about the definition and properties of the generalized hypergeometric function can be found in standard textbooks (Andrews, Askey and Roy, 1999), (Beals and Wong, 2016), (Luke, 1969) and reference books (Olver., Lozier, Boisvert and Clark (Eds.), 2010). The argument $x=1$ of the generalized hypergeometric function will be routinely omitted. In what follows we also will use the notation

$$\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{N}^r, \quad m = m_1 + m_2 + \dots + m_r, \quad \mathbf{f} = (f_1, \dots, f_r), \quad (\mathbf{f})_{\mathbf{m}} = (f_1)_{m_1} \cdots (f_r)_{m_r},$$

$$(\mathbf{f} - \mathbf{b})_{\mathbf{m}} = (f_1 - b_1)_{m_1} \cdots (f_r - b_r)_{m_r}.$$

In 1970 Minton (Minton, 1970) proved the summation formula

$${}_{r+2}F_{r+1}\left(\begin{matrix} -k, b, \mathbf{f} + \mathbf{m} \\ b + 1, \mathbf{f} \end{matrix}\right) = \frac{k!}{(b+1)_k} \frac{(\mathbf{f} - \mathbf{b})_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}} \quad (2)$$

valid for $k \geq m$, k is a natural number. Soon thereafter, his result was generalized by Karlsson (Karlsson, 1971) who replaced $-k$ by an arbitrary complex number a satisfying $\operatorname{Re}(1-a-m) > 0$ to get

$${}_{r+2}F_{r+1}\left(\begin{matrix} a, b, \mathbf{f} + \mathbf{m} \\ b + 1, \mathbf{f} \end{matrix}\right) = \frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(b+1-a)} \frac{(\mathbf{f} - \mathbf{b})_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}. \quad (3)$$

Gasper (Gasper, 1981) deduced a q -analogue and a generalization of Minton's and Karlsson's formulas; Chu (Chu, 1994), (Chu, 1995) found extensions to bilateral hypergeometric and q -hypergeometric series; their results were re-derived by simpler means and further generalized by Schlosser (Schlosser, 2001), who also found multidimensional extensions to hypergeometric functions associated with root systems (Schlosser, 2003). For further developments in this directions, see also (Rosengren, 2004) and (Rosengren, 2005). Quite recently, Miller and Srivastava (Miller and Srivastava, 2010) found some alternative derivations of the original Karlsson and Minton formulas.

Some of the results mentioned above found applications in the theory of rook polynomials and related combinatorial problems as discovered by Haglund in (Haglund, 1996), who also noticed that some of the results can be obtained by specializing a very general transformation due to Slater. We also mention an interesting work (Letessier, Valent and Wimp, 1994), where the reduction of the differential equation satisfied by the generalized hypergeometric functions with some integer parameter differences was found.

Conditions $k \geq m$ in (2) and $\text{Re}(1-a-m) > 0$ in (3) can not be dispensed with: the formulas become false when they are violated (take for example $a=-4$, $b=33/17$, $\mathbf{f}=(4.2;-5/3)$, $\mathbf{m}=(7;8)$). The situation with (3) is thereby quite subtle: the function on the right hand side is continuous at $a=1-m$ while the function on the left hand side is finite but has a discontinuity at this point. Further, if $\text{Re}(1-a-m) < 0$ and a is not a negative integer or zero, the series on the left hand side diverges. At the same time, it reduces to a finite sum for $a=-k$, so that such points are "isolated summability points" and analytic continuation of the right hand side in a does not lead to the correct value of the left hand side at such point.

In this paper we discuss the correct value of the left hand side of Minton's formula (2) at these special points $k=0,1,\dots,m-1$. In particular, we have obtained the following results

$${}_{r+2}F_{r+1}\left(\begin{matrix} 1-m, b, \mathbf{f} + \mathbf{m} \\ b+1, \mathbf{f} \end{matrix}\right) = \frac{(m-1)!}{(b+1)_{m-1}} \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}} - \frac{(-1)^m(m-1)!b}{(\mathbf{f})_{\mathbf{m}}}$$

and

$$\begin{aligned} {}_{r+2}F_{r+1}\left(\begin{matrix} 2-m, b, \mathbf{f} + \mathbf{m} \\ b+1, \mathbf{f} \end{matrix}\right) &= \frac{(m-2)!}{(b+1)_{m-2}} \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}} \\ &+ \frac{(-1)^m(m-2)!b}{(\mathbf{f})_{\mathbf{m}}} \left(1-b+m(m-2+f_r) + \sum_{q=1}^{r-1} (f_q - f_{q+1} - m_{q+1}) \sum_{i=1}^q m_i \right). \end{aligned}$$

In addition we present an extension of Karlsson's formula (3) obtained by replacing the number b by a vector $\mathbf{b}$ and the number $b+1$ by the vector $\mathbf{b}+\mathbf{p}$, where $\mathbf{p}$ is a positive integer vector. Suppose $\mathbf{b}=(b_1,\dots,b_l)$ is a complex vector, $\mathbf{p}=(p_1,\dots,p_l)$ is a vector of positive integers, $p=p_1+\dots+p_l$, and all elements of the vector

$$\beta = (b_1, b_1+1, \dots, b_1+p_1-1, \dots, b_l, b_l+1, \dots, b_l+p_l-1) = (\beta_1, \beta_2, \dots, \beta_p)$$

are distinct. If $\text{Re}(p-a-m) > 0$, then

$$\frac{1}{\Gamma(1-a)^{r+l+1}} {}_{r+l+1}F_{r+l}\left(\begin{matrix} a, \mathbf{b}, \mathbf{f} + \mathbf{m} \\ \mathbf{b} + \mathbf{p}, \mathbf{f} \end{matrix}\right) = \frac{(\mathbf{b})_{\mathbf{p}}}{(\mathbf{f})_{\mathbf{m}}} \sum_{q=1}^p \frac{\Gamma(\beta_q)(\mathbf{f}-\beta_q)_{\mathbf{m}}}{B_q \Gamma(1+\beta_q-a)},$$

where

$$B_q = \prod_{v=1, v \neq q}^p (\beta_v - \beta_q).$$

In particular,

$$\frac{1}{\Gamma(1-a)^{r+2}} {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, \mathbf{f} + \mathbf{m} \\ b+p, \mathbf{f} \end{matrix}\right) = \frac{\Gamma(b+p)}{(\mathbf{f})_{\mathbf{m}}} \sum_{q=0}^{p-1} \frac{(b)_q (\mathbf{f}-b-q)_{\mathbf{m}}}{(p-q-1)! (-q)_q \Gamma(1+q+b-a)}.$$

Further, we apply some generalizations Miller-Paris transformations to obtain new formula for the generalized hypergeometric function with integral parameters differences in the unit. To formulate the result we will need the polynomials

$$Q_m(b, c, \mathbf{f}, \mathbf{m}; t) = \frac{(c-b-t-m)_m}{(c-b-m)_m} \sum_{k=0}^m {}_{r+1}F_r\left(\begin{matrix} -k, \mathbf{f} + \mathbf{m} \\ \mathbf{f} \end{matrix}\right) \frac{(t)_k (b)_k}{(1+t+b-c)_k k!}.$$

and

$$\hat{Q}_m(a, b, c, \mathbf{f}, \mathbf{m}; t) = \sum_{k=0}^m \frac{(-1)^k C_{k,r}(a)_k (b)_k (t)_k}{(c-a-m)_k (c-b-m)_k} {}_3F_2 \left(\begin{matrix} -m+k, t+k, c-a-b-m \\ c-a-m+k, c-b-m+k \end{matrix} \right),$$

$$C_{k,r} = \frac{(-1)^k}{k!} {}_{r+1}F_r \left(\begin{matrix} -k, \mathbf{f} + \mathbf{m} \\ \mathbf{f} \end{matrix} \right).$$

where

Suppose $\mathbf{h}=(h_1, \dots, h_l)$ is a complex vector and $\operatorname{Re}(c+e-a-b-d-p-m)>0$, $\operatorname{Re}(e-d-p)>0$. Then

$${}_{r+l+3}F_{r+l+2} \left(\begin{matrix} a, b, d, \mathbf{h} + \mathbf{p}, \mathbf{f} + \mathbf{m} \\ c, e, \mathbf{h}, \mathbf{f} \end{matrix} \right) = \mathcal{A}_{m+p+3} F_{m+p+2} \left(\begin{matrix} c-a-m, c-b-m, d, \nu+1, \eta+1 \\ c, e+\kappa, \nu, \eta \end{matrix} \right)$$

where

$$\mathcal{A} = \frac{(\zeta + \kappa)_1 \Gamma(e) \Gamma(e-d-p+\kappa)}{(\zeta)_1 \Gamma(e+\kappa) \Gamma(e-d-p)}.$$

$\kappa=c-$

$a-b-m$ are the roots of the polynomial $Q_p(d, e, \mathbf{h}, \mathbf{p}; t)$, ν are the roots of the polynomial

$$Q_p(e-d+\kappa-p, e+\kappa, \zeta+\kappa, 1; t)$$

η are the roots of the polynomial

$$\hat{Q}_m(a, b, c, \mathbf{f}, \mathbf{m}; t).$$

The presented results are partially included in the publications (Karp and Prilepkina, 2018) and (Karp and Prilepkina, 2019).

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ENERGY CONSUMPTION MODEL OF ELECTRICAL DRIVES FOR MACHINE TOOLS

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Abstract. The energy saving of machine tools is recently a hot issue in the manufacturing process. The prediction of the energy consumption is an important step to achieve the energy saving process of machine tools. This paper describes a development of the energy consumption model of the electrical motor drives for machine tools to predict the energy consumption during the machining process. The model can be generally represented as polynomial of the processing variables which can be obtained from the NC controller, and its coefficients can be determined by the curve fitting of the measured data. The energy consumption of the machine tools can further be optimized based on this model. The validity of the derived model is proved through the experimental works.

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ESTIMATES ON THE FIFTH COEFFICIENTS OF STRONGLY STARLIKE FUNCTIONS

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Abstract. Let A be the class of analytic functions in the open unit disk D which have the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For given $\alpha \in (0,1]$, let us define the class S_{α}^* of strongly starlike functions of order α which satisfy

$$\left| \operatorname{Arg} \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \alpha, \quad z \in D.$$

In this talk, we introduce a new method to find the sharp bound for the fifth coefficient a_5 of the functions in S_{α}^* by using the properties of Caratheodory functions. And similar coefficient problems for gamma starlike functions of order β ($\beta \geq 0$) and Bazilevic functions of order γ ($\gamma \geq 0$) are examined by this approach.

Key words. Univalent functions, Strongly starlike functions, Gamma starlike functions, Bazilevic functions, Fifth coefficients.

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OPTIMIZATION APPROACH FOR IDENTIFICATION PROBLEMS FOR STATIONARY EQUATION OF ADMIXTURE TRANSFER

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Abstract. The identification problem for the stationary admixture transfer equation considered in bounded domain with a Dirichlet condition imposed on the boundary of the domain is studied. By applying an optimization method this problem is reduced to the control problem in which the role of controls is played by variable coefficients of diffusion and chemical reaction. Solvability of the control problem is proved, the optimality system, which describes necessary conditions of an extremum is derived, the stability estimates for optimal solutions are established and the efficient numerical algorithm for solving the control problem is developed.

Keywords. Mass transfer model, diffusion-reaction equation, variable coefficients of diffusion and reaction, identification problem, control problem.

1. Introduction

In recent years, much attention has been paid to the formulation and study of new classes of problems for heat and admixture transfer models. Control problems can be stated as examples of such problems. Among theoretical studies of the indicated problems for linear and nonlinear models of heat and admixture transfer, we note [1–4].

Along with control problems, an important role in applications is played by identification problems for heat and mass transfer models. In the latter problems, unknown densities of boundary or distributed sources or the coefficients involved in the differential equations or the boundary conditions for the model under study are recovered from additional information about the solution of the original boundary value problem. Importantly, with a certain choice of the minimized cost functional, identification problems can be reduced to corresponding extremum problems. Following this approach, control problems arise there. They can be studied using well-known constrained minimization methods. This approach as applied to heat and mass transfer models was described in [5 – 11].

The goal of this paper is to develop a theory for the uniqueness and stability analysis of inverse extremum problems for a linear admixture transfer model defined by the variable diffusivity and quantity characterizing disintegration of pollutant by chemical reactions. This model is considered in a bounded domain Ω with a Dirichlet condition specified on the boundary Γ .

2. Statement of the boundary value problem

We study the following admixture transfer model:

$$- \operatorname{div}(\lambda \nabla \varphi) + k \varphi = f \quad \text{in } \Omega, \quad \varphi|_{\Gamma} = 0 \quad (1)$$

considered in a bounded domain Ω in the space $\mathbf{R}^d$, $d=2,3$ with Lipschitz boundary Γ . Here $\lambda \equiv \lambda(\mathbf{x}) > 0$ is a variable diffusion coefficient, $k \equiv k(\mathbf{x}) \geq 0$ is a quantity characterizing disintegration of pollutant by chemical reactions, $f \equiv f(\mathbf{x})$ is a volume source density.

As usual we will use the functional spaces $H^1(\Omega)$, $H^s(\Omega)$, $L^\infty(\Omega)$, $L^2_+(\Omega) = \{k \in L^2(\Omega): k \geq 0\}$, $V = \{\varphi \in H^1(\Omega): \varphi|_{\Gamma} = 0\}$ and sets $H^s_{\lambda_0}(\Omega)$, $L_{\lambda_0}(\Omega)$, $\lambda_0 = \text{const} > 0$ while studying problem 1 and corresponding identification problem. The properties of these spaces and denotations can be found in [6]. Assuming that $\Gamma \in C^{0,1}$, $\lambda \in H^s_{\lambda_0}(\Omega)$, $k \in L^2_+(\Omega)$, $f \in L^2(\Omega)$ we define the weak solution of problem (1) as the function $\varphi \in V$ satisfying the identity

$$(\lambda \nabla \varphi, \nabla h) + (k \varphi, h) = (f, h) \quad \forall h \in V. \quad (2)$$

3. Statement of the identification problem, reduction to an extremum problem, and necessary optimality conditions

The boundary value problem (1), considered in Section 2, involves three functional parameters, namely, the diffusivity λ , the decay coefficient k and the function f describing the source density. The boundary value problem (1) contains a number of parameters that must be given to ensure the uniqueness of the solution. In practice, situations can arise when some of the parameters are unknown. For this reason, we need additional information about the solution φ of problem (1). As this information we can use, for example, concentration $\varphi_d(x)$ measured in some subdomain $Q \subset \Omega$ (see fig. 1). Let coefficients λ and k be unknown functions and we must determine these functions along with the solution φ of problem (1). For the study of this identification problem we apply optimization method and reduce solution of this problem to the corresponding extremum problem (see [6]). Following this method, as distributed controls we will choose the functions λ and k , which are changed in some sets K_1 and K_2 .

Fig. 1. The geometry of the domain

More concretely, we assume that the following conditions take place:

- (i) $\Gamma \in C^{0,1}$, $f \in L^2(\Omega)$;
- (ii) $K_1 \subset H_{\lambda_0}^s(\Omega)$, $\lambda_0 > 0$, $s > d/2$, $K_2 \subset L^2_+(\Omega)$ are convex closed sets.

Setting $K = K_1 \times K_2$, $u = (\lambda, k)$, we introduce an operator $F: V \times K \times L^2(\Omega) \rightarrow V^*$ acting according to the formula

$$\langle F(\varphi, \lambda, k, f), h \rangle = (\lambda \nabla \varphi, \nabla h) + (k \varphi, h) - (f, h) \quad \forall h \in V \quad (3)$$

where $\varphi \in V$ is the weak solution of problem (1).

Let $I: V \rightarrow \mathbf{R}$ be a weakly lower semicontinuous cost functional. Consider the extremum problem:

$$\frac{\mu_0}{2} I(\varphi) + \frac{\mu_1}{2} \|\lambda\|_s^2 + \frac{\mu_2}{2} \|k\|^2 \rightarrow \inf, \quad F(\varphi, u, f) = 0, \quad (\varphi, u) \in V \times K. \quad (4)$$

Here, μ_0, μ_1, μ_2 are nonnegative parameters that specify the relative importance of each of the terms in (4). Another goal of introducing μ_1 is to ensure the uniqueness and stability of solutions of particular extremum problems.

As admissible cost functionals, we use

$$I_1(\varphi) = \|\varphi - \varphi_d\|_Q^2 = \int_Q |\varphi - \varphi_d|^2 d\mathbf{x} = \int_\Omega r(\varphi - \tilde{\varphi}_d)^2 d\mathbf{x}, \quad I_2(\varphi) = \|\varphi - \varphi_d\|_{L^2(Q)}^2. \quad (5)$$

Here $\varphi_d \in L^2(Q)$ is a given function in Q , $r = \chi_Q$ is the characteristic function of Q , and $\tilde{\varphi}_d \in L^2(\Omega)$ is a function that is equal to φ_d in Q and vanishes outside of Q . It is well known, that each of the functionals in (5) is weakly lower semicontinuous. The set of admissible pairs (φ, u) for problem (4) is defined as $Z_{ad} = \{(\varphi, u) \in V \times K : F(\varphi, u, f) = 0, \quad J(\varphi, u) < \infty\}$. In addition to (ii), let the following condition hold:

- (iii) $\mu_0 > 0$, $\mu_1 \geq 0$, $\mu_2 \geq 0$ and K_1 and K_2 are bounded sets or $\mu_l > 0$, $l=0,1,2$ and the functional I is bounded from below.

Theorem 1. Let $I:V \rightarrow \mathbf{R}$ be a weakly lower semicontinuous cost functional; conditions (i), (ii), and (iii) hold; and Z_{ad} be a nonempty set. Then problem (4) has at least one solution $(\varphi, u) \in V \times K$.

Note that Theorem 1 remains valid for the functionals I_1 and I_2 , since they are nonnegative and weakly lower semicontinuous, while Z_{ad} is not empty. Therefore, the following result holds.

Theorem 2. Let conditions (i) and (ii) be satisfied, $\mu_0 > 0$ and $\mu_l > 0$ or $\mu_0 > 0$ and $\mu_l \geq 0$; and K_l be bounded sets, $l=1,2$. Then problem (4) for $I=I_m$, $m=1,2$ has at least one solution $(\varphi, u) \in V \times K$.

Let us derive necessary optimality conditions for problem (4). For this purpose, the extremal principle is applied to smooth convex extremum problems. According to the general theory of extremum problems (see [13, 14]), we introduce a Lagrange multiplier $\eta \in V$ which is interpreted as an ‘‘adjoint’’ concentration. Additionally, the Lagrangian $\mathcal{L}: V \times K \times L^2(\Omega) \times V \rightarrow \mathbf{R}$ is defined as

$$\mathcal{L}(\varphi, u, f, \eta) \equiv J(\varphi, u) + \langle F(\varphi, u, f), \eta \rangle \equiv (\mu_0/2)I(\varphi) + (\mu_1/2)\|\lambda\|_s^2 + (\mu_2/2)\|k\|^2 + \langle F(\varphi, u, f), \eta \rangle. \quad (6)$$

The result of [13, p. 79] implies the following assertion.

Theorem 3. Under the conditions of Theorem 2, let a pair $(\hat{\varphi}, \hat{u}) \in V \times K$ be a local minimizer in problem (4), and let the functional $I(\cdot): V \rightarrow \mathbf{R}$ be continuously differentiable with respect to φ at the point $\hat{\varphi}$. Then there exists a unique Lagrange multiplier $\eta \in V$ such that the Euler-Lagrange equation

$$\mathcal{L}'_{\varphi}(\hat{\varphi}, \hat{u}, f, \eta) \equiv F'_{\varphi}(\hat{\varphi}, \hat{u}, f)^* \eta + (\mu_0/2)I'_{\varphi}(\hat{\varphi}) = 0 \quad \text{in } V^* \quad (7)$$

holds and the minimum principle $\mathcal{L}(\hat{\varphi}, \hat{u}, f, \eta) \leq \mathcal{L}(\varphi, u, f, \eta) \quad \forall u \in K$ is valid and is equivalent to the variational inequalities

$$\langle \mathcal{L}'_{\lambda}(\hat{\varphi}, \hat{u}, f, \eta), \lambda - \hat{\lambda} \rangle = \mu_1 \langle \hat{\lambda}, \lambda - \hat{\lambda} \rangle_{s, \Omega} + ((\lambda - \hat{\lambda}) \nabla \hat{\varphi}, \nabla \eta) \geq 0 \quad \forall \lambda \in K_1, \quad (8)$$

$$\langle \mathcal{L}'_k(\hat{\varphi}, \hat{u}, f, \eta), k - \hat{k} \rangle = \mu_2 \langle \hat{k}, k - \hat{k} \rangle_{\Omega} + ((k - \hat{k}) \nabla \hat{\varphi}, \nabla \eta) \geq 0 \quad \forall k \in K_2. \quad (9)$$

It follows from (6) that

$$\langle F'_{\varphi}(\hat{\varphi}, \hat{u}, f)^* \eta, \tau \rangle = \langle F'_{\varphi}(\hat{\varphi}, \hat{u}, f) \tau, \eta \rangle = (\hat{\lambda} \nabla \tau, \nabla \eta) + (\hat{k} \tau, \eta) \quad \forall \tau \in V. \quad (10)$$

From (10), we conclude that the Euler–Lagrange equation (7) is equivalent to the identity

$$(\hat{\lambda} \nabla \tau, \nabla \eta) + (\hat{k} \tau, \eta) = -(\mu_0/2) \langle I'_{\varphi}(\hat{\varphi}), \tau \rangle \quad \forall \tau \in V. \quad (11)$$

Identity (11) is a weak formulation of a boundary value problem for the adjoint concentration η . Its form depends on a type of the cost functional I . The problem (11) is referred formally as the adjoint problem below. We emphasize that the direct problem (2), adjoint problem (11), and inequalities (8), (9) comprise an optimality system describing the necessary conditions for a minimum in problem (4). Below, based on an analysis of this optimality system, we formulate sufficient conditions on the input data under which the solution of problem (4) is unique and stable for particular cost functionals.

4. Stability estimates for solutions of control problems

Assume that the function f for the equation of state $F(\varphi, u, f)=0$ ranges over a set $F_{ad} \subset L^2(\Omega)$. Denote by $(\varphi_1, u_1) = (\varphi_1, \lambda_1, k_1) \in V \times K$ an arbitrary solution of problem (4) with a given function $f=f_1 \in F_{ad}$. Let $(\varphi_2, u_2) = (\varphi_2, \lambda_2, k_2) \in V \times K$ denote the solution of the problem

$$\frac{\mu_0}{2} \tilde{I}(\varphi) + \frac{\mu_1}{2} \|\lambda\|_s^2 + \frac{\mu_2}{2} \|k\|^2 \rightarrow \inf, \quad F(\varphi, u, f_2) = 0, \quad (\varphi, u) \in V \times K, \quad (12)$$

which is obtained from (4) by replacing I with a close functional $\tilde{I}$ and by replacing f_1 with a close function $\tilde{f} = f_2 \in F_{ad}$. Assuming that K_1 and K_2 are bounded sets, the functions φ_i satisfy the estimates

$$\|\varphi_i\|_1 \leq M_{\varphi} = C_0 \sup_{f \in F_{ad}} \|f\|, \quad C_0 = (\delta_1 \lambda_0)^{-1}, \quad i = 1, 2, \quad (13)$$

where $\delta_1 = \text{const} > 0$, $\lambda_0 = \text{const} > 0$ are certain constants. Clearly, $M_{\varphi} < \infty$ if the sets K_1 , K_2 and F_{ad} are all bounded.

Relying on the results of [12], we now establish sufficient conditions for the uniqueness and stability of the solution $(\hat{\varphi}, \hat{\lambda}, \hat{k})$ of the following control problem corresponding to the cost functional $I_1(\varphi) = \|\varphi - \varphi_d\|_Q^2$:

$$\frac{\mu_0}{2} I_1(\varphi) + \frac{\mu_1}{2} \|\lambda\|_{s,\Omega}^2 + \frac{\mu_2}{2} \|k\|_\Omega^2 \rightarrow \inf, \quad F(\varphi, u, f) = 0, \quad (\varphi, u) \in V \times K. \quad (14)$$

Theorem 4. Assume that conditions (i) and (ii) hold and, additionally, K_1 , K_2 and $F_{ad} \subset L^2(\Omega)$ are bounded sets. Let the triple $(\varphi_i, \lambda_i, k_i) \in V \times K_1 \times K_2$ be a solution of problem (14) corresponding to given functions $\varphi_d^{(i)} \in L^2(Q)$ and $f_i \in F_{ad}$, $i=1,2$, where Q is an arbitrary open bounded set. Let $M_\varphi^0 = M_\varphi + \max(\|\varphi_d^{(1)}\|_Q, \|\varphi_d^{(2)}\|_Q)$ and let the following conditions

$$\mu_1(1-\varepsilon) > 5\mu_0\gamma_0^2 C_0^2 M_\varphi^0 M_\varphi, \quad \mu_2(1-\varepsilon) > 5\mu_0\gamma_1^2 C_0^2 M_\varphi^0 M_\varphi, \quad \mu_0 > 0, \quad (15)$$

be satisfied for some constant $\varepsilon \in (0,1)$, where M_φ is defined in (13). Then the following stability estimates hold:

$$\|\varphi_1 - \varphi_2\|_Q \leq \|\varphi_d^{(1)} - \varphi_d^{(2)}\|_Q + \beta(f_1 - f_2), \quad (16)$$

$$\|\lambda_1 - \lambda_2\|_{s,\Omega} \leq \sqrt{\mu_0 / \varepsilon \mu_1} \Delta, \quad \|k_1 - k_2\| \leq \sqrt{\mu_0 / \varepsilon \mu_2} \Delta, \quad (17)$$

$$\|\varphi_1 - \varphi_2\|_{1,\Omega} \leq M_\varphi (\gamma_0 \sqrt{\mu_0 / \varepsilon \mu_1} + \gamma_1 \sqrt{\mu_0 / \varepsilon \mu_2}) \Delta + \|f_1 - f_2\|_\Omega, \quad (18)$$

Here Δ and β are defined by

$$\Delta = \|\varphi_d^{(1)} - \varphi_d^{(2)}\|_Q + \beta(f_1 - f_2), \quad (19)$$

$$\beta(\|f\|_\Omega) = a\|f\|_\Omega + b\|f\|_\Omega^2 = 2C_0 M_\varphi^0 \|f\|_\Omega + 2C_0^2 M_\varphi^0 M_\varphi^{-1} \|f\|_\Omega^2. \quad (20)$$

5. Numerical algorithm

The optimality system plays an important role in investigating properties of solutions of the control problem. On the basis of the analysis of optimality systems sufficient conditions for the input data which provide the uniqueness and stability of solutions to individual extremum problems can be formulated. Optimality system derived above can be used to design effective numerical algorithms for solving control problem (14). The simplest numerical algorithm can be obtained by applying simple iteration method for solving the optimality system. The n -th iteration of the algorithm consists in finding values φ^n , η^n , λ^{n+1} and k^{n+1} for given λ^n and k^n by sequentially solving following problems

$$\begin{aligned} (\lambda^n \nabla \varphi^n, \nabla h) + (k^n \varphi^n, h) &= (f, h) \quad \forall h \in V, \\ (\lambda^n \nabla \tau, \nabla \eta^n) + (k^n \eta^n, \tau) &= -(\mu_0 / 2) \langle I'_\varphi, \tau \rangle \quad \forall \tau \in V, \\ \mu_1 (\lambda^{n+1}, \lambda - \lambda^n)_{s,\Omega} + ((\lambda - \lambda^{n+1}) \nabla \varphi^n, \nabla \eta^n) &\geq 0 \quad \forall \lambda \in K_1, \\ \mu_2 (k^{n+1}, k - k^n)_\Omega + ((k - k^{n+1}) \nabla \varphi^n, \nabla \eta^n) &\geq 0 \quad \forall k \in K_2. \end{aligned} \quad (21)$$

Direct and adjoint problems in (21) can be solved by finite element method using FreeFem++.

The authors plan to devote a separate paper to studying the convergence of the algorithm and to the analysis of results of numerical experiments.

6. Conclusion

Summarizing, we have formulated identification problems for the linear diffusion–reaction equation. Using optimization method these problems were reduced to corresponding control problems. We have proved the solvability of these control problems, have derived the optimality system, which describes the necessary conditions of extremum, have established sufficient conditions on the input data under which the solutions of particular extremum problems are unique and stable. Based on analysis of

optimality system we also developed the numerical algorithm of solving control problems. The properties of such an algorithm and numerical results will be analyzed elsewhere. The separate paper of the authors will be devoted to the analysis of results of numerical experiments.

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PRIMITIVE NORMALITY OF THE CLASS OF PRINCIPALLY WEAKLY INJECTIVE S-ACTS

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Abstract. One of the standard problems in the model theory of S-acts is to describe the monoids over which a certain class of S-acts has some model-theoretic properties. Primitive normality of the class of all S-acts and the class of all injective S-acts was studied in works of Stepanova A.A. and Efremov E.L. Namely, it is proved that the class of all injective S-acts is primitive normal. The notion of principally weakly injective S-act can be considered as a natural generalization of the notion of injective S-act. In this article we describe monoids S such that the class of principally weakly injective S-acts is primitive normal. It is shown that the class of principally weakly injective S-acts is primitive normal if and only if S is linearly ordered monoid.

Keywords. Theory, primitive normal theory, monoid, S-act, principally weakly injective S-act.

This theme relates to model-theoretic algebra, that is a branch of mathematics that connects the model theory with other areas of mathematics. The subject of the research is the class of principally weakly injective S-acts.

The model theory of modules over a ring has long been a respectable branch of both model theory and ring theory. The concept of an act over a monoid S (S-act) is a generalization of the concept of a module over a ring. The model theory of S-acts is rather less developed but again exhibits a nice interplay between algebra and model theory, with its own distinct flavor. Many problems in model theory of S-acts came from the module model theory. For example, the problem of an algebraic description of monoids S over which some classes of S-acts have some model-theoretic properties (axiomatizability, completeness, stability, primitive normality, primitive connectivity, etc.). These properties for classes of projective, flat, regular, free S-acts were studied in the works of Gould V., Poizat B., Mustafin T.G., Stepanova A.A., Ovchinnikova E.V. and others.

The characterization of the monoids S with axiomatizable and model complete class of regular S-acts was given in [8]. In this article the authors describe the monoids with complete class of regular S-acts which satisfy the additional conditions. They studied the monoids S , all regular S-acts over which have the stable and superstable theory. The authors proved the stability of the axiomatizable model complete class of regular S-acts. They also described the monoids S with the superstable and ω -stable class of regular S-acts when this class is axiomatizable and model complete.

In [10] the authors investigate the commutative monoids over which the axiomatizable class of regular S-acts is primitive normal and antiadditive. They proved that the primitive normality of an axiomatizable class of regular S-acts over the commutative monoid S is equivalent to the antiadditivity of this class and it is equivalent to the linearity of the order of a semigroup R such that an S-act ${}_S R$ is a maximal (under the inclusion) regular subact of the S-act ${}_S S$.

The monoids S over which the class of all regular S-acts is axiomatizable and primitive connected were studied in [13]. Stepanova A.A. proved that the axiomatizable class of all regular S-acts is primitive connected if and only if the semigroup R is rectangular band of groups and $R = eR$ for some idempotent $e \in R$, where ${}_S R$ is the inclusion maximal regular S-subact in the S-act ${}_S S$.

Questions of axiomatizability, completeness, model completeness and stability for classes of free, projective and flat S-acts were considered in [5].

The difficulty of studying the model-theoretical properties of classes of injective S-acts, in contrast to projective, flat, free S-acts, is the absence of an algebraic description of injective S-acts. The first work in this direction can be considered the work of Stepanova A.A. [9], where she described commutative monoids over which the class of all injective S-acts is axiomatizable, complete and model

complete. The continuation of these studies is the work of Stepanova A.A. and Efremov E.L. where the structure of monoids S with axiomatizable classes of weakly injective S -acts, finitely generated weakly injective S -acts and principally weakly injective S -acts has been described [3].

Complete, model-complete, stable and superstable classes of injective and weakly injective S -acts were studied in [1]. It is shown that for a right-reversible monoid S the class of injective (weakly injective, finitely generated, weakly injective) S -acts is complete only if S is trivial. For an arbitrary monoid S it is shown that the completeness of the class of principally weakly injective S -acts is equivalent to the triviality of the monoid S . For a finite monoid S it is proved that the stability (superstability) of the theory of any injective, weakly injective, finitely generated weakly injective S -act is equivalent to the fact that S is a linearly ordered monoid (fully ordered monoid). In addition, it is shown that for an arbitrary monoid S a class of principally weakly injective S -acts is stable (superstable) if and only if S is a linearly ordered monoid (fully ordered monoid).

In this article, we describe the monoids S such that the class of principally weakly injective S -acts is primitive normal.

Below we recall some facts of the theory of S -acts, model theory and universal algebra [4, 6-7].

Throughout this article, S is a monoid with neutral element 1.

An algebraic system $\langle A; s \rangle_{s \in S}$ of signature $L_S = \{s \mid s \in S\}$ is called a (left) S -act, or an act over S , or simply an S -act, whenever $s_1(s_2a) = (s_1s_2)a$ and $1a = a$ for all $s_1, s_2 \in S$, $a \in A$. We denote an S -act $\langle A; s \rangle_{s \in S}$ by ${}_S A$. All S -acts in this article are left S -acts. Each subsystem ${}_S B$ of an S -act ${}_S A$ is called a S -subact of ${}_S A$ and denoted by ${}_S B \subseteq {}_S A$.

We say that a homomorphism $\varphi: {}_S B \rightarrow {}_S C$ of S -acts extends a homomorphism $f: {}_S A \rightarrow {}_S C$ of S -acts, where ${}_S A \subseteq {}_S B$ whenever $\varphi|_A = f$, i.e. $\varphi(a) = f(a)$ for all $a \in A$.

An injective S -act is an S -act ${}_S Q$ such that for any S -act ${}_S A$, for any S -subact ${}_S B$ of ${}_S A$ and for any homomorphism $\varphi: {}_S A \rightarrow {}_S Q$ there exists a homomorphism $\bar{\varphi}: {}_S B \rightarrow {}_S Q$ which extends φ , i.e., $\bar{\varphi}$ is such that the diagram

$$\begin{array}{ccc} {}_S A & \subseteq & {}_S B \\ \downarrow \varphi & \nearrow \bar{\varphi} & \\ {}_S Q & & \end{array}$$

is commutative.

A principally weakly injective S -act is an S -act ${}_S Q$ such that for any principal left ideal Ss of S and for any homomorphism $\varphi: {}_S Ss \rightarrow {}_S Q$ there exists a homomorphism $\bar{\varphi}: {}_S S \rightarrow {}_S Q$ which extends φ .

We denote by **S-PWInj** the class of all principally weakly injective S -acts.

A monoid S is called a linear ordered monoid if the set $\{Ss \mid s \in S\}$ is linear ordered with respect to inclusion.

Let T be a complete theory of signature L , $\mathcal{A} = \langle A; L \rangle$ be an algebraic system.

A tuple $\langle a_1, \dots, a_n \rangle$ of elements of A and a tuple $\langle x_1, \dots, x_n \rangle$ of variables are denoted by $\bar{a}$ and $\bar{x}$, respectively. Let $\bar{v}$ be tuple of elements or variables. We introduce the following notation: $l(\bar{v})$ is the length of $\bar{v}$, $\bar{v}(i)$ is the i -th element of $\bar{v}$. Instead of $\bar{a} \in A^n$ we write $\bar{a} \in A$.

If $\Phi(\bar{x}, \bar{y})$ is a formula of signature L , $\bar{a} \in A$, $l(\bar{a}) = l(\bar{y})$, then we denote by $\Phi(\mathcal{A}, \bar{a})$ the set $\{\bar{b} \in A \mid \mathcal{A} \models \Phi(\bar{b}, \bar{a})\}$.

A formula $\exists \bar{x}(\Phi_0 \wedge \dots \wedge \Phi_k)$, where Φ_i ($i \leq k$) are atomic formulas of signature L , is said to be primitive. Let $\Phi(\bar{x}, \bar{y})$ be a primitive formula of signature L , $\bar{a} \in A$, $l(\bar{a}) = l(\bar{y})$. Then we say that a set $\Phi(\mathcal{A}, \bar{a})$ is primitive. If $\bar{b} \in A$ and $l(\bar{b}) = l(\bar{y})$, then the sets $\Phi(\mathcal{A}, \bar{a})$ and $\Phi(\mathcal{A}, \bar{b})$ are called primitive copies.

A theory T is said to be primitive normal if $X = Y$ or $X \cap Y = \emptyset$ for any primitive copies X and Y . A class $\mathcal{K}$ of algebraic systems of signature L is called primitive normal if a theory $\text{Th}(\mathcal{A})$ is primitive normal for any algebraic system $\mathcal{A} \in \mathcal{K}$.

Theorem. *The class **S-PWInj** of all principally weakly injective S-acts is primitive normal if and only if S is a linearly ordered monoid.*

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NATURAL CONVECTION IN AN ENCLOSURE BY APPLYING A MAGNETIC FIELD

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Abstract. Natural convection of an electrically conducting fluid in an enclosure by applying magnetic field is studied numerically. The two vertical side walls are held isothermally at temperatures θ_h and θ_c , while the horizontal top and bottom walls are adiabatic. A numerical method of finite difference scheme consisting of modified ADI method and SLOR method is used to solve the vorticity-stream function formulation of the problem. Numerical predictions are obtained for a wide range of Gr and Ha at the $Pr=0.733$. The numerical results show that by applying magnetic field has decreased the rate of convective heat transfer. The average Nusselt number decreases with an increase of Hartmann number.

Keywords. Natural Convection, Electrically Conducting Fluid, Magnetic Field, Aspect Ratio, Grashof number, Prandtl number, Nusselt number, Hartmann number.

REMOTE SENSING PROBLEM IN THE EXISTENCE OF ACOUSTIC NOISE IN THE OCEAN

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Abstract. The paper considers the problem of remote sensing of the ocean by a point isotropic sound source. The inverse problem is formulated, which consists in determining the coefficient of volume scattering in a weakly scattering medium. A formula is obtained for the calculation of the received signal taking into account acoustic noise, which is caused by a distributed sound source. Computational experiments were performed to analyze the solution of the inverse problem in the presence of acoustic noise in the medium.

Key words. Radiation transfer equation, volumetric scattering, sea bottom sonar, remote sensing.

In the single scattering approximation and point source the solution of the inverse problem (remote sensing problem) is obtained in the paper [1]. In this paper authors consider the remote sensing problem taking into account acoustical noise in the ocean.

Echo signal propagates in the medium $G \equiv \mathbb{R}^2$. Let J_0 describe a point isotropic sound source located at the origin O and emitted a pulse in the initial time. J_1 is the round distributed source located at the point r_0 .

$$J_0(\mathbf{r}, \mathbf{k}, t) = P_0 \delta(\mathbf{r}) \delta(t), J_1(\mathbf{r}, \mathbf{k}, t) = \begin{cases} P_1, & |\mathbf{r} - \mathbf{r}_0| < R \\ 0, & \text{иначе.} \end{cases} \quad (1)$$

Here, δ denotes the Dirac delta-function and P_0, P_1 are the powers of the sources J_0, J_1 , respectively. Authors assume that there are no sound sources in the medium up to the moment $t = 0$. The point receiver located at the point O . In the case of [1], the source was point and isotropic ($J = J_0$). Thus, the solution of the initial-boundary value problem:

$$I(\mathbf{O}, \mathbf{k}, t) = I_1(\mathbf{O}, \mathbf{k}, t) = \frac{J_1}{|\Omega|} \exp(-\mu ct) \sigma \left(\frac{ct}{2} \mathbf{k} \right) \frac{1}{t}. \quad (2)$$

Redefined $\mathbf{r} = ct\mathbf{k}/2$ and received signal $\bar{I}(\mathbf{r}, t) := I(\mathbf{O}, \mathbf{k}, t)$ in the equation (2) authors obtain the relation for determining the coefficient of volume scattering [1]:

$$\sigma(\mathbf{r}) = \frac{|\Omega| t \exp(\mu ct)}{P_0} \bar{I}(\mathbf{r}, t), \mathbf{r} \in \mathbb{R}^2. \quad (3)$$

Equation (3) is a solution of the inverse problem for determining the volume scattering coefficient σ in the single scattering approximation, a point source, emitted a pulse at the initial time, and in the case of a receiver concentrated at the origin.

The signal measured by the receiver in the single scattering approximation with a combined source $J = J_0 + J_1$ takes the form of

$$I(\mathbf{O}, \mathbf{k}, t) = I_0(\mathbf{O}, \mathbf{k}, t) + I_1(\mathbf{O}, \mathbf{k}, t) = \int_0^{ct} \exp(\mu t') J_1 \left(-t' \mathbf{k}, \mathbf{k}, t - \frac{t'}{c} \right) dt' + \frac{P_0}{|\Omega|} \exp(-\mu ct) \sigma \left(\frac{ct}{2} \mathbf{k} \right) \frac{1}{t} + \frac{1}{|\Omega|} \int_0^{ct} \exp(-\mu t') \sigma(-t' \mathbf{k}) \int_0^R \int_0^{2\pi} J_1 \left(\mathbf{x} - t' \mathbf{k}, -\frac{\mathbf{x}}{|\mathbf{x}|}, t - \frac{t' + |\mathbf{x}|}{c} \right) \frac{\exp(-\mu |\mathbf{x}|)}{|\mathbf{x}|} \rho d\rho d\varphi dt', \quad (4)$$

where $\mathbf{x} = \mathbf{r}_0 + t' \mathbf{k} + \rho(\cos\varphi, \sin\varphi)$, and the first term is responsible for the signal that does not undergo scattering in the medium.

The goal of the computational experiment is analyzing of the influence of the additional source J_1 in the solution of the remote sensing problem which is obtained in the case of a point isotropic source. From the practical point

of view, authors consider the problem of remote sensing in the ocean. By scattering coefficient σ , they simulate biological objects (plankton clusters, schools of fish). J_1 describe an active source in the ocean (acoustic buoy, autonomous unmanned vehicle or submarine).

The received signal is calculated by the formula (4). Then the inverse problem of determining the coefficient of volume scattering is solved as equation (3). To solve the problem of numerical integration, the Monte Carlo method is used with the number of nodes equal to N . The volume scattering coefficient σ is set as

$$\sigma(y_1, y_2) = \begin{cases} 0.04, & \text{if } \sqrt{(y_1 - 40)^2 + (y_2 + 40)^2} < 10 \\ 0.02, & \text{if } \sqrt{(y_1 - 40)^2 + (y_2 - 40)^2} < 20 \\ 0.01, & \text{else.} \end{cases}$$

Numerical experiments are carried out with the following sensing parameters: $\mu = 0.9[\text{m}^{-1}]$, $c = 1500[\text{m/s}]$, $x_1 \in (-80, 80)[\text{m}]$, $x_2 \in [0, 40][\text{m}]$, $P_0 = 1$, $P_1 = \frac{10^{-4}}{\pi R^2}$, $R = 15[\text{m}]$, $r_0 = (0; 56)$.

Fig. 1 Distribution of volume scattering coefficient. Left: exact solution and position of sources (yellow color). Right: recovered function σ .

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MIRIMANOFF POLYNOMIALS: APPROACH TO HYPERGEOMETRIC AND GENERATING FUNCTIONS

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Abstract. The aim of the this presentation is to survey and study some fundamental properties of the Mirimanoff polynomials with the aid of hypergeometric functions (hypergeometric series) and Apostol type and Frobenius type polynomials and numbers. We give some formulas and identities for these polynomials. The results of this presentation will be discussed comparatively with the results given previously studies and works.

Key words. Mirimanoff polynomials, Frobenius-Euler numbers and polynomials, Bernoulli and Euler numbers and polynomials, Generating functions, Hypergeometric functions.

Introduction

Hypergeometric series and generating functions are among the most widely used subjects of physics, engineering and other sciences, especially in almost all fields of mathematics. Using the techniques and methods of these functions, the results of this presentation will be presented.

The Mirimanoff polynomials are defined by the following formula:

$$f_m(x, n) = \sum_{j=0}^{n-1} j^m x^j \quad (1)$$

(cf. Carlitz 1959; Vandiver 1942).

Equation (1) is also modified as follows:

$$f_m(x, n, k) = \sum_{j=0}^{n-1} (j+k)^m x^j \quad (2)$$

(cf. Carlitz 1959; Vandiver 1942). By using (2), we have

$$f_m(x, n, k) = \sum_{v=0}^m \binom{m}{v} k^{m-v} f_v(x, n).$$

In [Simsek 2019], we gave

$$\left(\frac{1}{\lambda}\right)^{1-n} f_m\left(\frac{1}{\lambda}, n, x\right) = \frac{1}{1-\lambda} (H_m(x+n; \lambda) - \lambda^n H_m(x; \lambda))$$

where $H_m(x; \lambda)$ denotes the Frobenius-Euler polynomials which are defined by means of the following generating function:

$$G_F(t, x; u) = \frac{1-u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(x; u) \frac{t^n}{n!} \quad (3)$$

(cf. Apostol 1951-Vandiver 1942).
Substituting $u = -1$ into (3), we get

$$H_n(x; -1) = E_n(x)$$

where $E_n(x)$ denotes the Euler polynomials.

The polynomials $P(x; m, n; \lambda, p)$ including sums of higher power of binomial coefficients are defined by means of the following generating function:

$$\begin{aligned} G(t, x; \lambda, n, p) &= \frac{1}{n!} {}_pF_{p-1} \left[\begin{matrix} -n, -n, \dots, -n \\ 1, 1, \dots, 1 \end{matrix}; (-1)^p \lambda e^t \right] {}_pF_p \left[\begin{matrix} 1, 1, \dots, 1 \\ 1, 1, \dots, 1 \end{matrix}; xt \right] \\ &= \sum_{m=0}^{\infty} P(x; m, n; \lambda, p) \frac{t^m}{m!} \end{aligned}$$

where $n, p \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ or $\mathbb{C}$ (cf. Djordjevic, Milovanovic 2014; Koepf 2014; Temme 1996).

In [Simsek 2019], we gave

$$f_m\left(\frac{1}{\lambda}, n, x\right) = n! P\left(x; m, n-1; \frac{1}{\lambda}, 0\right).$$

Substituting $x = a$ into (1), we have

$$f_m(a, n) = \sum_{j=0}^{n-1} j^m a^j = \frac{a^{m+1} \mathfrak{B}_{m+1}(n; a) - \mathfrak{B}_{m+1}(a)}{m+1}$$

where $\mathfrak{B}_m(x; \lambda)$ denotes the Apostol Bernoulli polynomials which are defined by means of the following generating function:

$$\frac{t}{\lambda e^t - 1} e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x; \lambda) \frac{t^n}{n!}$$

where $\mathfrak{B}_n(\lambda) = \mathfrak{B}_n(0; \lambda)$ denotes the Apostol Bernoulli numbers (cf. Apostol 1951-Vandiver 1942).

Substituting $a = 1$ into the above equation, we have

$$f_m(1, n) = \sum_{j=0}^{n-1} j^m = \frac{B_{m+1}(n) - B_{m+1}}{m+1},$$

where $B_{m+1}(n)$ denotes Bernoulli polynomials (cf. Apostol 1951-Vandiver 1942).

Substituting $x = -1$ into (1), we have

$$f_m(-1, n) = \sum_{j=0}^{n-1} (-1)^j j^m = \frac{(-1)^{n-1} E_{m+1}(n) - E_{m+1}}{2}$$

(cf. Kim 2005-Simsek 2019) where $E_m(x)$ denotes the Euler polynomials which are defined by means of the following generating function:

$$\frac{2}{e^t + 1} e^{xt} = \sum_{m=0}^{\infty} E_m(x) \frac{t^m}{m!}$$

where $E_m = E_m(0)$ denotes the Euler numbers (cf. Apostol 1951-Vandiver 1942).

Conclusion

The result of this paper is related to the polynomials $P(x; m, n; \lambda, p)$ which related to the various kind of the special numbers and polynomials and also special functions including the Mirimanoff polynomials, the Frobenius-Euler numbers and polynomials, the Bernoulli numbers and polynomials,

the Euler numbers and polynomials. The results of this presentation may give potentials applications for other areas.

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